



On the Fourier–Jacobi expansion of the unitary Kudla lift

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ABSTRACT

We study a theta lift from a cusp form f on $U(1, 1)$ to a cusp form $\mathcal{L}f$ on $U(2, 1)$ (the unitary Kudla lift of f). We give an explicit expression of the Fourier–Jacobi expansion of $\mathcal{L}f$ in terms of periods and Hecke eigenvalues of f . As an application, we give a criterion for the nonvanishing of $\mathcal{L}f$.

1. Introduction

1.1 Let f be a holomorphic cusp form on $U(1, 1)$. In [Kud81], Kudla constructed a holomorphic cusp form $\mathcal{L}f$ on $U(2, 1)$ by integrating f against a theta kernel on $U(2, 1) \times U(1, 1)$ (see also [Kud79]). Furthermore, he showed that $\mathcal{L}f$ is a Hecke eigenform if so is f , and gave a relation between an automorphic L -function of $\mathcal{L}f$ and that of f . The main object of the paper is to study the Fourier–Jacobi expansion of $\mathcal{L}f$. In particular, we show that the *primitive components* of $\mathcal{L}f$ are expressed in terms of certain $U(1)$ -periods of f , and that *nonprimitive components* of $\mathcal{L}f$ satisfy certain recursion relations if f is a Hecke eigenform. As an application of these results, we obtain a criterion for the nonvanishing of $\mathcal{L}f$ (for a nonvanishing criterion in a representation theoretic setting, we refer to [GRS97] and [Tan99]; see the remark after Main Theorem III in § 1.4).

1.2 To be more precise, let K be an imaginary quadratic field of discriminant D . Let $G = U(2, 1) = U(S)$ and $H = U(1, 1) = U(T)$, where

$$S = \begin{bmatrix} & & 1/\sqrt{D} \\ & 1 & \\ -1/\sqrt{D} & & \end{bmatrix}, \quad T = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

As is well known, (G, H) forms a dual reductive pair. Let \mathcal{X} be the set of Hecke characters χ of K satisfying $\chi|_{\mathbb{Q}_{\mathbf{A}}^{\times}} = \omega$, where ω denotes the quadratic Hecke character of \mathbb{Q} corresponding to K/\mathbb{Q} . For $\chi \in \mathcal{X}$, let $w_{\infty}(\chi)$ be the integer such that $\chi(z_{\infty}) = (z_{\infty}/|z_{\infty}|)^{w_{\infty}(\chi)}$ for $z_{\infty} \in K_{\infty}^{\times} = \mathbb{C}^{\times}$. Fix a positive integer l divisible by $w(K)$, the number of roots of unity in K . We also fix a character $\chi_0 \in \mathcal{X}$ with $w_{\infty}(\chi_0) = -1$. The character χ_0 determines a splitting \mathcal{M}_{χ_0} of metaplectic representations of $G_{\mathbf{A}} \times H_{\mathbf{A}}$ on $\mathcal{S}(K_{\mathbf{A}}^2) \otimes \mathcal{S}(K_{\mathbf{A}})$, where $\mathcal{S}(K_{\mathbf{A}}^m)$ denotes the space of Schwartz–Bruhat functions on $K_{\mathbf{A}}^m$. We then construct a theta kernel

$$\theta(g, h) = \chi_0^{-1}(\det g)\chi_0^{-2}(\det h) \sum_{X \in K^2, \xi \in K} \mathcal{M}_{\chi_0}(g \times h)(\Phi_0 \otimes \varphi_0)(X, \xi)$$

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with a suitably chosen test function $\Phi_0 \otimes \varphi_0 \in \mathcal{S}(K_{\mathbf{A}}^2) \otimes \mathcal{S}(K_{\mathbf{A}})$ depending on l . Let f be a holomorphic cusp form on $H_{\mathbf{A}}$ of weight $l - 1$, level D and character $\tilde{\chi}_0$ (cf. § 3.3). Set

$$\mathcal{L}f(g) = \int_{H_{\mathbb{Q}} \backslash H_{\mathbf{A}}} f(h)\theta(g, h) dh \quad (g \in G_{\mathbf{A}}).$$

The theta lift $\mathcal{L}f$ was first studied by Kudla and we call $\mathcal{L}f$ the *unitary Kudla lift* of f . Kudla showed the following [Kud79, Kud81]:

- (1.1) $\mathcal{L}f$ is a holomorphic cusp form on $G_{\mathbf{A}}$ of weight l and level 1;
- (1.2) if f is a Hecke eigenform, so is $\mathcal{L}f$.

He also gave a relation between an automorphic L -function of $\mathcal{L}f$ and that of f .

1.3 In [Shi79], Shintani investigated the theory of (refined) Fourier–Jacobi expansions of automorphic forms on G . It should be noted that Piatetski–Shapiro independently developed a similar theory from a more representation theoretic point of view (see [Pia80] and [BPR03]). In this subsection, we briefly recall Shintani’s theory. For details, see §§ 4 and 5. Let R be an algebraic subgroup of G given by

$$R_{\mathbb{Q}} = \left\{ (w, x)t := \begin{bmatrix} 1 & \sqrt{D}w^{\sigma} & x + \sqrt{D}ww^{\sigma}/2 \\ 0 & 1 & w \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & t & \\ & & 1 \end{bmatrix} \mid w \in K, x \in \mathbb{Q}, t \in K^1 \right\},$$

where σ stands for the nontrivial automorphism of K/\mathbb{Q} and $K^1 = \{t \in K^{\times} \mid tt^{\sigma} = 1\}$. Denote by ψ the additive character of $\mathbb{Q}_{\mathbf{A}}/\mathbb{Q}$ determined by $\psi(x_{\infty}) = \exp(2\pi\sqrt{-1}x_{\infty})$ ($x_{\infty} \in \mathbb{R}$). Let F be a holomorphic cusp form on $G_{\mathbf{A}}$ of weight l on $\mathcal{K}_f = \prod_{p < \infty} G(\mathbb{Z}_p)$ (cf. § 3.1). Then F admits the Fourier–Jacobi expansion

$$F(g) = \sum_{m \in \mathbb{Q}, m > 0} F^m(g) \quad (g \in G_{\mathbf{A}}),$$

where

$$F^m(g) = \int_{\mathbb{Q} \backslash \mathbb{Q}_{\mathbf{A}}} \psi(-mx)F((0, x)g) dx.$$

Let \mathfrak{a} be a nonzero fractional ideal of K and take an element α_f of $K_{\mathbf{A},f}^{\times}$ (the finite part of $K_{\mathbf{A}}^{\times}$) corresponding to \mathfrak{a} (cf. § 2.2). Let $F_{\mathfrak{a}}^m$ be the function on $R_{\mathbb{Q}} \backslash R_{\mathbf{A}}$ given by $F_{\mathfrak{a}}^m(r) = F^m(r \cdot \text{diag}(\alpha_f^{\sigma}, 1, \alpha_f^{-1}))$ ($r \in R_{\mathbf{A}}$). Let $\chi \in \mathcal{X}$ with $w_{\infty}(\chi) = -1$ and suppose that χ satisfies certain conditions on its Artin conductor and epsilon factor. Then there exists a nonzero theta function $\Theta_{\mathfrak{a},\chi}^m: R_{\mathbb{Q}} \backslash R_{\mathbf{A}} \rightarrow \mathbb{C}$, which is *primitive* and an eigenfunction with respect to the action of $U(1)$ with eigenvalues determined by χ (for a more precise statement, see § 4.6). It is known that $\Theta_{\mathfrak{a},\chi}^m$ is unique up to constant multiples (see Theorem 4.4). For a fractional ideal \mathfrak{b} of K with $\mathfrak{b} \subset \mathfrak{a}$, put

$$C_F(\mathfrak{b}; \Theta_{\mathfrak{a},\chi}^m) := \int_{R_{\mathbb{Q}} \backslash R_{\mathbf{A}}} F_{\mathfrak{b}}^m(r) \overline{\Theta_{\mathfrak{a},\chi}^m(r)} dr.$$

We call $\{C_F(\mathfrak{b}; \Theta_{\mathfrak{a},\chi}^m) \mid m, \mathfrak{a}, \mathfrak{b}, \chi\}$ the *Fourier–Jacobi components* of F . If $\mathfrak{b} = \mathfrak{a}$, the Fourier–Jacobi component $C_F(\mathfrak{a}; \Theta_{\mathfrak{a},\chi}^m)$ is said to be *primitive*. In his pioneering work [Shi79], Shintani showed the following results (cf. § 5):

- (1.3) F is completely determined by its Fourier–Jacobi components;
- (1.4) F is a Hecke eigenform if and only if, for each (m, \mathfrak{a}, χ) , $\{C_F(\mathfrak{b}; \Theta_{\mathfrak{a},\chi}^m) \mid \mathfrak{b} \subset \mathfrak{a}\}$ satisfies certain recursion relations;
- (1.5) assume that F is a Hecke eigenform, then $F \neq 0$ if and only if at least one of the primitive components of F is nonzero.

1.4 Let $\mathcal{L}f$ be the unitary Kudla lift of a holomorphic cusp form f on $H_{\mathbf{A}}$ of weight $l - 1$, level D and character $\tilde{\chi}_0$. Let (m, \mathfrak{a}, χ) be as in § 1.3. Set

$$W_{\chi, f}^m(h) = \int_{K^1 \backslash K_{\mathbf{A}}^1} (\chi/\chi_0)^1(t^{-1})f(\iota_m(t)h) d^\times t \quad (h \in H_{\mathbf{A}}),$$

where ι_m is an embedding of K^1 into H depending on m defined by (7.6) (for the definition of $(\chi/\chi_0)^1$, see § 2.5). The function $W_{\chi, f}^m$ on $H_{\mathbf{A}}$ can be seen as a spherical function for $(U(1, 1), U(1))$. We now state the main results of the paper.

MAIN THEOREM I (Theorem 7.5). *The primitive Fourier–Jacobi components of $\mathcal{L}f$ are given as follows:*

$$C_{\mathcal{L}f}(\mathfrak{a}; \Theta_{\mathfrak{a}, \chi}^m) = c(m, \mathfrak{a}, \chi) \cdot \overline{I(\Theta_{\mathfrak{a}, \chi}^m)} \cdot J(m, \mathfrak{a}, \chi, f),$$

where $c(m, \mathfrak{a}, \chi)$ is a nonzero explicit constant depending only on (m, \mathfrak{a}, χ) ,

$$I(\Theta_{\mathfrak{a}, \chi}^m) = \int_{K^1 \backslash K_{\mathbf{A}}^1} \Theta_{\mathfrak{a}, \chi}^m(t) d^\times t$$

is the period of $\Theta_{\mathfrak{a}, \chi}^m$ and $J(m, \mathfrak{a}, \chi, f)$ is a linear combination of $W_{\chi, f}^m(h_{j, m, \mathfrak{a}})$ with some $h_{j, m, \mathfrak{a}} \in H_{\mathbf{A}}$. (Note that $J(m, \mathfrak{a}, \chi, f)$ is a linear combination of ‘CM-values’ of f ; see the remark in § 7.6.)

MAIN THEOREM II (Theorem 7.6). *Suppose that f is a Hecke eigenform. Then, for each (m, \mathfrak{a}, χ) , $\{C_{\mathcal{L}f}(\mathfrak{b}; \Theta_{\mathfrak{a}, \chi}^m) \mid \mathfrak{b} \subset \mathfrak{a}\}$ satisfies certain recursion relations. This implies that $\mathcal{L}f$ is a Hecke eigenform and $L(\mathcal{L}f, \xi; s) = L(\xi; s)L(f, \xi; s)$ for a Hecke character ξ of K (Corollary 7.7).*

MAIN THEOREM III (Theorem 7.8). *Suppose that f is a Hecke eigenform. Then $\mathcal{L}f \neq 0$ if and only if there exists a triplet (m, \mathfrak{a}, χ) such that $L(\chi; 1/2)J(m, \mathfrak{a}, \chi, f) \neq 0$.*

Remark. It should be noted that Gelbart, Rogawski and Soudry [GRS97] studied the nonvanishing of a theta lift from $U(2)$ to $U(3)$ in a representation theoretic setting (see also [Tan99]). To be more precise, let $\sigma = \bigotimes_v \sigma_v$ be a cuspidal representation of $U(2)$, and suppose that the theta lift of σ to $U(1)$ is zero. Then they showed that the theta lift of σ given by theta integrals is nonzero precisely when the local Howe lift of each component σ_v is nonzero. On the other hand, our criterion is concerned with the nonvanishing of the theta lift of an automorphic form (an *element* of the space of the automorphic representation), and is derived from the study of the Fourier–Jacobi expansion of the theta lift.

Remark. The inner product formula for the unitary Kudla lift has been given in [MS06].

1.5 The paper is organized as follows. The first five sections are of a preliminary nature. After fixing the notation in § 2, we define automorphic forms on $U(1, 1)$ and $U(2, 1)$ in § 3. In § 4, we recall several fundamental properties of theta functions on R after Shintani’s work [Shi79] and our preceding papers [MS00, MS02]. In § 5, we review Shintani’s theory of the Fourier–Jacobi expansion of holomorphic automorphic forms on $U(2, 1)$ and reformulate Shintani’s result mentioned above. In § 6, we study metaplectic representations of $U(2, 1) \times U(1, 1)$. We use a mixed model of the metaplectic representation, which is more convenient in the calculation of the Fourier–Jacobi components than the Schrödinger model. The intertwining operator between two models is given in § 6.6. In § 7, we define the unitary Kudla lift and state the main results of the paper (Theorems 7.5, 7.6 and 7.8), whose proofs occupy the remaining sections. The key of the proofs is Lemma 8.3, in which we use the properties of metaplectic representations in an essential way. By using this, we obtain a formula relating the Fourier–Jacobi components of $\mathcal{L}f$ to certain integrals of spherical functions $W_{\chi, f}^m$ in § 8 (Proposition 8.4). Using this and the results of § 5, we give another proof of (1.1) in § 9. In § 10, we reduce the proof of Theorems 7.5 and 7.6 to the calculation of certain local integrals introduced in this section (see Theorems 10.1 and 10.2). The calculations are carried out in § 11 for the primitive case and in § 12 for the nonprimitive case.

2. Notation

2.1 An index of notation is given in Table 1. As usual, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and \mathbb{C} denote the ring of rational integers, the rational number field, the field of real and complex numbers, respectively. For $z \in \mathbb{C}$ and $m \in \mathbb{Q}$, put $\mathbf{e}[z] = \exp(2\pi\sqrt{-1}z)$ and $\mathbf{e}_m[z] = \mathbf{e}[mz]$. For a finite prime p , let \mathbb{Z}_p be the p -adic integer ring and \mathbb{Q}_p the p -adic number field. We put $\mathbb{Z}_f = \prod_{p < \infty} \mathbb{Z}_p$. Let $\mathbb{Q}_{\mathbf{A}}$ denote the adèle ring of \mathbb{Q} . For a prime v of \mathbb{Q} , $|\cdot|_v$ stands for the absolute value of \mathbb{Q}_v^\times . When $v = p < \infty$, the normalized additive valuation $\text{ord}_p: \mathbb{Q}_p^\times \rightarrow \mathbb{Z}$ is given by $|a|_p = p^{-\text{ord}_p a}$ ($a \in \mathbb{Q}_p^\times$). We denote by $|a|_{\mathbf{A}} = \prod_v |a_v|_v$ the idèle norm of $a = (a_v)_v \in \mathbb{Q}_{\mathbf{A}}^\times$. Let ψ be the additive character of $\mathbb{Q}_{\mathbf{A}}/\mathbb{Q}$ satisfying $\psi(x_\infty) = \mathbf{e}[x_\infty]$ ($x_\infty \in \mathbb{R}$). For $m \in \mathbb{Q}$, we put $\psi_m(x) = \psi(mx)$ for $x \in \mathbb{Q}_{\mathbf{A}}$. Let ψ_v (respectively $\psi_{m,v}$) be the restriction of ψ (respectively ψ_m) to \mathbb{Q}_v for each prime v of \mathbb{Q} .

Let X be a linear algebraic group defined over \mathbb{Q} . For a prime v of \mathbb{Q} , X_v stands for the group of \mathbb{Q}_v -rational points of X . Denote by $X_{\mathbf{A}}$ and $X_{\mathbf{A},f}$ the adélization of X and its finite part, respectively.

Let $\text{diag}(a_1, \dots, a_n)$ be the diagonal matrix of degree n with the i th diagonal component a_i for $i = 1, \dots, n$. For $x \in \mathbb{R}$, we write $[x]$ for the integer with $0 \leq x - [x] < 1$. If P is a condition, we put $\delta(P) = 1$ if P holds, and $\delta(P) = 0$ otherwise. For a set C , char_C denotes the characteristic function of C .

2.2 Let K be an imaginary quadratic field of discriminant D with integer ring \mathcal{O}_K . We fix an embedding of K into \mathbb{C} and let $\kappa = \sqrt{D}$ so that $\text{Im}(\kappa) > 0$. Denote by σ the nontrivial automorphism of K/\mathbb{Q} . For $z \in K$, we write $\text{Tr}(z)$ and $\text{N}(z)$ for $\text{Tr}_{K/\mathbb{Q}}(z) = z + z^\sigma$ and $\text{N}_{K/\mathbb{Q}}(z) = zz^\sigma$, respectively, if there is no fear of confusion. We put $\psi_K(z) = \psi(\text{Tr}(z))$ ($z \in K_{\mathbf{A}}$). For a prime v of \mathbb{Q} , let $K_v = K \otimes_{\mathbb{Q}} \mathbb{Q}_v$. For a finite prime p , put $\mathcal{O}_{K,p} = \mathcal{O}_K \otimes_{\mathbb{Z}} \mathbb{Z}_p$. We set $\mathcal{O}_{K,f} = \prod_{p < \infty} \mathcal{O}_{K,p}$. By an ideal of K , we always mean a nonzero fractional ideal of K . For an ideal \mathfrak{a} of K and a finite prime p , we put $\mathfrak{a}_p = \mathfrak{a} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,p}$. Let $\mathfrak{a}_f = \mathfrak{a} \otimes_{\mathcal{O}_K} \mathcal{O}_{K,f} = \prod_{p < \infty} \mathfrak{a}_p$. For $\alpha = \alpha_\infty \alpha_f \in K_{\mathbf{A}}^\times$, we denote by $\text{id}(\alpha)$ the ideal \mathfrak{a} of K determined by $\mathfrak{a}_f = \alpha_f \mathcal{O}_{K,f}$. Let $K^1 = \{t \in K^\times \mid tt^\sigma = 1\}$. We put $\mathcal{O}_{K,f}^1 = K_{\mathbf{A},f}^1 \cap \mathcal{O}_{K,f}^\times = \prod_{p < \infty} \mathcal{O}_{K,p}^1$, where $\mathcal{O}_{K,p}^1 = K_p^1 \cap \mathcal{O}_{K,p}^\times$. When p splits in K/\mathbb{Q} , we fix an identification K_p with $\mathbb{Q}_p \oplus \mathbb{Q}_p$, and put $\Pi_{p,1} = (p, 1)$ and $\Pi_{p,2} = (1, p)$. When p ramifies in K/\mathbb{Q} (namely $p|D$), we fix a prime element Π_p of K_p and put $\pi_p = \text{N}(\Pi_p)$. When p is inert or splits in K/\mathbb{Q} , we take and fix a prime element π_p of \mathbb{Q}_p .

Let θ be an element of \mathcal{O}_K such that $\mathcal{O}_K = \mathbb{Z} + \mathbb{Z}\theta, \theta - \theta^\sigma = \kappa$ and $\text{ord}_p \text{N}_{K/\mathbb{Q}}(\theta) = 1$ for every p ramified in K/\mathbb{Q} . For example, we can take θ as

$$\theta = \frac{D' + \sqrt{D}}{2}, \quad D' = \begin{cases} D & \text{if } D \text{ is odd,} \\ D/2 & \text{if } D \text{ is even.} \end{cases}$$

For $A \in M_{m,n}(K)$, put $A^* = {}^t A^\sigma$. Let $A \in M_n(K)$ be a hermitian (respectively an anti-hermitian) matrix of degree n , namely $A^* = A$ (respectively $A^* = -A$). Let $W = K^n$ be the K -vector space of column vectors of dimension n coefficients in K . For $w, w' \in W$, we put $A(w, w') = w^* A w'$ and $A[w] = A(w, w)$. If A is anti-hermitian, we define an alternating \mathbb{Q} -form $\langle \cdot, \cdot \rangle_A$ on W by $\langle w, w' \rangle_A = \text{Tr}(w^* A w')$ ($w, w' \in W$).

For a prime v of \mathbb{Q} , we denote by $\mathcal{S}(K_v^m)$ the space of Schwartz–Bruhat functions on K_v^m .

2.3 Let

$$S = \begin{bmatrix} & & \kappa^{-1} \\ & 1 & \\ -\kappa^{-1} & & \end{bmatrix}, \quad T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

TABLE 1. Index of notation.

§	Notation
2.1	$e[z], \mathbf{e}_m[z], \psi, \psi_m, \psi_v, [x], \delta(P), \text{char}_C$
2.2	$K, D, \mathcal{O}_K, \kappa, \sigma, \text{Tr}(z), \text{N}(z), \psi_K(z), K_v, \mathcal{O}_{K,p}, \mathcal{O}_{K,f}, \mathbf{a}_p, \mathbf{a}_f, \text{id}(\alpha), 1, \mathcal{O}_{K,p}^1, \mathcal{O}_{K,f}^1, \theta$
2.3	$S, T, G, H, \mathcal{K}_p, \mathcal{U}_p, \mathcal{U}'_p, \mathcal{K}_f, \mathcal{U}_f, \mathcal{U}'_f, \mathcal{D}, J, \mathcal{K}_\infty, \mathfrak{H}, j, \mathbf{d}_G(a, t), \mathbf{d}_G(a), (w, x), N, R, A, \mathbf{n}_H(x), \bar{\mathbf{n}}_H(x), \mathbf{d}_H(a), w_0, N^H, A^H$
2.4	$dx_v, dz_v, d_{\mathbf{a}}z, dx, dz, d^\times a, h(K^1), w(K)$
2.5	$\mathcal{X}, \omega, w_\infty(\chi), \mathcal{X}_v, \xi^1$
3.1	$\mathfrak{A}_l(\mathcal{K}_f), \mathfrak{S}_l(\mathcal{K}_f), \mathfrak{Y}_l, \mathfrak{S}_l(\mathcal{K}_f; \Omega^{-1})$
3.2	$\mathcal{H}_p^G, L(\xi; s), L(F, \xi; s)$
3.3	$\tilde{\chi}_{0,p}, \tilde{\chi}_0, j', A_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0), S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0), S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0; \chi_0\Omega)$
3.4	$\tilde{\mathcal{H}}_p^H, \mathcal{H}_p^H, L(f, \xi; s)$
4.1	$\mathbf{T}^m, \langle\langle, \rangle\rangle_R$
4.2	$\mathcal{L}, V_p^m, V_\infty^m, V_{L^2}^m, \rho^m$
4.3	$\lambda_{K_v}(\psi_{m,v}), \mathcal{M}_\chi^{\kappa,m}$
4.4	$\theta_{\chi,\Phi}^m, \mathbf{T}_\chi^m$
4.5	$\mathbf{T}^{m,k}, \mathbf{T}_\chi^{m,k}, \mathbf{T}_{\text{hol}}^m$
4.6	$N(\mathbf{a})_f, R(\mathbf{a})_f, \mathbf{T}^m(\mathbf{a}), \mathbf{T}_{\text{hol}}^m(\mathbf{a}), \mathbf{T}_{\text{hol}}^m(\mathbf{a}, \chi), \mathcal{P}'_{\mathbf{b}}, \mathbf{T}_{\text{hol,prim}}^m(\mathbf{a}), \mathbf{T}_{\text{hol,prim}}^m(\mathbf{a}, \chi)$
4.7	$A_p(\chi), \mathfrak{P}_p, \epsilon(s, \chi, \psi_{m,K_p}), \mu_p(m, \mathbf{a}), \delta_p$
4.8	$\mathcal{X}_{\text{prim}}(m, \mathbf{a}), \mathcal{X}_{\text{prim}}^+(m, \mathbf{a}), \mathcal{E}$
4.9	$\mathcal{R}(\mathbf{a})$
4.10	\mathcal{Q}_p
4.11	$I_\Theta(r), I(\Theta), C(\chi)$
5.1	F^m, F_α^m
5.2	$C_F(\mathbf{b}, \Theta_{\mathbf{a},\chi}^m)$
6.2	$\mathbf{H}, \mathbf{T}, g \otimes h$
6.3	$\mathcal{N}_T, \mathcal{N}_{\mathbf{T}}, \rho_T, \rho_{\mathbf{T}}$
6.4	$\mathcal{M}_{\chi_0}^T, \widetilde{\mathcal{M}}_{\chi_0}^{\mathbf{T}}$
6.5	$i, \tilde{\rho}_{\mathbf{T}}, \widetilde{\mathcal{M}}_{\chi_0}^{\mathbf{T}}, I$
6.6	$\Phi_{0,p}, \varphi_{0,p}, \Phi_{0,\infty}, \varphi_{0,\infty}, \Phi_0, \varphi_0$
7.1	$\mathcal{M}_{\chi_0}^{\mathbf{T}}, \theta$
7.2	$\mathcal{L}f$
7.3	$\eta_m, H_{\eta_m}, \iota_m$
7.4	$W_{\chi,f}^m$
7.6	$c_\infty, c_p, E(\chi), w_0^\epsilon, B(\chi, \epsilon), J(m, \mathbf{a}, \chi, f)$
8.3	$\Lambda_{\beta,\Theta}^m$
9.1	$\mathcal{W}_\infty^m(k)$
9.2	C_m
9.3	$k(\chi)$
9.4	$\tau, \mathcal{I}_{\beta_\infty}(W, \tau)$
10.1	$X_{\text{unr}}(K^1), \mathcal{W}^m(\chi), \mathcal{W}^m(\chi; \chi_0\Omega), T^m(\mathbf{a}, \chi), \mathcal{I}_{\mathbf{b}}(W, \tau)$
10.2	$v_0, \delta_e(\chi)$
10.3	$C^\infty(H/\mathcal{U}', \tilde{\chi}_0), \phi^\vee$

Then S (respectively $\sqrt{-1}^{-1}T$) is a hermitian matrix of signature $(2, 1)$ (respectively $(1, 1)$). Denote by G and H the unitary groups of S and T , respectively. By definition, $G_{\mathbb{Q}} = \{g \in \mathrm{GL}_3(K) \mid g^*Sg = S\}$ and $H_{\mathbb{Q}} = \{h \in \mathrm{GL}_2(K) \mid h^*Th = T\}$. We put

$$\mathcal{K}_p = G_p \cap \mathrm{GL}_3(\mathcal{O}_{K,p}), \quad \mathcal{U}_p = H_p \cap \mathrm{GL}_2(\mathcal{O}_{K,p}), \quad \mathcal{U}'_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{U}_p \mid c \in D\mathcal{O}_{K,p} \right\}.$$

Then \mathcal{K}_p (respectively \mathcal{U}_p) is a maximal open compact subgroup of G_p (respectively H_p), and \mathcal{U}'_p is an open subgroup of \mathcal{U}_p . Note that $\mathcal{U}_p = \mathcal{U}'_p$ unless $p \mid D$. We set

$$\mathcal{K}_f = \prod_{p < \infty} \mathcal{K}_p, \quad \mathcal{U}_f = \prod_{p < \infty} \mathcal{U}_p, \quad \mathcal{U}'_f = \prod_{p < \infty} \mathcal{U}'_p.$$

Let

$$\mathcal{D} = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in \mathbb{C}^2 \mid \frac{z - \bar{z}}{\kappa} - w\bar{w} > 0 \right\}$$

be a hermitian symmetric domain isomorphic to the complex two ball. The real points G_{∞} of G acts on \mathcal{D} as follows. For $g \in G_{\infty}$ and $Z = {}^t(z, w) \in \mathcal{D}$, there exist $g\langle Z \rangle \in \mathcal{D}$ and $J(g, Z) \in \mathbb{C}^{\times}$ such that $g \cdot Z = J(g, Z) \cdot (g\langle Z \rangle)^{\sim}$, where $Z^{\sim} = {}^t(z, w, 1) \in \mathbb{C}^3$. Then $(g, Z) \mapsto g\langle Z \rangle$ defines a holomorphic action of G_{∞} on \mathcal{D} and $J: G_{\infty} \times \mathcal{D} \rightarrow \mathbb{C}^{\times}$ is a holomorphic automorphic factor. Let $\mathcal{K}_{\infty} = \{g \in G_{\infty} \mid g_{\infty}\langle Z_0 \rangle = Z_0\}$ be the stabilizer of $Z_0 = {}^t(\kappa/2, 0) \in \mathcal{D}$ in G_{∞} . It is known that $\mathcal{K}_{\infty} = G_{\infty} \cap U(S_0)$ with $S_0 = \mathrm{diag}(-2/D, 1, 1/2)$.

We define the action of H_{∞} on the upper half plane $\mathfrak{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ and the holomorphic automorphic factor $j: H_{\infty} \times \mathfrak{H} \rightarrow \mathbb{C}^{\times}$ in a usual way:

$$h\langle z \rangle = \frac{az + b}{cz + d}, \quad j(h, z) = cz + d$$

for $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_{\infty}$ and $z \in \mathfrak{H}$. Denote by \mathcal{U}_{∞} the stabilizer of $z_0 = \sqrt{-1} \in \mathfrak{H}$ in H_{∞} .

We put $\mathbf{d}_G(a, t) = \mathrm{diag}(a^{\sigma}, t, a^{-1})$ and $\mathbf{d}_G(a) = \mathbf{d}_G(a, 1)$ for $a \in K^{\times}, t \in K^1$, and

$$(w, x) = \begin{bmatrix} 1 & \kappa w^{\sigma} & x + \frac{\kappa}{2} w w^{\sigma} \\ & 1 & w \\ & & 1 \end{bmatrix} \quad (w \in K, x \in \mathbb{Q}).$$

Note that $(w, x)(w', x') = (w + w', x + x' + \frac{1}{2}\langle w, w' \rangle_{\kappa})$ and $\mathbf{d}_G(a, t) \cdot (w, x) \cdot \mathbf{d}_G(a, t)^{-1} = (atw, \mathbf{N}(a)x)$, where $\langle w, w' \rangle_{\kappa} = \mathrm{Tr}(\kappa w^{\sigma} w')$. We write nt for $n \cdot \mathbf{d}_G(1, t)$ if there is no fear of confusion (the scalar matrix $\mathrm{diag}(t, t, t)$ is always denoted by $t1_3$ throughout the paper). Define subgroups N, R and A of G by $N_{\mathbb{Q}} = \{(w, x) \mid w \in K, x \in \mathbb{Q}\}$, $R_{\mathbb{Q}} = \{nt \mid n \in N_{\mathbb{Q}}, t \in K^1\}$ and $A_{\mathbb{Q}} = \{\mathbf{d}_G(a) \mid a \in K^{\times}\}$. We see that $N_p \cap \mathcal{K}_p = \{(w, x + x_w) \mid w \in \mathcal{O}_{K,p}, x \in \mathbb{Z}_p\}$, where $x_w = 2^{-1} \mathrm{Tr} \theta w w^{\sigma}$ ($w \in K$).

Put

$$\mathbf{n}_H(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad \bar{\mathbf{n}}_H(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \quad (x \in \mathbb{Q})$$

and

$$\mathbf{d}_H(a) = \begin{pmatrix} a^{\sigma} & 0 \\ 0 & a^{-1} \end{pmatrix} \quad (a \in K^{\times}), \quad w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Define subgroups N^H and A^H of H by $N_{\mathbb{Q}}^H = \{\mathbf{n}_H(x) \mid x \in \mathbb{Q}\}$ and $A_{\mathbb{Q}}^H = \{\mathbf{d}_H(a) \mid a \in K^{\times}\}$.

We have the Iwasawa decompositions:

$$G_{\mathbf{A}} = R_{\mathbf{A}} A_{\mathbf{A}} \mathcal{K}_f \mathcal{K}_{\infty}, \quad H_{\mathbf{A}} = N_{\mathbf{A}}^H A_{\mathbf{A}}^H \mathcal{U}_f \mathcal{U}_{\infty}.$$

2.4 In this subsection, we normalize Haar measures on various groups. Let v be a prime of \mathbb{Q} . Let dx_v be the Haar measure on \mathbb{Q}_v self-dual with respect to the pairing $(x_v, x'_v) \mapsto \psi_v(x_v x'_v)$. Note that $\int_{\mathbb{Z}_p} dx_p = 1$ for $p < \infty$ and that dx_∞ is the usual Lebesgue measure on \mathbb{R} . Let dz_v be the Haar measure on K_v self-dual with respect to the pairing $(z_v, z'_v) \mapsto \psi_K(z_v^\sigma z'_v)$. Note that $\int_{\mathcal{O}_{K,p}} dz_p = |D|_p^{1/2}$ and $dz_\infty = 2d_L z_\infty$, where $d_L z_\infty$ is the usual Lebesgue measure on \mathbb{C} . For an ideal \mathfrak{a} of K_p , $d_{\mathfrak{a}}z$ stands for the Haar measure on K_p normalized by $\text{vol}(\mathfrak{a}) = 1$.

Let $dx = \prod_v dx_v$ and $dz = \prod_v dz_v$. Then dx and dz are the Haar measures on $\mathbb{Q}_{\mathbf{A}}$ and $K_{\mathbf{A}}$ with $\text{vol}(\mathbb{Q} \backslash \mathbb{Q}_{\mathbf{A}}) = 1$ and $\text{vol}(K \backslash K_{\mathbf{A}}) = 1$, respectively. Let $d^\times a = \prod_v d^\times a_v$ be the Haar measure on $K_{\mathbf{A}}^\times$, where $d^\times a_v$ is the Haar measure on K_v^\times normalized by $\text{vol}(\mathcal{O}_{K,p}^\times) = 1$ if $v = p < \infty$, and $d^\times a_\infty = da_\infty / N(a_\infty)$. Let $d^\times t = \prod_v d^\times t_v$ be the Haar measure on $K_{\mathbf{A}}^1$, where $d^\times t_v$ is the Haar measure on K_v^1 normalized by $\text{vol}(\mathcal{O}_{K,p}^1) = 1$ if $v = p < \infty$, and $\text{vol}(K_\infty^1) = 1$ if $v = \infty$. Under this normalization, we have $\text{vol}(K^1 \backslash K_{\mathbf{A}}^1) = h(K^1) / w(K)$, where $h(K^1) = \#(K^1 \backslash K_{\mathbf{A}}^1 / K_\infty^1 \mathcal{O}_{K,f}^1)$ is the class number of K^1 and $w(K)$ is the number of roots of unity in K .

The Haar measures on N_v and R_v are given as follows:

$$\begin{aligned} dn_v &= dw_v dx_v \quad (n_v = (w_v, x_v), w_v \in K_v, x_v \in \mathbb{Q}_v), \\ dr_v &= dn_v d^\times t_v \quad (r_v = n_v t_v, n_v \in N_v, t_v \in K_v^1). \end{aligned}$$

When $v = p < \infty$, we normalize Haar measures dg_p and dh_p on G_p and H_p by

$$\int_{\mathcal{K}_p} dg_p = 1, \quad \int_{\mathcal{U}_p} dh_p = 1.$$

When $v = \infty$, we normalize Haar measures dg_∞ and dh_∞ on G_∞ and H_∞ by

$$\int_{G_\infty} \Phi(g_\infty) dg_\infty = \int_{R_\infty} \int_{\mathbb{C}^\times} \int_{\mathcal{K}_\infty} |N(a_\infty)|_\infty^{-2} \Phi(r_\infty \mathbf{d}_G(a_\infty) k_\infty) dk_\infty d^\times a_\infty dr_\infty \quad (\Phi \in L^1(G_\infty))$$

and

$$\int_{H_\infty} \varphi(h_\infty) dh_\infty = \int_{\mathbb{R}} \int_{\mathbb{C}^\times} \int_{\mathcal{U}_\infty} |N(a_\infty)|_\infty^{-1} \varphi(\mathbf{n}_H(x) \mathbf{d}_H(a_\infty) u_\infty) du_\infty d^\times a_\infty dx_\infty \quad (\varphi \in L^1(H_\infty)).$$

Here dk_∞ (respectively du_∞) is normalized by $\text{vol}(\mathcal{K}_\infty) = 1$ (respectively $\text{vol}(\mathcal{U}_\infty) = 1$). Finally, let $dn = \prod_v dn_v, dr = \prod_v dr_v, dg = \prod_v dg_v$ and $dh = \prod_v dh_v$ be the Haar measures on $N_{\mathbf{A}}, R_{\mathbf{A}}, G_{\mathbf{A}}$ and $H_{\mathbf{A}}$, respectively.

2.5 Let \mathcal{X} be the set of unitary Hecke characters χ of K satisfying $\chi|_{\mathbb{Q}_{\mathbf{A}}^\times} = \omega$, where $\omega = \omega_{K/\mathbb{Q}}$ is the quadratic Hecke character of \mathbb{Q} corresponding to K/\mathbb{Q} . For $\chi \in \mathcal{X}$, let $w_\infty(\chi)$ be the integer such that $\chi(z_\infty) = (z_\infty / |z_\infty|)^{w_\infty(\chi)}$ for $z_\infty \in K_\infty^\times$. Note that $w_\infty(\chi)$ is always odd. For a prime v of \mathbb{Q} , let \mathcal{X}_v be the set of the set of unitary characters χ of K_v^\times with $\chi|_{\mathbb{Q}_v^\times} = \omega_v$. For a Hecke character ξ of K trivial on $\mathbb{Q}_{\mathbf{A}}^\times$, we define a character ξ^1 of $K_{\mathbf{A}}^1 / K^1$ by $\xi^1(z^\sigma / z) = \xi(z)$ ($z \in K_{\mathbf{A}}^\times$).

3. Automorphic forms on $U(2, 1)$ and $U(1, 1)$

3.1 Automorphic forms on $U(2, 1)$

We henceforth fix a positive integer l divisible by $w(K)$. We first recall the definition of holomorphic automorphic forms on $G = U(S)$. Let $\mathfrak{A}_l(\mathcal{K}_f)$ be the space of smooth functions F on $G_{\mathbb{Q}} \backslash G_{\mathbf{A}}$ satisfying:

- (i) $F(gk_f k_\infty) = J(k_\infty, Z_0)^{-l} F(g)$ ($g \in G_{\mathbf{A}}, k_f \in \mathcal{K}_f, k_\infty \in \mathcal{K}_\infty$);
- (ii) for any $g_f \in G_{\mathbf{A},f}, g_\infty \langle Z_0 \rangle \mapsto J(g_\infty, Z_0)^l F(g_f g_\infty)$ is holomorphic on \mathcal{D} .

Denote by $\mathfrak{S}_l(\mathcal{K}_f)$ the space of $F \in \mathfrak{A}_l(\mathcal{K}_f)$ satisfying

$$\int_{\mathbb{Q} \setminus \mathbb{Q}_{\mathbf{A}}} F((0, x)g) dx = 0 \quad (g \in G_{\mathbf{A}}).$$

We call $\mathfrak{A}_l(\mathcal{K}_f)$ (respectively $\mathfrak{S}_l(\mathcal{K}_f)$) the space of *holomorphic automorphic forms* (respectively *holomorphic cusp forms*) of weight l on \mathcal{K}_f . It is known that $F \in \mathfrak{S}_l(\mathcal{K}_f)$ is bounded on $G_{\mathbf{A}}$.

Let \mathcal{Y}_l be the set of unitary characters Ω of $K_{\mathbf{A}}^1/K^1$ satisfying $\Omega|_{\mathcal{O}_{K,f}^1} = 1$ and $\Omega(z_{\infty}) = z_{\infty}^l$ for $z_{\infty} \in K_{\infty}^1$. For $\Omega \in \mathcal{Y}_l$, let $\mathfrak{S}_l(\mathcal{K}_f; \Omega^{-1})$ be the space of $F \in \mathfrak{S}_l(\mathcal{K}_f)$ satisfying $F(t1_3 \cdot g) = \Omega^{-1}(t)F(g)$ for $t \in K_{\mathbf{A}}^1$. We then have $\mathfrak{S}_l(\mathcal{K}_f) = \bigoplus_{\Omega \in \mathcal{Y}_l} \mathfrak{S}_l(\mathcal{K}_f; \Omega^{-1})$.

3.2 Hecke operators and L -functions for $U(2, 1)$

For $p < \infty$, let $\mathcal{H}_p^G = \mathcal{H}(G_p, \mathcal{K}_p)$ be the Hecke algebra of (G_p, \mathcal{K}_p) . By definition, \mathcal{H}_p^G is the space of compactly supported bi- \mathcal{K}_p -invariant functions on G_p and the product is given by

$$(\Phi * \Phi')(g) = \int_{G_p} \Phi(gx^{-1})\Phi'(x) dx \quad (\Phi, \Phi' \in \mathcal{H}_p^G).$$

The \mathbb{C} -algebra \mathcal{H}_p^G acts on $\mathfrak{S}_l(\mathcal{K}_f; \Omega^{-1})$ by

$$(F * \Phi)(g) = \int_{G_p} F(gx^{-1})\Phi(x) dx \quad (F \in \mathfrak{S}_l(\mathcal{K}_f; \Omega^{-1}), \Phi \in \mathcal{H}_p^G).$$

The following facts are elementary (cf. [Shi79, Lemma 2] and [Kud81, p. 333]).

LEMMA 3.1. *Let $F \in \mathfrak{S}_l(\mathcal{K}_f; \Omega^{-1})$.*

- (i) *Suppose that p is inert in K/\mathbb{Q} . Then we have $\mathcal{H}_p^G = \mathbb{C}[\Phi_{1,p}]$ with $\Phi_{1,p} = \text{char}_{\mathcal{K}_p \mathbf{d}_G(p) \mathcal{K}_p}$ and*

$$\begin{aligned} (F * \Phi_{1,p})(g) &= F(g \mathbf{d}_G(p^{-1})) + \sum_{x \in (\mathbb{Z}_p - p\mathbb{Z}_p)/p\mathbb{Z}_p} F(g(0, p^{-1}x)) \\ &+ \sum_{w \in \mathcal{O}_{K,p}/p\mathcal{O}_{K,p}} \sum_{x \in \mathbb{Z}_p/p^2\mathbb{Z}_p} F(g(w, x + x_w) \mathbf{d}_G(p)). \end{aligned}$$

- (ii) *Suppose that p ramifies in K/\mathbb{Q} and let Π be a prime element of K_p . Then we have $\mathcal{H}_p^G = \mathbb{C}[\Phi_{1,p}]$ with $\Phi_{1,p} = \text{char}_{\mathcal{K}_p \mathbf{d}_G(\Pi) \mathcal{K}_p}$ and*

$$\begin{aligned} (F * \Phi_{1,p})(g) &= F(g \mathbf{d}_G(\Pi^{-1})) + \sum_{w \in (\mathcal{O}_{K,p} - \Pi \mathcal{O}_{K,p})/\Pi \mathcal{O}_{K,p}} F(g(\Pi^{-1}w, x_{\Pi^{-1}w})) \\ &+ \sum_{w \in \mathcal{O}_{K,p}/\Pi \mathcal{O}_{K,p}} \sum_{x \in \mathbb{Z}_p/p^2\mathbb{Z}_p} F(g(w, x + x_w) \mathbf{d}_G(\Pi)). \end{aligned}$$

- (iii) *Suppose that p splits in K/\mathbb{Q} and put $\Pi_1 = (p, 1)$ and $\Pi_2 = (1, p)$. Then we have $\mathcal{H}_p^G = \mathbb{C}[\Phi_{1,p}, \Phi_{2,p}, \Phi_{0,p}^{\pm}]$, where $\Phi_{i,p} = \text{char}_{\mathcal{K}_p \mathbf{d}_G(\Pi_i) \mathcal{K}_p}$ ($i = 1, 2$), $\Phi_{0,p}^{\pm} = \text{char}_{\mathcal{K}_p(\Pi_1/\Pi_2)^{\pm 1} 1_3 \mathcal{K}_p}$, and*

$$\begin{aligned} (F * \Phi_{i,p})(g) &= F(g \mathbf{d}_G(\Pi_i^{-1})) + \sum_{w \in \mathcal{O}_{K,p}/\Pi_i \mathcal{O}_{K,p}} F(g(w, x_w)(\Pi_i/\Pi_i^{\sigma})) \\ &+ \sum_{w \in \mathcal{O}_{K,p}/\Pi_i^{\sigma} \mathcal{O}_{K,p}} \sum_{x \in \mathbb{Z}_p/p^2\mathbb{Z}_p} F(g(w, x + x_w) \mathbf{d}_G(\Pi_i^{\sigma})) \quad (i = 1, 2), \\ (F * \Phi_{0,p}^{\pm})(g) &= F(g \cdot (\Pi_1/\Pi_2)^{\mp 1} 1_3) \quad (= \Omega^{\pm 1}(\Pi_1/\Pi_2)F(g)). \end{aligned}$$

If, for every $p < \infty$, there exists a \mathbb{C} -algebra homomorphism Λ_p of \mathcal{H}_p^G to \mathbb{C} such that $F * \Phi = \Lambda_p(\Phi)F$ holds for any $\Phi \in \mathcal{H}_p^G$, we say that F is a *Hecke eigenform* with eigenvalues $\{\Lambda_p\}$.

Let $F \in \mathfrak{S}_l(\mathcal{K}_f; \Omega^{-1})$ be a Hecke eigenform with eigenvalues $\{\Lambda_p\}$ and ξ a Hecke character of K . Let $L(\xi; s) = \prod_{p < \infty} L_p(\xi; s)$ be the Hecke L -function of ξ . We define an L -function

$$L(F, \xi; s) = \prod_{p < \infty} L_p(F, \xi; s),$$

where the local factor $L_p(F, \xi; s)$ is given as follows.

(i) If p is inert in K/\mathbb{Q} , put

$$L_p(F, \xi; s) = L_p(\xi; s)Q_p(\xi_p(p)p^{-2s})^{-1},$$

where

$$Q_p(X) = 1 - p^{-2}\{\Lambda_p(\Phi_{1,p}) - p + 1\}X + X^2.$$

(Here we make a convention that $\xi_p(p) = 0$ if $\xi|_{\mathcal{O}_{K,p}^\times} \neq 1$. We follow a similar convention in the later discussions.)

(ii) If p ramifies in K/\mathbb{Q} , put

$$L_p(F, \xi; s) = L_p(\xi; s)Q_p(\xi_p(\Pi)p^{-s})^{-1},$$

where

$$Q_p(X) = 1 - p^{-1}\{\Lambda_p(\Phi_{1,p}) - p + 1\}X + X^2.$$

(iii) If p splits in K/\mathbb{Q} , put

$$L_p(F, \xi; s) = Q_{1,p}(\xi_p(\Pi_1)p^{-s})^{-1}Q_{2,p}(\xi_p(\Pi_2)p^{-s})^{-1},$$

where

$$\begin{aligned} Q_{1,p}(X) &= 1 - p^{-1}\Lambda_2 X + p^{-1}\epsilon\Lambda_1 X^2 - \epsilon X^3, \\ Q_{2,p}(X) &= 1 - p^{-1}\Lambda_1 X + p^{-1}\epsilon^{-1}\Lambda_2 X^2 - \epsilon^{-1} X^3, \end{aligned}$$

with $\Lambda_i = \Lambda_p(\Phi_{i,p})$ ($i = 1, 2$) and $\epsilon = \Lambda_p(\Phi_{0,p}^+) = \Omega_p(\Pi_1/\Pi_2)$.

3.3 Automorphic forms on $U(1, 1)$

We next recall the definition of automorphic forms on $H = U(T)$. In what follows, we fix once and for all an element χ_0 of \mathcal{X} with $w_\infty(\chi_0) = -1$ (cf. § 2.5). For $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{U}'_p$, put

$$\tilde{\chi}_{0,p}(u) = \begin{cases} \chi_{0,p}(a) & \text{if } \in p\mathcal{O}_{K,p}, \\ \chi_{0,p}(c) & \text{otherwise.} \end{cases}$$

Note that $c \in \mathcal{O}_{K,p}^\times$ if $c \notin p\mathcal{O}_{K,p}$.

LEMMA 3.2. For any $p < \infty$, $\tilde{\chi}_{0,p}$ is a character of \mathcal{U}'_p .

Proof. The lemma is easily verified when $p|D$. Suppose that $p \nmid D$. For $u \in \mathcal{U}'_p = \mathcal{U}_p$, there exists a $t_u \in \mathcal{O}_{K,p}^\times$ such that $\det u = t_u^\sigma/t_u$. It is straightforward to see that $\tilde{\chi}_{0,p}(u) = \chi_{0,p}^{-1}(t_u)$, from which the lemma follows. \square

Define a character $\tilde{\chi}_0$ of \mathcal{U}'_f by $\tilde{\chi}_0 = \prod_{p < \infty} \tilde{\chi}_{0,p}$ and put

$$j'(h_\infty, z) = \det h_\infty^{-1} \cdot j(h_\infty, z) \quad (h_\infty \in H_\infty, z \in \mathfrak{H}).$$

Let $A_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ be the space of smooth functions f on $H_\mathbb{Q} \backslash H_\mathbf{A}$ satisfying the following three conditions:

(i) $f(hu_f u_\infty) = j'(u_\infty, z_0)^{1-l} \tilde{\chi}_0(u_f) f(h)$ ($h \in H_\mathbf{A}, u_f \in \mathcal{U}'_f, u_\infty \in \mathcal{U}_\infty$);

- (ii) for any $h_f \in H_{\mathbf{A},f}$, $h_\infty \langle z_0 \rangle \mapsto j'(h_\infty, z_0)^{l-1} f(h_f h_\infty)$ is holomorphic on \mathfrak{H} ;
- (iii) f is holomorphic at each cusp.

We denote by $S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ the space of $f \in A_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ satisfying

$$\int_{\mathbb{Q} \setminus \mathbb{Q}_{\mathbf{A}}} f(\mathbf{n}_H(x)h) dx = 0 \quad (h \in H_{\mathbf{A}}).$$

We call $A_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ (respectively $S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$) the space of *holomorphic automorphic forms* (respectively *holomorphic cusp forms*) of weight $l - 1$ on \mathcal{U}'_f of character $\tilde{\chi}_0$. It is known that $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ is bounded on $H_{\mathbf{A}}$.

As in §3.1, we have $S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0) = \bigoplus_{\Omega \in \mathcal{Y}_l} S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0; \chi_0 \Omega)$, where $S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0; \chi_0 \Omega)$ is the space of $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ satisfying $f(t1_2 \cdot h) = (\chi_0 \Omega)(t)f(h)$ for $t \in K_{\mathbf{A}}^1$.

3.4 Hecke operators and L-functions for $U(1, 1)$

For $p < \infty$, let $\tilde{\mathcal{H}}_p^H = \mathcal{H}(H_p, \mathcal{U}'_p; \tilde{\chi}_{0,p})$ be the space of compactly supported functions ϕ on H_p satisfying $\phi(u_1 h u_2) = \tilde{\chi}_{0,p}(u_1 u_2) \phi(h)$ ($h \in H_p, u_1, u_2 \in \mathcal{U}'_p$). Then $\tilde{\mathcal{H}}_p^H$ forms a \mathbb{C} -algebra with the product

$$(\phi * \phi')(h) = \text{vol}(\mathcal{U}'_p)^{-1} \int_{H_p} \phi(hx^{-1}) \phi'(x) dx \quad (\phi, \phi' \in \tilde{\mathcal{H}}_p^H, h \in H_p),$$

and acts on $S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0; \chi_0 \Omega)$ by

$$(f * \phi)(h) = \text{vol}(\mathcal{U}'_p)^{-1} \int_{H_p} f(hx^{-1}) \phi(x) dx \quad (f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0; \chi_0 \Omega), \phi \in \tilde{\mathcal{H}}_p^H).$$

First suppose that p is inert in K/\mathbb{Q} . Then $\tilde{\mathcal{H}}_p^H = \mathbb{C}[\phi_{1,p}]$, where $\phi_{1,p}$ is the element of $\tilde{\mathcal{H}}_p^H$ satisfying $\text{Supp } \phi_{1,p} \subset \mathcal{U}'_p \mathbf{d}_H(p) \mathcal{U}'_p$ and $\phi_{1,p}(\mathbf{d}_H(p)) = \chi_{0,p}^{-1}(p) = -1$.

Next suppose that p splits in K/\mathbb{Q} . Then $\tilde{\mathcal{H}}_p^H = \mathbb{C}[\phi_{1,p}, \phi_{2,p}, \phi_{0,p}^\pm]$, where $\phi_{1,p}, \phi_{2,p}, \phi_{0,p}^\pm$ are the elements of $\tilde{\mathcal{H}}_p^H$ satisfying $\text{Supp } \phi_{i,p} \subset \mathcal{U}'_p \mathbf{d}_H(\Pi_i) \mathcal{U}'_p$, $\phi_{i,p}(\mathbf{d}_H(\Pi_i)) = \chi_{0,p}^{-1}(\Pi_i)$ and $\text{Supp } \phi_{0,p}^\pm \subset \mathcal{U}'_p \mathbf{d}_H((\Pi_1/\Pi_2)^{\pm 1}) \mathcal{U}'_p = \mathcal{U}'_p (\Pi_1/\Pi_2)^{\mp 1} 1_2$, $\phi_{0,p}^\pm(\mathbf{d}_H((\Pi_1/\Pi_2)^{\pm 1})) = \chi_{0,p}^{-1}((\Pi_1/\Pi_2)^{\pm 1})$. We note that $\phi_{1,p} * \phi_{0,p}^- = \phi_{2,p}$. In both cases, $\tilde{\mathcal{H}}_p^H$ is commutative and we set $\mathcal{H}_p^H = \tilde{\mathcal{H}}_p^H$.

Finally, suppose that p ramifies in K/\mathbb{Q} . In this case, $\tilde{\mathcal{H}}_p^H$ is *not* commutative in general. Let ϕ_p^\pm be the elements of $\tilde{\mathcal{H}}_p^H$ satisfying $\text{Supp } \phi_p^\pm \subset \mathcal{U}'_p \mathbf{d}_H(\Pi^{\pm 1}) \mathcal{U}'_p$ and $\phi_p^\pm(\mathbf{d}_H(\Pi^{\pm 1})) = \chi_{0,p}^{-1}(\Pi^{\pm 1})$. We let \mathcal{H}_p^H the subalgebra of $\tilde{\mathcal{H}}_p^H$ generated by $\phi_p^+ + \phi_p^-$. Then \mathcal{H}_p^H is commutative.

LEMMA 3.3. *Let $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0; \chi_0 \Omega)$.*

- (i) *Suppose that p is inert in K/\mathbb{Q} . Then we have*

$$(f * \phi_{1,p})(h) = -f(h \mathbf{d}_H(p^{-1})) - \sum_{x \in (\mathbb{Z}_p - p\mathbb{Z}_p)/p\mathbb{Z}_p} f(h \mathbf{n}_H(p^{-1}x)) - \sum_{x \in \mathbb{Z}_p/p^2\mathbb{Z}_p} f(h \mathbf{n}_H(x) \mathbf{d}_H(p)).$$

- (ii) *Suppose that p ramifies in K/\mathbb{Q} . Then*

$$(f * (\phi_p^+ + \phi_p^-))(h) = \chi_{0,p}^{-1}(\Pi) \sum_{x \in \mathbb{Z}_p/p\mathbb{Z}_p} f(h \bar{\mathbf{n}}_H(Dx) \mathbf{d}_H(\Pi^{-1})) + \chi_{0,p}(\Pi) \sum_{x \in \mathbb{Z}_p/p\mathbb{Z}_p} f(h \mathbf{n}_H(x) \mathbf{d}_H(\Pi)).$$

(iii) Suppose that p splits in K/\mathbb{Q} . Then we have

$$\begin{aligned} (f * \phi_{1,p})(h) &= \chi_{0,p}^{-1}(\Pi_1) \left\{ f(h\mathbf{d}_H(\Pi_1^{-1})) + \sum_{x \in \mathbb{Z}_p/p\mathbb{Z}_p} f(h\mathbf{n}_H(x)\mathbf{d}_H(\Pi_2)) \right\}, \\ (f * \phi_{2,p})(h) &= \chi_{0,p}^{-1}(\Pi_2) \left\{ f(h\mathbf{d}_H(\Pi_2^{-1})) + \sum_{x \in \mathbb{Z}_p/p\mathbb{Z}_p} f(h\mathbf{n}_H(x)\mathbf{d}_H(\Pi_1)) \right\}, \\ (f * \phi_{0,p}^\pm)(h) &= \chi_{0,p}^{-1}((\Pi_1/\Pi_2)^{\pm 1}) \cdot f(h\mathbf{d}_H(\Pi_1/\Pi_2)^{\mp 1}) \quad (= \Omega_p^{\pm 1}(\Pi_1/\Pi_2)f(h)). \end{aligned}$$

If, for every $p < \infty$, there exists a \mathbb{C} -algebra homomorphism λ_p of \mathcal{H}_p^H to \mathbb{C} such that $f * \phi = \lambda_p(\phi)f$ holds for any $\phi \in \mathcal{H}_p^H$, we say that f is a Hecke eigenform with eigenvalues $\{\lambda_p\}$.

Let $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0; \chi_0\Omega)$ be a Hecke eigenform with eigenvalues $\{\lambda_p\}$ and ξ a Hecke character of K . We define an L -function

$$L(f, \xi; s) = \prod_{p < \infty} L_p(f, \xi; s),$$

where the local factor $L_p(f, \xi; s)$ is given as follows.

(i) If p is inert in K/\mathbb{Q} , put

$$L_p(f, \xi; s) = R_p(\xi_p(p)p^{-2s})^{-1},$$

where

$$R_p(X) = 1 - \{p^{-1}\lambda_p(\phi_{1,p}) + 1 - p^{-1}\}X + X^2.$$

(ii) If p ramifies in K/\mathbb{Q} , put

$$L_p(f, \xi; s) = R_p(\xi_p(\Pi)p^{-s})^{-1},$$

where

$$R_p(X) = 1 - p^{-1/2}\lambda_p(\phi_p^+ + \phi_p^-)X + X^2.$$

(iii) If p splits in K/\mathbb{Q} , put

$$L_p(f, \xi; s) = R_{1,p}(\xi_p(\Pi_1)p^{-s})^{-1}R_{2,p}(\xi_p(\Pi_2)p^{-s})^{-1},$$

where

$$R_{1,p}(X) = 1 - p^{-1/2}\lambda_p(\phi_{1,p})X + \epsilon X^2, \quad R_{2,p}(X) = 1 - p^{-1/2}\lambda_p(\phi_{2,p})X + \epsilon^{-1}X^2$$

with $\epsilon = \lambda_p(\phi_{0,p}^+) = \Omega_p(\Pi_1/\Pi_2)$. Note that $\lambda_p(\phi_{2,p}) = \epsilon^{-1}\lambda_p(\phi_{1,p})$ and, hence, $R_{2,p}(X) = R_{1,p}(\epsilon^{-1}X)$.

Remark. Suppose that K has class number 1. For $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$, put $f_{dm}(h_\infty(i)) = j'(h_\infty, i)^{l-1}f(h_\infty)(h_\infty \in H_\infty)$. Then $f \mapsto f_{dm}$ gives rise to an isomorphism between $S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ and the space of holomorphic cusp forms of weight $l - 1$ and character ω on $\Gamma_0(D)$. Assume that $f_{dm}(z) = \sum_{n=0}^\infty c(n)e[nz]$ is a normalized newform in the sense of [Li75]. Define a twisted L -function

$$Z(f_{dm}; s) = \sum_{\mathfrak{a}} c(\mathfrak{N}\mathfrak{a})\alpha^l \mathfrak{N}\mathfrak{a}^{-s},$$

$\mathfrak{a} = \alpha\mathcal{O}_K$ running over the nonzero integral ideals of K . A straightforward calculation shows that

$$L(f, \mathbf{1}; s) = \zeta(2s)Z(f_{dm}; s + l - 1) \prod_{p|D} (1 - p^{-2s})(1 - \overline{c(p)}\eta_p^l p^{-s-l+1})^{-1}.$$

Here, for $p|D$, η_p is an element of K such that $\eta_p\mathcal{O}_K$ is a prime ideal of K dividing p .

4. Theta functions

4.1 The space of theta functions

In this section, we recall several facts about theta functions on R (for details, see [MS02, § 2]). For $m \in \mathbb{Q}^\times$, let \mathbf{T}^m be the space of smooth functions Θ on $R_{\mathbb{Q}} \backslash R_{\mathbf{A}}$ satisfying:

- (i) $\Theta((0, x)r) = \psi_m(x)\Theta(r)$ ($x \in \mathbb{Q}_{\mathbf{A}}, r \in R_{\mathbf{A}}$);
- (ii) for any $r \in R_{\mathbf{A}}, t_\infty \mapsto \Theta(rt_\infty)$ is K_∞^1 -finite.

Let ρ' be the smooth representation of $R_{\mathbf{A},f}$ on \mathbf{T}^m given by right translations. Define an inner product $\langle\langle \cdot, \cdot \rangle\rangle_R$ on \mathbf{T}^m by

$$\langle\langle \Theta, \Theta' \rangle\rangle_R = \int_{R_{\mathbb{Q}} \backslash R_{\mathbf{A}}} \Theta(r) \overline{\Theta'(r)} dr.$$

4.2 Lattice model

In this and the following subsections, we recall the definition of metaplectic representations on a lattice model (for the details, see [MVW87, ch. 2, I. 4], [MS00, § 1] and [MS02, § 2]). Let $m \in \mathbb{Q}^\times$ and $\mathcal{L} = (mD)^{-1}\mathbb{Z} + 2^{-1}\kappa\mathbb{Z}$ be a lattice in K . For a rational prime p , let V_p^m be the space of $\Phi_p \in \mathcal{S}(K_p)$ satisfying

$$\Phi_p(z_p + l_p) = \psi_{m,p}(\frac{1}{2}\langle z_p, l_p \rangle_\kappa + \frac{1}{4}\langle l_p, l_p^\sigma \rangle_\kappa) \Phi_p(z_p)$$

for $z_p \in K_p$ and $l_p \in \mathcal{L}_p = \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let V_∞^m be the space consisting of functions on $K_\infty = \mathbb{C}$ of the form $P(z_\infty) \mathbf{e}[2^{-1}m\kappa z_\infty \overline{z_\infty}]$ (respectively $P(z_\infty) \mathbf{e}[-2^{-1}m\kappa z_\infty \overline{z_\infty}]$) if $m > 0$ (respectively if $m < 0$), where P is a polynomial in z_∞ . Define an inner product $(\cdot, \cdot)_v$ on V_v^m by

$$(\Phi_v, \Phi'_v)_v = \int_{K_v} \Phi_v(z_v) \overline{\Phi'_v(z_v)} d^m z_v \quad (\Phi_v, \Phi'_v \in V_v^m),$$

where $d^m z_v$ is the Haar measure on K_v self-dual with respect to the pairing $(z, w) \mapsto \psi_{m,v}(\langle z, w \rangle_\kappa)$ (note that $d^m z_v = |\mathbf{N}(m\kappa)|_v^{1/2} dz_v$).

Let V^m be the restricted tensor product of V_v^m over primes v of \mathbb{Q} with respect to $\{\Phi_{0,p}\}$, where $\Phi_{0,p} \in V_p^m$ is given by

$$\Phi_{0,p}(z_p) = \text{char}_{\mathcal{L}_p}(z_p) \psi_{m,p}(\frac{1}{4}\langle z_p, z_p^\sigma \rangle_\kappa) \quad (z_p \in K_p).$$

We denote by $V_{L^2}^m$ the completion of V^m with respect to the inner product

$$(\Phi, \Phi') = \int_{K_{\mathbf{A}}} \Phi(z) \overline{\Phi'(z)} d^m z,$$

where $d^m z = \prod_v d^m z_v$. Let ρ^m be a unitary representation of $N_{\mathbf{A}}$ on $V_{L^2}^m$ defined by

$$\rho^m(w, x)\Phi(z) = \psi_m(\frac{1}{2}\langle z, w \rangle_\kappa + x)\Phi(z + w) \quad (\Phi \in V_{L^2}^m, (w, x) \in N_{\mathbf{A}}, z \in K_{\mathbf{A}}).$$

Then ρ^m is irreducible.

4.3 Metaplectic representation of $K_{\mathbf{A}}^1$

Let $\chi \in \mathcal{X}$ and v be a prime of \mathbb{Q} . Let $\lambda_{K_v}(\psi_{m,v})$ be the Weil constant attached to $(K_v/\mathbb{Q}_v, \psi_{m,v})$ (cf. [Wei64, Théorème 2]). By definition, we have

$$\int_{K_v} \varphi(z) \psi_v(mzz^\sigma) dz = \lambda_{K_v}(\psi_{m,v}) |m|_v^{-1} \int_{K_v} \widehat{\varphi}(z) \psi_v(-m^{-1}zz^\sigma) dz$$

for $\varphi \in \mathcal{S}(K_v)$. Here the Fourier transform $\widehat{\varphi}$ of φ is defined by

$$\widehat{\varphi}(z) = \int_{K_v} \varphi(w) \psi_v(\text{Tr}(w^\sigma z)) dw \quad (z \in K_v).$$

For $t_v \in K_v^1$, we define an endomorphism $\mathcal{M}_\chi^{\kappa,m}(t_v)$ of $V_{L^2}^m$ as follows. If $t_v = 1$, we put $\mathcal{M}_\chi^{\kappa,m}(t_v) = \text{Id}_{V_{L^2}^m}$. Suppose that $t_v \neq 1$. For $\Phi \in V^m$, we put

$$\begin{aligned} \mathcal{M}_\chi^{\kappa,m}(t_v)\Phi(z) &= \lambda_{K_v}(\psi_{m,v})^{-1}\chi_v\left(\frac{1-t_v}{\kappa}\right)|N(1-t_v)|_v^{-1/2} \\ &\quad \times \int_{K_v} \psi_{m,v}\left(\frac{1}{2}\text{Tr}\left(\frac{\kappa}{1-t_v}\right)w_v w_v^\sigma\right)\rho^m(w_v, 0)\Phi(z) d^m w_v \quad (z \in K_{\mathbf{A}}). \end{aligned}$$

It is known that $\mathcal{M}_\chi^{\kappa,m}(t_v)$ preserves V^m and extends uniquely to a unitary operator of $V_{L^2}^m$. For $t = (t_v)_v \in K_{\mathbf{A}}^1$, we put $\mathcal{M}_\chi^{\kappa,m}(t) = \bigotimes_v \mathcal{M}_\chi^{\kappa,m}(t_v)$. Then $\mathcal{M}_\chi^{\kappa,m}$ defines a unitary representation of $K_{\mathbf{A}}^1$ on $V_{L^2}^m$ satisfying $\mathcal{M}_\chi^{\kappa,m}(t)\rho^m(w, x)\mathcal{M}_\chi^{\kappa,m}(t^{-1}) = \rho^m(tw, x)$ ($t \in K_{\mathbf{A}}^1, (w, x) \in N_{\mathbf{A}}$). Hence, we can extend $\mathcal{M}_\chi^{\kappa,m}$ to a unitary representation of $R_{\mathbf{A}}$ on $V_{L^2}^m$ by

$$\mathcal{M}_\chi^{\kappa,m}((w, x)t) = \rho^m(w, x) \circ \mathcal{M}_\chi^{\kappa,m}(t) \quad ((w, x) \in N_{\mathbf{A}}, t \in K_{\mathbf{A}}^1).$$

4.4 Theta series

For $\chi \in \mathcal{X}$ and $\Phi \in V^m$, we set

$$\theta_{\chi,\Phi}^m(r) = \sum_{X \in K} (\mathcal{M}_\chi^{\kappa,m}(r)\Phi)(X) \quad (r \in R_{\mathbf{A}}).$$

The theta series $\theta_{\chi,\Phi}^m$ is absolutely convergent and belongs to \mathbf{T}^m . Let $\mathbf{T}_\chi^m = \{\theta_{\chi,\Phi}^m \mid \Phi \in V^m\}$.

PROPOSITION 4.1 ([Shi79, Proposition 2]; see also [MS02, Theorem 2.22]). *We have the following.*

(i) *We have an orthogonal decomposition*

$$\mathbf{T}^m = \bigoplus_{\chi \in \mathcal{X}} \mathbf{T}_\chi^m.$$

(ii) *For $\chi \in \mathcal{X}$ and $\Phi, \Phi' \in V^m$, we have $\langle\langle \theta_{\chi,\Phi}^m, \theta_{\chi,\Phi'}^m \rangle\rangle_R = c_m(\Phi, \Phi')$ with a positive constant c_m depending only on m .*

4.5 Holomorphic theta functions

For $k \in \mathbb{Z}$, set $\mathbf{T}^{m,k} = \{\Theta \in \mathbf{T}^m \mid \Theta(rt_\infty) = t_\infty^k \Theta(r) (r \in R_{\mathbf{A}}, t_\infty \in K_\infty^1)\}$ and $\mathbf{T}_\chi^{m,k} = \mathbf{T}_\chi^m \cap \mathbf{T}^{m,k}$. Let $\mathbf{T}_{\text{hol}}^m$ be the space of $\Theta \in \mathbf{T}^{m,0}$ such that

$$w_\infty \mapsto f_{\Theta,r_f}(w_\infty) := \mathbf{e}_m \left[-\frac{\kappa}{2} w_\infty \overline{w_\infty} \right] \Theta((w_\infty, 0)r_f)$$

is holomorphic on \mathbb{C} for any $r_f \in R_{\mathbf{A},f}$. We call $\mathbf{T}_{\text{hol}}^m$ the space of *holomorphic theta functions*.

LEMMA 4.2. *We have the following.*

(i) *We have*

$$\mathbf{T}_\chi^m = \begin{cases} \bigoplus_{k \geq (w_\infty(\chi)+1)/2} \mathbf{T}_\chi^{m,k} & \text{if } m > 0, \\ \bigoplus_{k \leq (w_\infty(\chi)-1)/2} \mathbf{T}_\chi^{m,k} & \text{if } m < 0. \end{cases}$$

(ii) *We have*

$$\mathbf{T}_{\text{hol}}^m = \begin{cases} \bigoplus_{\chi \in \mathcal{X}, w_\infty(\chi) = -1} \mathbf{T}_\chi^{m,0} & \text{if } m > 0, \\ \{0\} & \text{if } m < 0. \end{cases}$$

Proof. If $m > 0$, the assertions (i) and (ii) are direct consequences of Lemma 2.14 and Proposition 2.26 in [MS02]. Suppose that $m < 0$. Observe that $\overline{V^m} = V^{-m}$ and that $\rho^m(w, x)\Phi = \overline{\rho^{-m}(w, x)\overline{\Phi}}$ for $\Phi \in V^m$. This implies that $\mathcal{M}_\chi^{\kappa, m}(r)\Phi = \overline{\mathcal{M}_{\chi^{-1}}^{\kappa, -m}(r)\overline{\Phi}}$ ($r \in R_{\mathbf{A}}$) and, hence, $\theta_{\chi, \Phi}^m = \overline{\theta_{\chi^{-1}, \overline{\Phi}}^{-m}}$. It follows that

$$\mathbf{T}_\chi^m = \overline{\mathbf{T}_{\chi^{-1}}^{-m}} = \bigoplus_{k \geq (-w_\infty(\chi)+1)/2} \mathbf{T}_{\chi^{-1}}^{-m, k} = \bigoplus_{k \leq (w_\infty(\chi)-1)/2} \mathbf{T}_\chi^{m, k},$$

which proves assertion (i). To show assertion (ii), let $\Theta \in \mathbf{T}_{\text{hol}}^m$ and $r_f \in R_{\mathbf{A}, f}$. It is easily verified that $f_{\Theta, r_f}(w_\infty)$ is holomorphic, bounded on \mathbb{C} and tends to zero when $|w_\infty| \rightarrow \infty$. This implies that $f_{\Theta, r_f}(w_\infty)$ is identically equal to zero and, hence, $\Theta = 0$. \square

4.6 Primitive theta functions

Let \mathfrak{a} be an ideal of K . Define an open compact subgroup $N(\mathfrak{a})_f$ of $N_{\mathbf{A}, f}$ by

$$\begin{aligned} N(\mathfrak{a})_f &= \left\{ (w, x) \in N_{\mathbf{A}, f} \mid w \in \mathfrak{a}_f, x + \frac{\kappa}{2}ww^\sigma \in \mathfrak{a}_f\mathfrak{a}_f^\sigma \right\} \\ &= \{(w, x + x_w) \in N_{\mathbf{A}, f} \mid w \in \mathfrak{a}_f, x \in \mathfrak{a}_f\mathfrak{a}_f^\sigma\} \end{aligned}$$

(recall that $x_w = 2^{-1}\text{Tr}(\theta)ww^\sigma$). We put $R(\mathfrak{a})_f = \{nt \mid n \in N(\mathfrak{a})_f, t \in \mathcal{O}_{K, f}^1\}$. Let $\mathbf{T}^m(\mathfrak{a}) = \{\Theta \in \mathbf{T}^m \mid \rho'(r_0)\Theta = \Theta \text{ for any } r_0 \in R(\mathfrak{a})_f\}$ and $\mathbf{T}_{\text{hol}}^m(\mathfrak{a}) = \mathbf{T}_{\text{hol}}^m \cap \mathbf{T}^m(\mathfrak{a})$. It is easy to see that $\mathbf{T}_{\text{hol}}^m(\mathfrak{a}) = \{0\}$ unless $m > 0$ and $mN(\mathfrak{a})$ is integral. We henceforth assume that $m > 0$ and $mN(\mathfrak{a})$ is integral. Then

$$\mathbf{T}_{\text{hol}}^m(\mathfrak{a}) = \bigoplus_{\chi \in \mathcal{X}, w_\infty(\chi) = -1} \mathbf{T}_{\text{hol}}^m(\mathfrak{a}, \chi),$$

where $\mathbf{T}_{\text{hol}}^m(\mathfrak{a}, \chi) = \mathbf{T}_{\text{hol}}^m(\mathfrak{a}) \cap \mathbf{T}_\chi^m$.

To define the primitive part of $\mathbf{T}_{\text{hol}}^m(\mathfrak{a})$, let \mathfrak{b} be an ideal of K with $\mathfrak{a} \subset \mathfrak{b}$. Define an endomorphism $\mathcal{P}'_{\mathfrak{b}}$ of $\mathbf{T}_{\text{hol}}^m(\mathfrak{a})$ by

$$\mathcal{P}'_{\mathfrak{b}}\Theta = \int_{N(\mathfrak{b})_f} \rho'(n_f)\Theta d_{\mathfrak{b}}n_f \quad (\Theta \in \mathbf{T}_{\text{hol}}^m(\mathfrak{a})),$$

where $d_{\mathfrak{b}}n_f$ is the Haar measure on $N_{\mathbf{A}, f}$ normalized by $\text{vol}(N(\mathfrak{b})_f) = 1$. Note that $\mathcal{P}'_{\mathfrak{b}} = 0$ unless $mN(\mathfrak{b})$ is integral. Let

$$\mathbf{T}_{\text{hol, prim}}^m(\mathfrak{a}) = \{\Theta \in \mathbf{T}_{\text{hol}}^m(\mathfrak{a}) \mid \mathcal{P}'_{\mathfrak{b}}\Theta = 0 \text{ for any ideal } \mathfrak{b} \text{ of } K \text{ with } \mathfrak{b} \supsetneq \mathfrak{a}\}$$

and

$$\mathbf{T}_{\text{hol, prim}}^m(\mathfrak{a}, \chi) = \mathbf{T}_{\text{hol, prim}}^m(\mathfrak{a}) \cap \mathbf{T}_\chi^m.$$

We call $\mathbf{T}_{\text{hol, prim}}^m(\mathfrak{a}, \chi)$ the space of holomorphic *primitive theta functions* with respect to (\mathfrak{a}, χ) .

THEOREM 4.3 (Shintani [Shi79, § 2.11] and Glaubermann–Rogawski [GR89, Theorem 4.2]; see also [MS00, Corollary 6.5] and [MS02, Theorem 3.4]). *The space $\mathbf{T}_{\text{hol, prim}}^m(\mathfrak{a}, \chi)$ is at most one-dimensional.*

4.7 Artin conductor and epsilon factor

Let $p < \infty$ and $m \in \mathbb{Q}_p^\times$. For $\chi \in \mathcal{X}_p$, we put $A_p(\chi) = \text{Min}\{a \geq 0 \mid \chi|_{(1+\mathfrak{P}_p^a) \cap \mathcal{O}_{K, p}^\times} = 1\}$, where

$$\mathfrak{P}_p = \begin{cases} p\mathcal{O}_{K, p} & \text{if } p \text{ splits in } K/\mathbb{Q}, \\ \text{the maximal ideal of } \mathcal{O}_{K, p} & \text{otherwise.} \end{cases}$$

Let $\epsilon_p(s, \chi, \psi_{m, K_p})$ be Tate’s epsilon factor [Tat79, § 3], where $\psi_{m, K_p} = \psi_m \circ \text{Tr}_{K_p/\mathbb{Q}_p}$. It is known that $\epsilon_p(1/2, \chi, \psi_{m, K_p}) = \pm \chi(\kappa^{-1})$. Let \mathfrak{a} be an ideal of K_p such that $\mu_p(m, \mathfrak{a}) := \text{ord}_p m N(\mathfrak{a}) \geq 0$. Denote by $\mathcal{X}_{p, \text{prim}}(m, \mathfrak{a})$ the set of $\chi \in \mathcal{X}_p$ which satisfies

$$A_p(\chi) = \begin{cases} \mu_p(m, \mathfrak{a}) & \text{if } \delta_p = 0, \\ 2(\mu_p(m, \mathfrak{a}) + \delta_p) & \text{if } \delta_p > 0 \text{ and } \mu_p(m, \mathfrak{a}) > 0, \\ 2\delta_p \text{ or } 2\delta_p - 1 & \text{if } \delta_p > 0 \text{ and } \mu_p(m, \mathfrak{a}) = 0. \end{cases}$$

where $\delta_p = \text{ord}_p D$.

4.8 Epsilon dichotomy

Going back to the global situation, we let $m \in \mathbb{Q}^\times$, \mathfrak{a} an ideal of K and $\chi \in \mathcal{X}$ with $w_\infty(\chi) = -1$. Assume that $m > 0$ and $m N(\mathfrak{a})$ is integral. For $p < \infty$, we write χ_p for the p -component of χ . Set

$$\mathcal{X}_{\text{prim}}(m, \mathfrak{a}) = \{\chi \in \mathcal{X} \mid w_\infty(\chi) = -1, \chi_p \in \mathcal{X}_{p, \text{prim}}(m, \mathfrak{a}_p) \text{ for every } p < \infty\}$$

and

$$\mathcal{X}_{\text{prim}}^+(m, \mathfrak{a}) = \{\chi \in \mathcal{X}_{\text{prim}}(m, \mathfrak{a}) \mid \epsilon_p(1/2, \chi_p, \psi_{m, K_p}) = \chi_p(\kappa^{-1}) \text{ for every } p < \infty\}.$$

The following result gives a criterion for the existence of primitive theta functions.

THEOREM 4.4 ([MS00, Corollary 6.7] and [MS02, Theorem 3.8]). *For $\chi \in \mathcal{X}$, the space $\mathbf{T}_{\text{hol, prim}}^m(\mathfrak{a}, \chi)$ defined in § 4.6 is one-dimensional if and only if $\chi \in \mathcal{X}_{\text{prim}}^+(m, \mathfrak{a})$.*

Remark. Theorem 4.4 is a refined form of ‘epsilon dichotomy’ for $U(1)$ (for epsilon dichotomy for unitary groups, we refer to [Moe91], [Rog92] and [HKS96]).

Let \mathcal{E} be the set of the triplets (m, \mathfrak{a}, χ) , where m is a positive rational number, \mathfrak{a} an ideal of K with $m N(\mathfrak{a}) \in \mathbb{Z}$ and $\chi \in \mathcal{X}_{\text{prim}}^+(m, \mathfrak{a})$. By Theorem 4.4, we have $\dim \mathbf{T}_{\text{hol, prim}}^m(\mathfrak{a}, \chi) = 1$ for $(m, \mathfrak{a}, \chi) \in \mathcal{E}$.

4.9 The structure of $\mathbf{T}_{\text{hol}}^m(\mathfrak{a})$

We keep the notation and assumptions of § 4.8.

THEOREM 4.5 ([Shi79, Proposition 2]; see also [MS02, Theorem 3.9]). *We have a direct sum decomposition:*

$$\mathbf{T}_{\text{hol}}^m(\mathfrak{a}) = \sum_{\mathfrak{a}' \in \mathcal{R}(\mathfrak{a})} \sum_{\chi \in \mathcal{X}_{\text{prim}}^+(m, \mathfrak{a}')} \mathbf{T}_{\text{hol, prim}}^m(\mathfrak{a}', \chi),$$

where $\mathcal{R}(\mathfrak{a})$ is the set of ideals \mathfrak{a}' of K containing \mathfrak{a} with $m N(\mathfrak{a}')$ integral.

Remark. Let $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathcal{R}(\mathfrak{a})$ and $\chi_1 \in \mathcal{X}_{\text{prim}}^+(m, \mathfrak{a}_1), \chi_2 \in \mathcal{X}_{\text{prim}}^+(m, \mathfrak{a}_2)$. Then $\mathbf{T}_{\text{hol, prim}}^m(\mathfrak{a}_1, \chi_1)$ and $\mathbf{T}_{\text{hol, prim}}^m(\mathfrak{a}_2, \chi_2)$ are orthogonal to each other if at least one of the following conditions is satisfied (see [MS00], § 10):

- (i) $\chi_1 \neq \chi_2$;
- (ii) $m N(\mathfrak{a}_1) \neq m N(\mathfrak{a}_2)$;
- (iii) there exists a prime factor of $m N(\mathfrak{a}_1)$ which splits in K/\mathbb{Q} .

4.10 Exceptional case

Suppose that p ramifies in K/\mathbb{Q} and define an endomorphism \mathcal{Q}_p of $\mathbf{T}_{\text{hol}}^m(\mathfrak{a})$ by

$$\mathcal{Q}_p \Theta(r) = \int_{\Pi_p^{-1} \mathfrak{a}_p} \Theta(r(w, x_w)) d_{\Pi_p^{-1} \mathfrak{a}_p} w \quad (\Theta \in \mathbf{T}_{\text{hol}}^m(\mathfrak{a}))$$

(for the definition of $d_{\Pi_p^{-1}\mathfrak{a}_p} w$, see § 2.4). Note that \mathcal{Q}_p coincides with $\mathcal{P}'_{\Pi_p^{-1}\mathfrak{a}_p}$ if $\mu_p(\mathfrak{a}, m) > 0$. We say that $\chi_p \in \mathcal{X}_p$ is *ordinary* (respectively *exceptional*) if $A_p(\chi_p) \geq 2\delta_p$ (respectively $A_p(\chi_p) = 2\delta_p - 1$).

PROPOSITION 4.6 [MS02, Theorem 3.8]. *Let $\chi \in \mathcal{X}_{\text{prim}}^+(m, \mathfrak{a})$ and $\Theta \in \mathbf{T}_{\text{hol,prim}}^m(\mathfrak{a}, \chi)$. Suppose that p ramifies in K/\mathbb{Q} . Then*

$$\mathcal{Q}_p \Theta = \begin{cases} 0 & \text{if } \chi_p \text{ is ordinary,} \\ \Theta & \text{if } \chi_p \text{ is exceptional.} \end{cases}$$

4.11 Periods of theta functions

For $\Theta \in \mathbf{T}^m$, we put

$$I_{\Theta}(r) = \int_{K^1 \backslash K_{\mathbf{A}}^1} \Theta(tr) d^{\times} t \quad (r \in R_{\mathbf{A}})$$

and

$$I(\Theta) = I_{\Theta}(1) = \int_{K^1 \backslash K_{\mathbf{A}}^1} \Theta(t) d^{\times} t.$$

We call $I(\Theta)$ the *period* of Θ . The following fact is proved by Yang [Yan97, Theorem 0.3] (see also [MS02, Theorem 5.2]).

THEOREM 4.7. *Let $(m, \mathfrak{a}, \chi) \in \mathcal{E}$ and $\Theta \in \mathbf{T}_{\text{hol,prim}}^m(\mathfrak{a}, \chi) - \{0\}$.*

(i) *We have*

$$|I(\Theta)|^2 = C(\chi)L(\chi; \frac{1}{2})\langle\langle \Theta, \Theta \rangle\rangle_R,$$

where

$$C(\chi) = \frac{1}{2L(\omega; 1)} \prod_{p \nmid D, A_p(\chi_p) > 0} (1 + \omega_p(\pi_p)p^{-1})^{-1}$$

and $\langle\langle \cdot, \cdot \rangle\rangle_R$ is defined in § 4.1. (Recall that ω is the quadratic Hecke character of \mathbb{Q} associated with the extension K/\mathbb{Q} , ω_p the p -component of ω and π_p a prime element of \mathbb{Q}_p .)

(ii) *We have $I(\Theta) \neq 0$ if and only if $L(\chi; 1/2) \neq 0$.*

4.12 The following result will be needed in the next section.

PROPOSITION 4.8. *Suppose that p splits in K/\mathbb{Q} and $\mu_p(m, \mathfrak{a}) = 0$. Let $\chi \in \mathcal{X}_{\text{prim}}^+(m, \mathfrak{a})$ and $\Theta \in \mathbf{T}_{\text{hol,prim}}^m(\mathfrak{a}, \chi)$. For $t \in K_p^1 - \mathcal{O}_{K,p}^1$, we have*

$$\int_{\mathfrak{a}_p} \Theta(r(w, x_w)t^{-1}) d_{\mathfrak{a}_p} w = |N(1-t)|_p^{-1/2} \chi_p\left(\frac{1-t^{-1}}{\kappa}\right) \Theta(r) \quad (r \in R_{\mathbf{A}}).$$

Proof. First note that we may (and do) let $\theta = (1, 0)$ and $x_w = \frac{1}{2}N(w)$. Take $\Phi' = \otimes_v \Phi'_v \in V^m$ so that $\Theta = \theta_{\chi, \Phi'}$. It is sufficient to show that

$$\int_{\mathfrak{a}_p} \mathcal{M}_{\chi}^{\kappa, m}((w, x_w)t^{-1}) \Phi'_p d_{\mathfrak{a}_p} w = |N(1-t)|_p^{-1/2} \chi_p\left(\frac{1-t^{-1}}{\kappa}\right) \Phi'_p. \tag{4.1}$$

To simplify the notation, we omit the superscript m and the subscript p , and write \mathcal{M}_{χ} for $\mathcal{M}_{\chi}^{\kappa, m}$. Let $V(\mathfrak{a}) = \{\Phi \in V \mid \rho(w, x_w)\Phi = \Phi(w \in \mathfrak{a})\} = \{\Phi \in V \mid \mathcal{P}_{\mathfrak{a}}\Phi = \Phi\}$, where

$$\mathcal{P}_{\mathfrak{a}}\Phi = \int_{\mathfrak{a}} \rho(w, x_w)\Phi d_{\mathfrak{a}} w.$$

Observe that $V(\mathfrak{a})$ is one-dimensional since $\mu(m, \mathfrak{a}) = 0$ (cf. [MS00, Proposition 5.10]), and that both Φ'_p and the integral of (4.1) belong to $V(\mathfrak{a})$. This implies

$$\int_{\mathfrak{a}} \mathcal{M}_\chi((w, x_w)t^{-1})\Phi'_p d_{\mathfrak{a}}w = \gamma(t)\Phi'_p$$

with $\gamma(t) = \text{Tr}(\mathcal{P}_{\mathfrak{a}}\mathcal{M}_\chi(t^{-1})|_{V(\mathfrak{a})})$. To calculate $\gamma(t)$, recall that $\Phi(z) = (\rho(z, 0)\Phi, \Phi_0)$ ($\Phi \in V, z \in K$), where $\Phi_0(z) = \text{char}_{\mathcal{L}}(z)\psi_m(\frac{1}{4}\langle z, z^\sigma \rangle_\kappa) \in V$. It follows that $\mathcal{M}_\chi(t^{-1})\Phi(z)$ is equal to

$$(\rho(z, 0)\mathcal{M}_\chi(t^{-1})\Phi, \Phi_0) = (\Phi, \mathcal{M}_\chi(t)\rho(-z, 0)\Phi_0) = c_\chi(t) \int_K \eta_t(z, z')\Phi(z') d^m z',$$

where

$$c_\chi(t) = \overline{\lambda_K(\psi_m)^{-1}\chi\left(\frac{1-t}{\kappa}\right)|\mathbb{N}(1-t)|^{-1/2}} = \chi\left(\frac{1-t^{-1}}{\kappa}\right)|\mathbb{N}(1-t)|^{-1/2}$$

and

$$\begin{aligned} \eta_t(z, z') &= \int_K \psi_m\left(-\frac{1}{2}\text{Tr}\left(\frac{\kappa}{1-t}\right)\mathbb{N}(w')\right)\overline{\rho(w', 0)\rho(-z, 0)\Phi_0(z')} d^m w' \\ &= \int_K \psi_m\left(-\frac{1}{2}\text{Tr}\left(\frac{\kappa}{1-t}\right)\mathbb{N}(w') + \frac{1}{2}\langle w', z \rangle_\kappa - \frac{1}{2}\langle z', w' - z \rangle_\kappa\right)\overline{\Phi_0(z' - z + w')} d^m w'. \end{aligned}$$

This implies that

$$\mathcal{P}_{\mathfrak{a}}\mathcal{M}_\chi(t^{-1})\Phi(z) = c_\chi(t) \int_K \eta_{t,\mathfrak{a}}(z, z')\Phi(z') d^m z'$$

with

$$\eta_{t,\mathfrak{a}}(z, z') = \int_{\mathfrak{a}} \psi_m\left(\frac{1}{2}\langle z, w \rangle_\kappa + x_w\right)\eta_t(z + w, z') d_{\mathfrak{a}}w.$$

We thus have

$$\begin{aligned} \gamma(t) &= c_\chi(t) \int_K \eta_{t,\mathfrak{a}}(z, z) d^m z \\ &= c_\chi(t) \int_{\mathfrak{a}} \int_K \int_K \overline{\Phi_0(w' - w)} \\ &\quad \times \psi_m\left(\langle z, w - w' \rangle_\kappa + x_w - \frac{1}{2}\text{Tr}\left(\frac{\kappa}{1-t}\right)\mathbb{N}(w') + \frac{1}{2}\langle w', w \rangle_\kappa\right) d^m w' d^m z d_{\mathfrak{a}}w \\ &= c_\chi(t)\overline{\Phi_0(0)} \int_{\mathfrak{a}} \psi_m\left(\frac{1 - \kappa - (1 + \kappa)t}{2(1-t)}\mathbb{N}(w)\right) d_{\mathfrak{a}}w \\ &= c_\chi(t) \int_{\mathfrak{a}} \psi_m\left(\frac{1 - \kappa - (1 + \kappa)t}{2(1-t)}\mathbb{N}(w)\right) d_{\mathfrak{a}}w. \end{aligned}$$

Since

$$\frac{1 - \kappa - (1 + \kappa)t}{2(1-t)} \in \mathbb{Z}_p \quad \text{for } t \notin \mathcal{O}_{K,p}^1,$$

the last integral is equal to 1 and we have proved $\gamma(t) = c_\chi(t)$. □

5. Fourier–Jacobi component

5.1 Fourier–Jacobi expansion

In this section, keeping the notation of §4, we recall several basic facts about Fourier–Jacobi expansion of automorphic forms on $G = U(S)$ after [Shi79] (see also [MS02]).

Let F be a smooth function on $G_{\mathbb{Q}} \backslash G_{\mathbf{A}} / \mathcal{K}_f$. Then F admits the Fourier–Jacobi expansion

$$F(g) = \sum_{m \in \mathbb{Q}} F^m(g),$$

where

$$F^m(g) = \int_{\mathbb{Q} \backslash \mathbb{Q}_{\mathbf{A}}} F((0, x)g) \psi_m(-x) dx.$$

For $\alpha \in K_{\mathbf{A}}^{\times}$, we define a function F_{α}^m on $R_{\mathbb{Q}} \backslash R_{\mathbf{A}}$ by

$$F_{\alpha}^m(r) = F^m(\mathbf{rd}_G(\alpha)) \quad (r \in R_{\mathbf{A}}).$$

Note that $F_{\alpha}^m \in \mathbf{T}^m(\mathfrak{a})$ with $\mathfrak{a} = \text{id}(\alpha)$ (for the definition of $\text{id}(\alpha)$, see § 2.2).

PROPOSITION 5.1. *Let F be as above. Then F is in $\mathfrak{S}_l(\mathcal{K}_f)$ if and only if the following four conditions are satisfied:*

- (5.1) $F(gk_{\infty}) = J(k_{\infty}, Z_0)^{-l} F(g)$ ($k_{\infty} \in \mathcal{K}_{\infty}$);
- (5.2) $F^m = 0$ if $m \leq 0$;
- (5.3) for any $m \in \mathbb{Q}, m > 0$ and $\alpha = \alpha_{\infty} \alpha_f \in K_{\mathbf{A}}^{\times}$, we have $F_{\alpha}^m = \alpha_{\infty}^l \mathbf{e}_m[(\kappa/2)(\alpha_{\infty} \overline{\alpha_{\infty}} - 1)] F_{\alpha_f}^m$;
- (5.4) for any $m \in \mathbb{Q}, m > 0$ and $\alpha_f \in K_{\mathbf{A},f}^{\times}$, we have $F_{\alpha_f}^m \in \mathbf{T}_{\text{hol}}^m$.

Proof. The necessity is easily verified. Suppose that the conditions (5.1)–(5.4) are satisfied. For $g_f \in G_{\mathbf{A},f}$, we put $F_{dm}(g_{\infty} \langle Z_0 \rangle; g_f) = J(g_{\infty}, Z_0)^l F(g_f g_{\infty})$ ($g_{\infty} \in G_{\infty}$). It is sufficient to show that $Z \mapsto F_{dm}(Z; g_f)$ is holomorphic on \mathcal{D} for any $g_f \in G_{\mathbf{A},f}$. We may (and do) suppose that $g_f = r_f \mathbf{d}_G(\alpha_f)$ ($r_f \in R_{\mathbf{A},f}, \alpha_f \in K_{\mathbf{A},f}^{\times}$). For $Z = {}^t(z, w) \in \mathcal{D}$, put $g_z = (w, x) \mathbf{d}_G(\sqrt{y})$ with $z = x + (\kappa/2)(y + w\bar{w})$ ($x, y \in \mathbb{R}, y > 0$). Then $g_z \langle Z_0 \rangle = Z$. By (5.2) and (5.3), we obtain

$$\begin{aligned} F_{dm}(Z; g_f) &= J(g_z, Z_0)^l F(g_f g_z) \\ &= y^{-l/2} \sum_{m \in \mathbb{Q}, m > 0, m \mathbf{N}(\mathfrak{a}) \in \mathbb{Z}} F_{\alpha_f \sqrt{y}}^m(r_f(w, 0)) \mathbf{e}_m[x] \\ &= \sum_{m \in \mathbb{Q}, m > 0, m \mathbf{N}(\mathfrak{a}) \in \mathbb{Z}} F_{\alpha_f}^m(r_f(w, 0)) \cdot \mathbf{e}_m \left[-\frac{\kappa}{2}(w\bar{w} + 1) \right] \mathbf{e}_m[z]. \end{aligned}$$

The condition (5.4) implies that each term in the sum is holomorphic in Z and we are done. □

5.2 Primitive and non-primitive components

Recall that \mathcal{E} is the set of the triplets (m, \mathfrak{a}, χ) , where m is a positive rational number, \mathfrak{a} an ideal of K with $m \mathbf{N}(\mathfrak{a}) \in \mathbb{Z}$ and $\chi \in \mathcal{X}_{\text{prim}}^+(m, \mathfrak{a})$. We put $\mu_p(m, \mathfrak{a}) = \text{ord}_p m \mathbf{N}(\mathfrak{a})$. For $F \in \mathfrak{A}_l(\mathcal{K}_f)$, $(m, \mathfrak{a}, \chi) \in \mathcal{E}$ and an ideal \mathfrak{b} of K , we put

$$C_F(\mathfrak{b}; \Theta_{\mathfrak{a},\chi}^m) = \langle\langle F_{\mathfrak{b}}^m, \Theta_{\mathfrak{a},\chi}^m \rangle\rangle_R,$$

where $\Theta_{\mathfrak{a},\chi}^m \in \mathbf{T}_{\text{hol,prim}}^m(\mathfrak{a}, \chi) - \{0\}$ and $F_{\mathfrak{b}}^m = F_{\beta_f}^m$ with $\beta_f \in K_{\mathbf{A},f}^{\times}, \text{id}(\beta_f) = \mathfrak{b}$. Note that $\Theta_{\mathfrak{a},\chi}^m$ is unique up to constant multiples by Theorem 4.4 and that $F_{\mathfrak{b}}^m$ does not depend on the choice of β_f . In view of Theorem 4.5, F is determined by $\{C_F(\mathfrak{b}; \Theta_{\mathfrak{a},\chi}^m) \mid (m, \mathfrak{a}, \chi) \in \mathcal{E}, \mathfrak{b} \subset \mathfrak{a}\}$, which we call the *Fourier–Jacobi components* of F . If $\mathfrak{b} = \mathfrak{a}$ (respectively $\mathfrak{b} \subsetneq \mathfrak{a}$), the component $C_F(\mathfrak{b}; \Theta_{\mathfrak{a},\chi}^m)$ is said to be *primitive* (respectively *non-primitive*).

LEMMA 5.2. *Let $F \in \mathfrak{A}_l(\mathcal{K}_f; \Omega^{-1})$ and write Θ for $\Theta_{\mathfrak{a},\chi}^m$.*

- (i) *Suppose that p is inert in K/\mathbb{Q} . Then $C_F(p^k \mathfrak{a}; \Theta) = 0$ unless $k \geq 0$.*
- (ii) *Suppose that p ramifies in K/\mathbb{Q} . Then $C_F(\Pi^k \mathfrak{a}; \Theta) = 0$ unless $k \geq 0$.*

- (iii) Suppose that p splits in K/\mathbb{Q} . If $\mu_p(m, \mathfrak{a}) > 0$, we have $C_F(\Pi_1^{k_1}\Pi_2^{k_2}\mathfrak{a}; \Theta) = 0$ unless $k_1, k_2 \geq 0$. If $\mu_p(m, \mathfrak{a}) = 0$, we have $C_F(\Pi_1^{k_1}\Pi_2^{k_2}\mathfrak{a}; \Theta) = 0$ unless $k_1 + k_2 \geq 0$.
- (iv) Suppose that p splits in K/\mathbb{Q} and $\mu_p(m, \mathfrak{a}) = 0$. Then we have

$$C_F(\Pi_1^{k_1}\Pi_2^{k_2}\mathfrak{a}; \Theta) = \begin{cases} (p^{1/2}\chi(\Pi_2)\Omega^{-1}(\Pi_1/\Pi_2))^{k_2} \cdot C_F(\Pi_1^{k_1+k_2}\mathfrak{a}; \Theta) & \text{if } k_1 \geq 0 \text{ and } k_2 \leq 0, \\ (p^{1/2}\chi(\Pi_1)\Omega(\Pi_1/\Pi_2))^{k_1} \cdot C_F(\Pi_2^{k_1+k_2}\mathfrak{a}; \Theta) & \text{if } k_1 \leq 0 \text{ and } k_2 \geq 0. \end{cases}$$

Proof. The assertions (i)–(iii) are easily verified. To prove assertion (iv), we may (and do) assume that $k_1 + k_2 \geq 0$. Suppose that $k_1 \geq 0, k_2 \leq 0$, and put $t = \Pi_1^{-k_2}\Pi_2^{k_2}$ and $\beta = \Pi_1^{k_1+k_2}\alpha$. By Proposition 4.8, we have

$$\begin{aligned} & \langle\langle F_{\Pi_1^{k_1}\Pi_2^{k_2}\mathfrak{a}}^m, \Theta \rangle\rangle_R \\ &= \int_{R_{\mathbb{Q}} \setminus R_{\mathbf{A}}} F^m(r\mathbf{d}_G(\beta, t)t^{-1}1_3)\overline{\Theta(r)} \, dr = \Omega(t) \int_{R_{\mathbb{Q}} \setminus R_{\mathbf{A}}} F^m(r\mathbf{d}_G(\beta))\overline{\Theta(rt^{-1})} \, dr \\ &= \Omega(t) \int_{R_{\mathbb{Q}} \setminus R_{\mathbf{A}}} dr \int_{\mathbb{Z}_p} F^m(r((0, \alpha_2 y), 0)\mathbf{d}_G(\beta)) \, dy \int_{\mathbb{Z}_p} \overline{\Theta(r((0, \alpha_2 x), 0)t^{-1})} \, dx \\ &= \Omega(t) \int_{R_{\mathbb{Q}} \setminus R_{\mathbf{A}}} F^m(r\mathbf{d}_G(\beta)) \, dr \int_{\mathfrak{a}_p} \overline{\Theta(r(w, x_w)t^{-1})} \, dx \\ &= \Omega(t)|N(1-t)|^{-1/2}\chi^{-1}\left(\frac{1-t^{-1}}{\kappa}\right) \int_{R_{\mathbb{Q}} \setminus R_{\mathbf{A}}} F^m(r\mathbf{d}_G(\beta))\overline{\Theta(r)} \, dr \\ &= (p^{1/2}\chi(\Pi_2)\Omega^{-1}(\Pi_1/\Pi_2))^{k_2} \langle\langle F_{\Pi_1^{k_1+k_2}\mathfrak{a}}^m, \Theta \rangle\rangle_R. \end{aligned}$$

This proves assertion (iv) in the case where $k_1 \geq 0$ and $k_2 \leq 0$. The proof in the remaining case is similar and omitted. □

5.3 Shintani’s criterion

In this subsection, we recall a criterion due to Shintani [Shi79, §3] for $F \in \mathfrak{S}_l(\mathcal{K}_f; \Omega^{-1})$ being a Hecke eigenform in terms of its Fourier–Jacobi components. (For the definitions of Hecke operators $\Phi_{i,p}$, see §3.2.)

PROPOSITION 5.3. Suppose that p is inert in K/\mathbb{Q} .

- (i) We have $F * \Phi_{1,p} = \Lambda_p F$ with $\Lambda_p \in \mathbb{C}$ if and only if the following recursion relations hold for any $(m, \mathfrak{a}, \chi) \in \mathcal{E}$ and $k \geq 0$:

$$p^4 c(k+1) + \{\delta(k + \mu_p(m, \mathfrak{a}) > 0)p - 1 - \Lambda_p\}c(k) + c(k-1) = 0,$$

where we put $c(k) = C_F(p^k \mathfrak{a}; \Theta_{\mathfrak{a}, \chi}^m)$.

- (ii) Suppose that part (i) holds. Then we have

$$\sum_{k=0}^{\infty} c(k)X^k = \frac{1 + \delta(\mu_p(m, \mathfrak{a}) = 0)p^{-3}X}{Q_p(p^{-2}X)} \cdot c(0),$$

where $Q_p(X) = 1 - p^{-2}(\Lambda_p - p + 1)X + X^2$.

PROPOSITION 5.4. Suppose that p ramifies in K/\mathbb{Q} .

- (i) We have $F * \Phi_{1,p} = \Lambda_p F$ with $\Lambda_p \in \mathbb{C}$ if and only if the following recursion relations hold for any $(m, \mathfrak{a}, \chi) \in \mathcal{E}$ and $k \geq 0$:

$$p^2 c(k+1) + \{\delta(k > 0 \text{ or } \chi \text{ is exceptional}) \cdot p - 1 - \Lambda_p\}c(k) + c(k-1) = 0,$$

where we put $c(k) = C_F(\Pi^k \mathfrak{a}; \Theta_{\mathfrak{a}, \chi}^m)$.

(ii) Suppose that part (i) holds. Then we have

$$\sum_{k=0}^{\infty} c(k)X^k = \frac{1 + \delta(\chi \text{ is ordinary})p^{-1}X}{Q_p(p^{-1}X)} \cdot c(0),$$

where $Q_p(X) = 1 - p^{-1}(\Lambda_p - p + 1)X + X^2$.

PROPOSITION 5.5. Suppose that p splits in K/\mathbb{Q} .

(i) We have $F * \Phi_{i,p} = \Lambda_{i,p}F$ with $\Lambda_{i,p} \in \mathbb{C}$ ($i = 1, 2$) if and only if the following recursion relations hold for any $(m, \mathbf{a}, \chi) \in \mathcal{E}$ and $k_1, k_2 \geq 0$:

$$\Lambda_{1,p} \cdot c(k_1, k_2) = p^2c(k_1, k_2 + 1) + p\Omega^{-1}(\Pi_1/\Pi_2) \cdot c(k_1 + 1, k_2 - 1) + c(k_1 - 1, k_2),$$

$$\Lambda_{2,p} \cdot c(k_1, k_2) = p^2c(k_1 + 1, k_2) + p\Omega(\Pi_1/\Pi_2) \cdot c(k_1 - 1, k_2 + 1) + c(k_1, k_2 - 1),$$

where we put $c(k_1, k_2) = C_F(\Pi_1^{k_1} \Pi_2^{k_2} \mathbf{a}; \Theta_{\mathbf{a}, \chi}^m)$.

(ii) Suppose that part (i) holds. Then we have

$$\sum_{k_1, k_2=0}^{\infty} c(k_1, k_2)X_1^{k_1} X_2^{k_2} = \frac{R(X_1, X_2)}{Q_1(p^{-1}X_1)Q_2(p^{-1}X_2)} \cdot c(0, 0),$$

where

$$Q_1(X) = 1 - p^{-1}\Lambda_{2,p}X + p^{-1}\Omega(\Pi_1/\Pi_2)\Lambda_{1,p}X^2 - \Omega(\Pi_1/\Pi_2)X^3,$$

$$Q_2(X) = 1 - p^{-1}\Lambda_{1,p}X + p^{-1}\Omega^{-1}(\Pi_1/\Pi_2)\Lambda_{2,p}X^2 - \Omega^{-1}(\Pi_1/\Pi_2)X^3,$$

$$R(X_1, X_2) = (1 - p^{-2}X_1X_2) \prod_{i=1,2} (1 - \delta(\mu_p(m, \mathbf{a}) = 0)\chi^{-1}(\Pi_i)p^{-3/2}X_i).$$

THEOREM 5.6 [Shi79, § 3]. Suppose that $F \in \mathfrak{A}_l(\mathcal{K}_f; \Omega^{-1})$ is a Hecke eigenform. Then $F = 0$ if and only if $C_F(\mathbf{a}; \Theta_{\mathbf{a}, \chi}^m) = 0$ for any $(m, \mathbf{a}, \chi) \in \mathcal{E}$.

Proof. This follows from Lemma 5.2, Propositions 5.3–5.5 and Theorem 4.5. □

6. Metaplectic representation

6.1 In this section, we let F be either the p -adic number field \mathbb{Q}_p or the real number field \mathbb{R} . When $F = \mathbb{Q}_p$, let K be a quadratic extension of F or $F \oplus F$. When $F = \mathbb{R}$, let $K = \mathbb{C}$. Denote by σ the nontrivial automorphism of K/F . For $z \in K$, put $\text{Tr}(z) = z + z^\sigma$ and $\text{N}(z) = zz^\sigma$. Denote by $|\cdot|_F$ the normalized valuation of F . Let ψ be the additive character of F given by

$$\psi(x) = \begin{cases} \mathbf{e}[-(\text{the fractional part of } x)] & \text{if } F = \mathbb{Q}_p, \\ \mathbf{e}[x] & \text{if } F = \mathbb{R}. \end{cases}$$

Recall that dx (respectively dz) is the Haar measure on F (respectively K) self-dual with respect to the pairing $(x, x') \mapsto \psi(xx')$ (respectively $(z, z') \mapsto \psi(\text{Tr}(z^\sigma z'))$). Denote by $\lambda_K(\psi)$ the Weil constant attached to $(K/F, \psi)$ (cf. § 4.3). We have $\lambda_K(\psi) = \sqrt{-1}$ if $F = \mathbb{R}$.

Let $F = \mathbb{Q}_p$. Denote by \mathcal{O}_F the integer ring of F and let

$$\mathcal{O}_K = \begin{cases} \text{the integer ring of } K & \text{if } K \text{ is a field,} \\ \mathcal{O}_F \oplus \mathcal{O}_F & \text{if } K = F \oplus F. \end{cases}$$

Let \mathcal{X} be the set of characters of K^\times whose restriction to F^\times coincides with $\omega = \omega_{K/F}$, the character of F^\times corresponding to K/F by local class field theory.

6.2 Dual reductive pair

Let $\mathbf{H} = U(\mathbf{T})$ be the unitary group of a non-degenerate anti-hermitian matrix

$$\mathbf{T} = \begin{bmatrix} & & \kappa^{-1}T \\ & T & \\ -\kappa^{-1}T & & \end{bmatrix}.$$

Define a homomorphism of $G \times H$ to \mathbf{H} by

$$(g, h) \mapsto g \otimes h := (g_{ij}1_2)_{1 \leq i, j \leq 3} \cdot \begin{bmatrix} h & & \\ & h & \\ & & h \end{bmatrix} \quad (g = (g_{ij}) \in G, h \in H).$$

Note that the kernel of the homomorphism is $\{(t1_3, t^{-1}1_2) \mid t \in K^\times\}$.

6.3 Heisenberg group and representation

Let $\langle \cdot, \cdot \rangle_T$ and $\langle \cdot, \cdot \rangle_{\mathbf{T}}$ be the F -alternating forms on $W = K^2$ and $\mathbf{W} = K^6$ attached to T and \mathbf{T} , respectively (see § 2.2). Let \mathcal{N}_T and $\mathcal{N}_{\mathbf{T}}$ be the Heisenberg groups attached to $(W, \langle \cdot, \cdot \rangle_T)$ and $(\mathbf{W}, \langle \cdot, \cdot \rangle_{\mathbf{T}})$, respectively:

$$\begin{aligned} \mathcal{N}_T &= W \times F, (w, x)(w', x') = (w + w', x + x' + \frac{1}{2}\langle w, w' \rangle_T), \\ \mathcal{N}_{\mathbf{T}} &= \mathbf{W} \times F, (\mathbf{w}, x)(\mathbf{w}', x') = (\mathbf{w} + \mathbf{w}', x + x' + \frac{1}{2}\langle \mathbf{w}, \mathbf{w}' \rangle_{\mathbf{T}}). \end{aligned}$$

Define a smooth representation ρ_T (respectively $\rho_{\mathbf{T}}$) of \mathcal{N}_T (respectively $\mathcal{N}_{\mathbf{T}}$) on $V = \mathcal{S}(K)$ (respectively $\mathbf{V} = \mathcal{S}(K^2) \otimes \mathcal{S}(K)$) as follows:

$$\rho_T \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, x \right) \varphi(z) = \psi \left(-\text{Tr}(w_1^\sigma z) - \frac{1}{2} \text{Tr}(w_1^\sigma w_2) + x \right) \varphi(z + w_2),$$

for $w_1, w_2, z \in K, x \in F, \varphi \in \mathcal{S}(K)$;

$$\begin{aligned} &\rho_{\mathbf{T}} \left(\begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \mathbf{w}_3 \end{bmatrix}, x \right) (\Phi \otimes \varphi)(Z, z) \\ &= \psi \left(-\frac{1}{2} \text{Tr}(\kappa^{-1}T(\mathbf{w}_1, \mathbf{w}_3)) - \text{Tr}(\kappa^{-1}T(\mathbf{w}_1, Z)) + x \right) \times \Phi(Z + \mathbf{w}_3)(\rho_T(\mathbf{w}_2, 0)\varphi)(z) \end{aligned}$$

for $\mathbf{w}_i \in K^2$ ($i = 1, 2, 3$), $x \in F, \Phi \in \mathcal{S}(K^2), \varphi \in \mathcal{S}(K), Z \in K^2, z \in K$. Then ρ_T and $\rho_{\mathbf{T}}$ are irreducible, and any smooth irreducible representation of \mathcal{N}_T (respectively $\mathcal{N}_{\mathbf{T}}$) with central character $(0, x) \mapsto \psi(x)$ is equivalent to ρ_T (respectively $\rho_{\mathbf{T}}$).

6.4 Metaplectic representations

Let $\chi_0 \in \mathcal{X}$. We can define a splitting $\mathcal{M}_{\chi_0}^T$ (respectively $\mathcal{M}_{\chi_0}^{\mathbf{T}}$) of metaplectic representation of $H = U(T)$ (respectively $\mathbf{H} = U(\mathbf{T})$) attached to (ρ_T, χ_0) (respectively $(\rho_{\mathbf{T}}, \chi_0)$) as in [Mur01]. The following two lemmas are proved by a straightforward calculation and we omit their proofs. We note that $\mathcal{M}_{\chi_0}^{\mathbf{T}}$ is a mixed model of the metaplectic representation of \mathbf{H} .

LEMMA 6.1. *Let $\varphi \in \mathcal{S}(K)$ and $z \in K$. Then*

$$\begin{aligned} \mathcal{M}_{\chi_0}^T(\mathbf{d}_H(a))\varphi(z) &= \chi_0^{-1}(a)|N(a)|_F^{1/2} \varphi(az) \quad (a \in K^\times), \\ \mathcal{M}_{\chi_0}^T(\mathbf{n}_H(b))\varphi(z) &= \psi(bzz^\sigma)\varphi(z) \quad (b \in F), \\ \mathcal{M}_{\chi_0}^T(w_0)\varphi(z) &= \lambda_K(\psi)\widehat{\varphi}(z), \end{aligned}$$

where

$$\widehat{\varphi}(z) = \int_K \psi(\text{Tr}(w^\sigma z))\varphi(w) dw.$$

LEMMA 6.2. Let $\Phi \in \mathcal{S}(K^2)$, $\varphi \in \mathcal{S}(K)$, $Z \in K^2$ and $z \in K$. Then

$$\begin{aligned} \mathcal{M}_{\chi_0}^{\mathbf{T}}(1_3 \otimes h)(\Phi \otimes \varphi)(Z, z) &= \chi_0(\det h)\Phi(h^{-1}Z)(\mathcal{M}_{\chi_0}^{\mathbf{T}}(h)\varphi)(z) \quad (h \in H), \\ \mathcal{M}_{\chi_0}^{\mathbf{T}}((w, x) \otimes 1_2)(\Phi \otimes \varphi)(Z, z) &= \psi(x \cdot \kappa^{-1}T[Z])\Phi(Z)(\rho_T(-wZ, 0)\varphi)(z) \quad (w \in K, x \in F), \\ \mathcal{M}_{\chi_0}^{\mathbf{T}}(\mathbf{d}_G(a, t) \otimes 1_2)(\Phi \otimes \varphi)(Z, z) &= \chi_0^{-2}(a)\chi_0(t)|N(a)|_F\Phi(aZ)\varphi(t^{-1}z) \quad (a \in K^\times, t \in K^1), \\ \mathcal{M}_{\chi_0}^{\mathbf{T}}\left(\begin{bmatrix} & & 1 \\ & 1 & \\ -1 & & \end{bmatrix} \otimes 1_2\right)(\Phi \otimes \varphi)(Z, z) &= \widehat{\Phi}(Z)\varphi(z), \end{aligned}$$

where

$$\widehat{\Phi}(Z) = \int_{K^2} \psi(\text{Tr}(\kappa^{-1}T(Z, Z')))\Phi(Z') dZ'.$$

Here dZ' is the Haar measure on K^2 self-dual with respect to the pairing $(Z, Z') \mapsto \psi(\text{Tr}(\kappa^{-1}T(Z, Z')))$.

LEMMA 6.3. Suppose that $F = \mathbb{Q}_p$ and let $\varphi_{0,p}$ be the characteristic function of \mathcal{O}_K . Then we have $\mathcal{M}_{\chi_0}^{\mathbf{T}}(u)\varphi_{0,p} = \widetilde{\chi}_0(u)\varphi_{0,p}$ for $u \in \mathcal{U}'_p$ (for the definition of $\widetilde{\chi}_0$, see § 3.3).

Proof. Put $V_0 = \{\varphi \in V \mid \rho_T(l, \alpha(l))\varphi = \varphi \text{ for any } l \in L'\}$, where $\alpha(w) = \frac{1}{2} \text{Tr}(w_1^\sigma w_2)$ for $w = {}^t(w_1, w_2) \in W$ and $L' = {}^t(\sqrt{D}^{-1}\mathcal{O}_K, \mathcal{O}_K)$ is a lattice of W . It is easily verified that $V_0 = \mathbb{C}\varphi_{0,p}$ and $\mathcal{M}_{\chi_0}^{\mathbf{T}}(u)V_0 \subset V_0$ for $u \in \mathcal{U}'_p$. Thus, $\mathcal{M}_{\chi_0}^{\mathbf{T}}(u)\varphi_{0,p} = c_{\chi_0}(u)\varphi_{0,p}$ ($u \in \mathcal{U}'_p$) with $c_{\chi_0}(u) \in \mathbb{C}^\times$. A direct calculation shows that $c_{\chi_0}(\mathbf{n}_H(b)) = 1$ ($b \in \mathcal{O}_F$), $c_{\chi_0}(\overline{\mathbf{n}}_H(b)) = 1$ ($b \in D\mathcal{O}_F$) and $c_{\chi_0}(\mathbf{d}_H(a)) = \chi_0^{-1}(a)$ ($a \in \mathcal{O}_K^\times$). This immediately implies that $c_{\chi_0}(u) = \widetilde{\chi}_0(u)$ ($u \in \mathcal{U}'_p$). \square

Remark. This lemma gives another proof of Lemma 3.2.

6.5 Intertwining operator

We now describe a relation between our mixed model and the Schrödinger model used in [Kud81]. Define a linear isomorphism $i: K^3 \times K^3 \rightarrow K^6$ by $i(z^+, z^-) = {}^t(z_1^+, z_1^-, z_2^+, z_2^-, z_3^+, z_3^-)$ ($z^\pm = {}^t(z_1^\pm, z_2^\pm, z_3^\pm) \in K^3$). We then have $\langle i(z^+, z^-), i(w^+, w^-) \rangle_{\mathbf{T}} = \text{Tr}(S(z^+, w^-) - S(z^-, w^+))$. Let $\widetilde{\rho}_{\mathbf{T}}$ be the smooth irreducible representation of $\mathcal{N}_{\mathbf{T}}$ on $\mathcal{S}(K^3)$ corresponding to a complete polarization $\mathbf{W} = i(K^3, \{0\}) \oplus i(\{0\}, K^3)$:

$$\widetilde{\rho}_{\mathbf{T}}(i(w^+, w^-), x)f(z) = \psi(x - \text{Tr} S(w^+, z) - \frac{1}{2} \text{Tr} S(w^+, w^-))f(z + w^-)$$

for $w^\pm, z \in K^3, x \in F, f \in \mathcal{S}(K^3)$. Let $\widetilde{\mathcal{M}}_{\chi_0}^{\mathbf{T}}: \mathbf{H} \rightarrow GL(\mathcal{S}(K^3))$ be the metaplectic representation attached to $(\widetilde{\rho}_{\mathbf{T}}, \chi_0)$. Then we have

$$\begin{aligned} \widetilde{\mathcal{M}}_{\chi_0}^{\mathbf{T}}(g \otimes 1_2)f(z) &= \chi_0(\det g)f(g^{-1}z) \quad (g \in G), \\ \widetilde{\mathcal{M}}_{\chi_0}^{\mathbf{T}}(1_3 \otimes \mathbf{d}_H(a))f(z) &= \chi_0^{-3}(a)|N(a)|_F^{3/2}f(az) \quad (a \in K^\times), \\ \widetilde{\mathcal{M}}_{\chi_0}^{\mathbf{T}}(1_3 \otimes \mathbf{n}_H(b))f(z) &= \psi(bS[z])f(z) \quad (b \in F), \\ \widetilde{\mathcal{M}}_{\chi_0}^{\mathbf{T}}(1_3 \otimes w_0)f(z) &= \lambda_K(\psi)\widehat{f}(z), \end{aligned}$$

where

$$\widehat{f}(z) = \int_{K^3} \psi(\text{Tr} S(z, w))f(w) d_S w$$

($d_S w$ is the Haar measure self-dual with respect to the pairing $(w, w') \mapsto \psi(\text{Tr } S(w, w'))$). Note that $\widetilde{\mathcal{M}}_{\chi_0}^{\mathbf{T}}$ is used in [Kud81] to construct the theta kernel. Define $I: \mathcal{S}(K^2) \otimes \mathcal{S}(K) \rightarrow \mathcal{S}(K^3)$ by

$$I(\Phi \otimes \varphi)(z) = \int_K \psi(\text{Tr}(\kappa^{-1} z_1^\sigma u)) \Phi \left(\begin{pmatrix} u \\ z_3 \end{pmatrix} \right) du \cdot \varphi(z_2) \quad (z = {}^t(z_1, z_2, z_3) \in K^3).$$

We easily see that $I \circ \rho_{\mathbf{T}}(\mathbf{w}, x) = \widetilde{\rho}_{\mathbf{T}}(\mathbf{w}, x) \circ I \quad (\mathbf{w} \in \mathbf{W}, x \in F)$, which directly implies

$$I \circ \mathcal{M}_{\chi_0}^{\mathbf{T}}(\mathbf{h}) = \widetilde{\mathcal{M}}_{\chi_0}^{\mathbf{T}}(\mathbf{h}) \circ I \quad (\mathbf{h} \in \mathbf{H}).$$

6.6 Test functions

When $F = \mathbb{Q}_p$, we set $\Phi_{0,p} = \text{char}_{\mathcal{O}_K^2}$ and $\varphi_{0,p} = \text{char}_{\mathcal{O}_K}$. When $F = \mathbb{R}$, we set

$$\begin{aligned} \Phi_{0,\infty} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &= (z_1 - \sqrt{-1}z_2)^l \mathbf{e} \left[\frac{\sqrt{-1}}{2}(z_1 \overline{z_1} + z_2 \overline{z_2}) \right] \quad (z_1, z_2 \in \mathbb{C}), \\ \varphi_{0,\infty}(z) &= \mathbf{e}[\sqrt{-1}z\overline{z}] \quad (z \in \mathbb{C}). \end{aligned}$$

Set $\Phi_0 = \bigotimes_v \Phi_{0,v} \in \mathcal{S}(K_{\mathbf{A}}^2)$ and $\varphi_0 = \bigotimes_v \varphi_{0,v} \in \mathcal{S}(K_{\mathbf{A}})$. A straightforward calculation shows the following.

LEMMA 6.4. We have $I(\Phi_{0,p} \otimes \varphi_{0,p}) = |D|_p^{1/2} \text{char}_{\mathcal{O}_{K,p}^3}$ for $p < \infty$, and

$$I(\Phi_{0,\infty} \otimes \varphi_{0,\infty})({}^t(z_1, z_2, z_3)) = 2\sqrt{-1} \left(\frac{2}{\sqrt{D}} z_1 + z_3 \right)^l \mathbf{e} \left[\sqrt{-1} \left\{ -\frac{2}{D} z_1 \overline{z_1} + z_2 \overline{z_2} + \frac{1}{2} z_3 \overline{z_3} \right\} \right].$$

Using Lemma 6.4, we obtain the following result.

PROPOSITION 6.5. We have the following.

(i) Let $F = \mathbb{Q}_p$. Then we have

$$\mathcal{M}_{\chi_0}^{\mathbf{T}}(k \otimes u)(\Phi_{0,p} \otimes \varphi_{0,p}) = \chi_0(\det k) \chi_0^2(\det u) \widetilde{\chi}_0(u)^{-1} \Phi_{0,p} \otimes \varphi_{0,p} \quad (k \in \mathcal{K}_p, u \in \mathcal{U}'_p).$$

(ii) Let $F = \mathbb{R}$ and put $\lambda = w_{\infty}(\chi_0)$. Then we have

$$\begin{aligned} \mathcal{M}_{\chi_0}^{\mathbf{T}}(k \otimes u)(\Phi_{0,\infty} \otimes \varphi_{0,\infty}) \\ = (\det k)^\lambda J(k, Z_0)^{-l} (\det u)^{(3\lambda+1)/2-l} j(u, z_0)^{l-1} \Phi_{0,\infty} \otimes \varphi_{0,\infty} \quad (k \in \mathcal{K}_{\infty}, u \in \mathcal{U}_{\infty}). \end{aligned}$$

7. Main results

7.1 Theta kernel

We now go back to the global situation. Recall that we have fixed a $\chi_0 \in \mathcal{X}$ with $w_{\infty}(\chi_0) = -1$. Let $\mathcal{M}_{\chi_0}^{\mathbf{T}}$ be the metaplectic representation of $\mathbf{H}_{\mathbf{A}}$ on $\mathcal{S}(K_{\mathbf{A}}^2) \otimes \mathcal{S}(K_{\mathbf{A}})$ given by

$$\mathcal{M}_{\chi_0}^{\mathbf{T}}(\mathbf{h}) = \bigotimes_v \mathcal{M}_{\chi_{0,v}}^{\mathbf{T}}(\mathbf{h}_v) \quad (\mathbf{h} = (\mathbf{h}_v)_v \in \mathbf{H}_{\mathbf{A}}),$$

where the local components $\mathcal{M}_{\chi_{0,v}}^{\mathbf{T}}$ are defined in §6.4. Define a theta kernel on $G_{\mathbf{A}} \times H_{\mathbf{A}}$ by

$$\theta(g, h) = \chi_0^{-1}(\det g) \chi_0^{-2}(\det h) \sum_{X \in K^2, \xi \in K} \mathcal{M}_{\chi_0}^{\mathbf{T}}(g \otimes h)(\Phi_0 \otimes \varphi_0)(X, \xi)$$

for $g \in G_{\mathbf{A}}, h \in H_{\mathbf{A}}$ (cf. §6.6).

LEMMA 7.1. We have

$$\theta(\gamma g k_f k_{\infty}, \gamma' h u_f u_{\infty}) = \widetilde{\chi}_0(u_f)^{-1} J(k_{\infty}, Z_0)^{-l} j'(u_{\infty}, z_0)^{l-1} \theta(g, h)$$

for $\gamma \in G_{\mathbb{Q}}, g \in G_{\mathbf{A}}, k_f \in \mathcal{K}_f, k_{\infty} \in \mathcal{K}_{\infty}, \gamma' \in H_{\mathbb{Q}}, h \in H_{\mathbf{A}}, u_f \in \mathcal{U}'_f, u_{\infty} \in \mathcal{U}_{\infty}$.

Proof. The left $G_{\mathbb{Q}} \times H_{\mathbb{Q}}$ -invariance is proved by a Poisson summation formula in the usual way. The remaining part directly follows from Proposition 6.5. \square

7.2 The unitary Kudla lift

For $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ (cf. § 3.3), we put

$$\mathcal{L}f(g) = \int_{H_{\mathbb{Q}} \backslash H_{\mathbf{A}}} f(h)\theta(g, h) dh \quad (g \in G_{\mathbf{A}}). \tag{7.1}$$

The integral is absolutely convergent since f is rapidly decreasing on $H_{\mathbb{Q}} \backslash H_{\mathbf{A}}$. The theta lift (7.1) was first studied by Kudla [Kud79, Kud81] and we call $\mathcal{L}f$ the *unitary Kudla lift* of f . Lemma 7.1 implies

$$\mathcal{L}f(\gamma g k_f k_{\infty}) = J(k_{\infty}, Z_0)^{-l} \mathcal{L}f(g) \tag{7.2}$$

for $\gamma \in G_{\mathbb{Q}}, g \in G_{\mathbf{A}}, k_f \in \mathcal{K}_f, k_{\infty} \in \mathcal{K}_{\infty}$. In [Kud79], it is proved that

$$\mathcal{L}f \in \mathfrak{S}_l(\mathcal{K}_f) \tag{7.3}$$

for $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ by showing that the lifts of Poincaré series in $S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ are holomorphic. Later we give another proof of (7.3) by studying the Fourier–Jacobi expansion of $\mathcal{L}f$ (see Theorem 7.4). Note that

$$\mathcal{L}f(t1_3 \cdot g) = \Omega^{-1}(t)\mathcal{L}f(g) \quad (t \in K_{\mathbf{A}}^1, g \in G_{\mathbf{A}}) \tag{7.4}$$

for $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0; \chi_0\Omega)$, since $\theta(t1_3 \cdot g, h) = \chi_0(t)\theta(g, t1_2 \cdot h)$ for $t \in K_{\mathbf{A}}^1$.

Remark. Although the realization of the metaplectic representation here is different from that in [Kud81], the theta kernels coincide up to a constant multiple in view of §§ 6.5 and 6.6.

7.3 For $m \in \mathbb{Q}^{\times}$, put

$$\eta_m = \begin{pmatrix} m\theta^{\sigma} \\ 1 \end{pmatrix} \tag{7.5}$$

(for the definition of θ , see § 2.2). Let $H_{\eta_m} = \{h \in H \mid h\eta_m = \eta_m\}$ be the stabilizer subgroup of η_m in H . For $t = z^{\sigma}/z \in K^1$ ($z = x + y\theta \in K^{\times}, x, y \in \mathbb{Q}$), we put

$$\iota_m(t) = \frac{1}{z} \begin{pmatrix} x & mN(\theta)y \\ -y/m & x + \text{Tr}(\theta)y \end{pmatrix}. \tag{7.6}$$

The following fact is easily verified.

LEMMA 7.2. *We have the following.*

- (i) For $t \in K^1$, we have $\iota_m(t) \in H_{\eta_m, \mathbb{Q}}$ and $\det \iota_m(t) = t$.
- (ii) The mapping ι_m defines an isomorphism of K^1 onto $H_{\eta_m, \mathbb{Q}}$.
- (iii) We have $\{X \in K^2 \mid \kappa^{-1}T[X] = m\} = \{\gamma^{-1}\eta_m \mid \gamma \in H_{\eta_m, \mathbb{Q}} \backslash H_{\mathbb{Q}}\}$.

7.4 Spherical functions

Let $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$. For $m \in \mathbb{Q}^{\times}$ and $\chi \in \mathcal{X}$, set

$$W_{\chi, f}^m(h) = \int_{K^1 \backslash K_{\mathbf{A}}^1} (\chi/\chi_0)^1(t^{-1})f(\iota_m(t)h) d^{\times}t \quad (h \in H_{\mathbf{A}}) \tag{7.7}$$

(for the definition of $(\chi/\chi_0)^1$, see § 2.5).

7.5 Holomorphy and cuspidality of the unitary Kudla lift

The following fact is proved in §§ 9.2 and 9.4.

PROPOSITION 7.3. *Let $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ and $F = \mathcal{L}f$.*

- (i) *If $m \leq 0$, we have $F^m = 0$.*
- (ii) *For any $m \in \mathbb{Q}, m > 0$ and any $\alpha = \alpha_\infty \alpha_f \in K_{\mathbf{A}}^\times$, we have $F_\alpha^m = \alpha_\infty^l \mathbf{e}_m [2^{-1} \kappa(\alpha_\infty \overline{\alpha_\infty} - 1)] F_{\alpha_f}^m$ and $F_{\alpha_f}^m \in \mathbf{T}_{\text{hol}}^m$.*

(For the definitions of F^m and F_α^m , see § 5.1.)

THEOREM 7.4 [Kud79, Theorem 5.3]. *For $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0; \chi_0 \Omega)$, we have $\mathcal{L}f \in \mathfrak{S}_l(\mathcal{K}_f; \Omega^{-1})$.*

Proof. This follows from Propositions 5.1 and 7.3 and (7.4). □

7.6 Main results: primitive components

Let $(m, \mathbf{a}, \chi) \in \mathcal{E}$ (cf. § 4.8). Take and fix an element $h_{m\theta, \infty}$ of H_∞ with $h_{m\theta, \infty} \langle z_0 \rangle = m\theta$. Set

$$c_\infty = 4\pi \sqrt{-1}^l (m|\kappa|)^{l-2} \mathbf{e} \left[\frac{m\kappa}{2} \right] j'(h_{m\theta, \infty}, z_0)^{l-1},$$

$$c_p = p^{\mu_p(m, \mathbf{a}) - \delta_p/2} \times \begin{cases} 1 & \text{if } \delta_p = 0, \mu_p(m, \mathbf{a}) = 0, \\ 1 - \omega_p(\pi_p) p^{-1} & \text{if } \delta_p = 0, \mu_p(m, \mathbf{a}) > 0, \\ \frac{2p}{1+p} & \text{if } \delta_p > 0. \end{cases}$$

Note that $c_p = 1$ except for a finite number of p . Denote by $E(\chi)$ the set of finite primes p such that $p|D$ and χ_p is exceptional (cf. § 4.10). For $\epsilon = (\epsilon_p)_{p \in E(\chi)} \in \{0, 1\}^{E(\chi)}$, we put

$$w_0^\epsilon = \prod_{p \in E(\chi)} w_{0,p}^{\epsilon_p} \left(w_{0,p} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in H_p \right), \quad B(\chi, \epsilon) = \prod_{p \in E(\chi)} (p^{-1/2} \chi_p^{-1}(m\theta))^{\epsilon_p}.$$

We set

$$J(m, \mathbf{a}, \chi, f) = \sum_{\epsilon \in \{0,1\}^{E(\chi)}} B(\chi, \epsilon) W_{\chi, f}^m(\mathbf{d}_H(\alpha_f^{-1}) w_0^\epsilon h_{m\theta, \infty}),$$

where we choose $\alpha_f \in K_{\mathbf{A}, f}^\times$ so that $\text{id}(\alpha_f) = \mathbf{a}$.

Remark. Suppose that K has class number 1. Let f_{dm} be as in the remark in § 3.4. Then we have, for any $h_f \in H_{\mathbf{A}, f}$,

$$W_{\chi, f}^m(h_f h_{m\theta, \infty}) = \sum_{j=1}^N a_j f_{dm}(\tau_j),$$

where $a_j \in \mathbb{C}$ (depending only on m, χ, h_f) and $\tau_j \in K \cap \mathfrak{H}$ (depending only on m, h_f). It follows that $J(m, \mathbf{a}, \chi, f)$ is a linear combination of CM-values of f_{dm} . A similar assertion holds (with suitable modifications) when $h_K > 1$.

THEOREM 7.5 (Main Theorem I). *Let $(m, \mathbf{a}, \chi) \in \mathcal{E}, \Theta \in \mathbf{T}_{\text{hol, prim}}^m(\mathbf{a}, \chi)$ and $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$. Then the primitive component (cf. § 5.2) of the unitary Kudla lift $\mathcal{L}f$ is given by*

$$C_{\mathcal{L}f}(\mathbf{a}; \Theta) = c_\infty \prod_{p < \infty} c_p \cdot \chi_0^{-1}(\alpha_f) |N(\alpha_f)|^{3/2} \cdot \overline{I(\Theta)} \cdot J(m, \mathbf{a}, \chi, f).$$

7.7 Main results: non-primitive components

THEOREM 7.6 (Main Theorem II). *Let $f \in S_{l-1}(\mathcal{U}'_p, \tilde{\chi}_0; \chi_0\Omega)$ be a Hecke eigenform with eigenvalues $\{\lambda_p\}$. Let p be a finite prime, $(m, \mathfrak{a}, \chi) \in \mathcal{E}$ and $\Theta \in \mathbf{T}_{\text{hol,prim}}^m(\mathfrak{a}, \chi)$. We write μ_p for $\mu_p(m, \mathfrak{a})$.*

(i) *Suppose that p is inert in K/\mathbb{Q} . Set $c(k) = C_{\mathcal{L}f}(p^k\mathfrak{a}; \Theta)$. Then*

$$p^4c(k+1) + \{\delta(k + \mu_p > 0)p - p^2 - p\lambda_p(\phi_{1,p})\}c(k) + c(k-1) = 0$$

holds for $k \geq 0$.

(ii) *Suppose that p ramifies in K/\mathbb{Q} . Set $c(k) = C_{\mathcal{L}f}(\Pi^k\mathfrak{a}; \Theta)$. Then*

$$p^2c(k+1) - \{p^{1/2}\lambda_p(\phi_p^+ + \phi_p^-) + p \cdot \delta(k = 0 \text{ and } \chi \text{ is ordinary})\}c(k) + c(k-1) = 0$$

holds for $k \geq 0$.

(iii) *Suppose that p splits in K/\mathbb{Q} . Set $c(k_1, k_2) = C_{\mathcal{L}f}(\Pi_1^{k_1}\Pi_2^{k_2}\mathfrak{a}; \Theta)$. Then*

$$p^2c(k_1, k_2 + 1) - (p^{1/2}\lambda_p(\phi_{2,p}) + p)c(k_1, k_2) + p\Omega^{-1}(\Pi_1/\Pi_2)c(k_1 + 1, k_2 - 1) + c(k_1 - 1, k_2) = 0$$

and

$$p^2c(k_1 + 1, k_2) - (p^{1/2}\lambda_p(\phi_{1,p}) + p)c(k_1, k_2) + p\Omega(\Pi_1/\Pi_2)c(k_1 - 1, k_2 + 1) + c(k_1, k_2 - 1) = 0$$

hold for $k_1, k_2 \geq 0$.

Combining Theorem 7.6, Propositions 5.3–5.5 and Theorem 5.6, we obtain the following results.

COROLLARY 7.7. *Let $f \in S_{l-1}(\mathcal{U}'_p, \tilde{\chi}_0; \chi_0\Omega)$ be a Hecke eigenform with eigenvalues $\{\lambda_p\}$.*

(i) *The unitary Kudla lift $\mathcal{L}f$ is a Hecke eigenform with eigenvalues $\{\Lambda_p\}$, where Λ_p is given as follows.*

(a) *If p is inert in K/\mathbb{Q} , we have $\Lambda_p(\Phi_{1,p}) = p\lambda_p(\phi_{1,p}) + p^2 - 1$.*

(b) *If p ramifies in K/\mathbb{Q} , we have $\Lambda_p(\Phi_{1,p}) = p^{1/2}\lambda_p(\phi_p^+ + \phi_p^-) + p - 1$.*

(c) *If p splits in K/\mathbb{Q} , we have $\Lambda_p(\Phi_{1,p}) = p^{1/2}\lambda_p(\phi_{2,p}) + p$ and $\Lambda_p(\Phi_{2,p}) = p^{1/2}\lambda_p(\phi_{1,p}) + p$.*

(ii) *For a Hecke character ξ of K , we have*

$$L(\mathcal{L}f, \xi; s) = L(\xi; s)L(f, \xi; s).$$

(For the definition of L -functions, see §§ 3.2 and 3.4.)

Remark. Corollary 7.7 was proved by Kudla [Kud81, Corollary 1] by using Eichler commutation relations.

THEOREM 7.8 (Main Theorem III). *Let $f \in S_{l-1}(\mathcal{U}'_p, \tilde{\chi}_0; \chi_0\Omega)$ be a Hecke eigenform. The unitary Kudla lift $\mathcal{L}f$ does not vanish if and only if there exists $(m, \mathfrak{a}, \chi) \in \mathcal{E}$ such that*

$$L(\chi; \frac{1}{2}) \cdot J(m, \mathfrak{a}, \chi, f) \neq 0.$$

Proof. This follows from Theorems 5.6, 7.5 and 4.7. □

8. Integral expression of the Fourier–Jacobi components

8.1 Let $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$. The object of this section is to give an integral expression of the Fourier–Jacobi components of $\mathcal{L}f$. By Lemma 6.2, we have

$$\begin{aligned} \mathcal{L}f((w, x)t\mathbf{d}_G(\alpha)) &= \chi_0^{-1}(t)|N(\alpha)|_{\mathbf{A}} \int_{H_{\mathbb{Q}} \backslash H_{\mathbf{A}}} \sum_{X \in K^2, \xi \in K} \chi_0^{-1}(\det h)\psi(\kappa^{-1}T[X]x) \\ &\quad \times \Phi_0(\alpha h^{-1}X)(\mathcal{M}_{\chi_0}^T(h)\rho_T(-wh^{-1}X, 0)\mathcal{M}_{\chi_0}^T(t\mathbf{1}_2)\varphi_0)(\xi)f(h) dh \end{aligned}$$

for $(w, x) \in N_{\mathbf{A}}, t \in K_{\mathbf{A}}^1$ and $\alpha \in K_{\mathbf{A}}^{\times}$ (for the definition of $\mathcal{M}_{\chi_0}^T$ and ρ_T , see §§ 6.4 and 6.3).

8.2 The constant term

PROPOSITION 8.1. We have $(\mathcal{L}f)^0 = 0$.

Proof. Let $\eta' = {}^t(1, 0) \in K^2$ and $N' = N^H$. Since $\{X \in K^2 \mid T[X] = 0\} = \{\gamma^{-1}\eta' \mid \gamma \in N'_\mathbb{Q} \backslash H_\mathbb{Q}\}$, we have

$$\begin{aligned} (\mathcal{L}f)^0((w, 0)t\mathbf{d}_G(\alpha)) &= \chi_0^{-1}(t)|N(\alpha)|_{\mathbf{A}} \int_{H_\mathbb{Q} \backslash H_{\mathbf{A}}} \sum_{\gamma \in N'_\mathbb{Q} \backslash H_\mathbb{Q}} \chi_0^{-1}(\det \gamma h) \Phi_0(\alpha(\gamma h)^{-1}\eta') \\ &\quad \times \sum_{\xi \in K} (\mathcal{M}_{\chi_0}^T(h) \rho_T(-w(\gamma h)^{-1}\eta', 0) \mathcal{M}_{\chi_0}^T(t1_2)\varphi_0)(\xi) f(h) dh. \end{aligned}$$

Since $h \mapsto \sum_{\xi \in K} \mathcal{M}_{\chi_0}^T(h)\varphi(\xi)$ is left $H_\mathbb{Q}$ -invariant for any $\varphi \in \mathcal{S}(K)$, we obtain

$$\begin{aligned} (\mathcal{L}f)^0((w, 0)t\mathbf{d}_G(\alpha)) &= \chi_0^{-1}(t)|N(\alpha)|_{\mathbf{A}} \int_{N'_\mathbb{Q} \backslash H_{\mathbf{A}}} \chi_0^{-1}(\det h) \Phi_0(\alpha h^{-1}\eta') \sum_{\xi \in K} (\rho_T(-w\eta', 0) \mathcal{M}_{\chi_0}^T(th)\varphi_0)(\xi) f(h) dh \\ &= \chi_0^{-1}(t)|N(\alpha)|_{\mathbf{A}} \int_{N'_\mathbb{Q} \backslash H_{\mathbf{A}}} \chi_0^{-1}(\det h) \Phi_0(\alpha h^{-1}\eta') \sum_{\xi \in K} \psi(\mathrm{Tr}(w^\sigma \xi)) (\mathcal{M}_{\chi_0}^T(th)\varphi_0)(\xi) f(h) dh \\ &= \chi_0^{-1}(t)|N(\alpha)|_{\mathbf{A}} \int_{N'_\mathbf{A} \backslash H_{\mathbf{A}}} \chi_0^{-1}(\det h) \Phi_0(\alpha h^{-1}\eta') \sum_{\xi \in K} \psi(\mathrm{Tr}(w^\sigma \xi)) (\mathcal{M}_{\chi_0}^T(th)\varphi_0)(\xi) c_f(-N(\xi); h) dh. \end{aligned}$$

Here we set

$$c_f(b; h) = \int_{\mathbb{Q} \backslash \mathbb{Q}_{\mathbf{A}}} \psi(-bx) f(\mathbf{n}_H(x)h) dx \quad (h \in H_{\mathbf{A}})$$

for $b \in \mathbb{Q}$ and $h \in H_{\mathbf{A}}$. We have $c_f(-N(\xi); h) = 0$ for $\xi \in K$ by holomorphy and cuspidality of f . This implies that $(\mathcal{L}f)^0((w, 0)t\mathbf{d}_G(\alpha)) = 0$ and we are done. \square

8.3 Non-constant terms

Let $m \in \mathbb{Q}^\times$ and $\beta \in K_{\mathbf{A}}^\times$. Then $(\mathcal{L}f)_\beta^m \in \mathbf{T}^m(\mathbf{b}) \cap \mathbf{T}^{m,0}$ with $\mathbf{b} = \mathrm{id}(\beta)$. Let $\chi \in \mathcal{X}$. For $\Theta \in \mathbf{T}_\chi^m$, we set

$$\Lambda_{\beta, \Theta}^m(h) = |N(\beta)|_{\mathbf{A}} \chi_0^{-1}(\det h) \Phi_0(\beta h^{-1}\eta_m) \int_{K_{\mathbf{A}}} (\mathcal{M}_{\chi_0}^T(h)\varphi_0)(-w) \overline{I_\Theta(w, x_w)} dw \quad (8.1)$$

(for the definition of I_Θ , see § 4.11).

LEMMA 8.2. For $\Theta \in \mathbf{T}_\chi^m$, we have

$$\langle (\mathcal{L}f)_\beta^m, \Theta \rangle_R = \int_{H_{\eta_m, \mathbb{Q}} \backslash H_{\mathbf{A}}} \Lambda_{\beta, \Theta}^m(h) f(h) dh.$$

Proof. By Lemma 7.2 and an argument similar to that in § 8.2, we have

$$\begin{aligned} (\mathcal{L}f)_\beta^m((w, 0)t) &= \chi_0^{-1}(t)|N(\beta)|_{\mathbf{A}} \int_{H_{\eta_m, \mathbb{Q}} \backslash H_{\mathbf{A}}} \chi_0^{-1}(\det h) \Phi_0(\beta h^{-1}\eta_m) \\ &\quad \times \sum_{\xi \in K} \psi(\mathrm{Tr}(m\theta\xi w^\sigma) - mx_w) (\mathcal{M}_{\chi_0}^T(th)\varphi_0)(\xi - w) f(h) dh \end{aligned}$$

for $w \in K_{\mathbf{A}}$ and $t \in K_{\mathbf{A}}^1$. The left $R_\mathbb{Q}$ -invariance of Θ implies that

$$\Theta((w, 0)t) = \psi\left(\frac{1}{2} \mathrm{Tr}(m\kappa\xi w^\sigma)\right) \Theta((w - \xi, 0)t)$$

for $\xi \in K$. We thus have

$$\begin{aligned} & \langle\langle (\mathcal{L}f)_\beta^m, \Theta \rangle\rangle_R \\ &= |\mathbf{N}(\beta)|_{\mathbf{A}} \int_{H_{\eta_m, \mathbb{Q}} \backslash H_{\mathbf{A}}} \int_{K \backslash K_{\mathbf{A}}} \int_{K^1 \backslash K_{\mathbf{A}}^1} \chi_0^{-1}(t \det h) \Phi_0(\beta h^{-1} \eta_m) f(h) \\ & \quad \times \sum_{\xi \in K} \psi \left(-\frac{m \operatorname{Tr}(\theta)}{2} (w - \xi)(w - \xi)^\sigma \right) (\mathcal{M}_{\chi_0}^T(th)\varphi_0)(\xi - w) \overline{\Theta((w - \xi, 0)t)} d^\times t dw dh \\ &= |\mathbf{N}(\beta)|_{\mathbf{A}} \int_{H_{\eta_m, \mathbb{Q}} \backslash H_{\mathbf{A}}} \int_{K_{\mathbf{A}}} \int_{K^1 \backslash K_{\mathbf{A}}^1} \chi_0^{-1}(t \det h) \Phi_0(\beta h^{-1} \eta_m) f(h) \\ & \quad \times (\mathcal{M}_{\chi_0}^T(th)\varphi_0)(-w) \overline{\Theta((w, x_w)t)} d^\times t dw dh \\ &= |\mathbf{N}(\beta)|_{\mathbf{A}} \int_{H_{\eta_m, \mathbb{Q}} \backslash H_{\mathbf{A}}} \int_{K_{\mathbf{A}}} \int_{K^1 \backslash K_{\mathbf{A}}^1} \chi_0^{-1}(\det h) \Phi_0(\beta h^{-1} \eta_m) f(h) \\ & \quad \times (\mathcal{M}_{\chi_0}^T(h)\varphi_0)(-t^{-1}w) \overline{\Theta(t(t^{-1}w, x_{t^{-1}w}))} d^\times t dw dh \\ &= \int_{H_{\eta_m, \mathbb{Q}} \backslash H_{\mathbf{A}}} \Lambda_{\beta, \Theta}^m(h) f(h) dh. \end{aligned} \quad \square$$

LEMMA 8.3 (Key lemma). *Let v be a prime of \mathbb{Q} . Let $\varphi \in \mathcal{S}(K_v)$ and $\Theta \in \mathbf{T}_\chi^m$. Then, for $t \in K_v^1$ and $r \in R_{\mathbf{A}}$, we have*

$$\int_{K_v} (\mathcal{M}_{\chi_0}^T(\iota_m(t))\varphi)(-w) \overline{\Theta(r(w, x_w))} dw = (\chi\chi_0)^1(t^{-1}) \int_{K_v} \varphi(-w) \overline{\Theta(rt(w, x_w))} dw.$$

Proof. If $t = 1$, the assertion is trivial. Suppose that $t \neq 1$. Take $\Phi \in V^m$ such that $\Theta = \theta_{\chi, \Phi}^m$ (cf. § 4.4). It is sufficient to show the equality

$$\int_{K_v} (\mathcal{M}_{\chi_0}^T(\iota_m(t))\varphi)(-w) \overline{\mathcal{M}_\chi^{\kappa, m}(r(w, x_w))\Phi} dw = (\chi\chi_0)^1(t^{-1}) \int_{K_v} \varphi(-w) \overline{\mathcal{M}_\chi^{\kappa, m}(rt(w, x_w))\Phi} dw. \quad (*)$$

Take an element $x + y\theta$ of K_v^\times with $x, y \in \mathbb{Q}_v$ ($y \neq 0$) such that $t = (x + y\theta)^\sigma / (x + y\theta)$. Observe that

$$T(\iota_m(t) - 1_2) = \frac{y}{x + y\theta} \begin{pmatrix} -m^{-1} & \theta^\sigma \\ \theta & -mN(\theta) \end{pmatrix} = \frac{y}{x + y\theta} (A^*)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & m\kappa^2 \end{pmatrix} A^{-1}$$

with

$$A = \begin{pmatrix} m\theta^\sigma & m\theta \\ 1 & 1 \end{pmatrix},$$

and that we can take $\{\zeta_m = {}^t(m\theta, 1)\}$ as a basis of $K_v^2 / \operatorname{Ker}(\iota_m(t) - 1_2)$. Note that $\iota_m(t)\zeta_m = t\zeta_m$. In view of [Mur01, Theorem 1.8], we have

$$\begin{aligned} & (\mathcal{M}_{\chi_0}^T(\iota_m(t))\varphi)(-w) \\ &= \gamma(t) \int_{K_v} \psi \left(\frac{1}{2} \langle z\zeta_m, z\iota_m(t)\zeta_m \rangle_T \right) \rho_T(z(1_2 - \iota_m(t))\zeta_m, 0) \varphi(-w) dz \\ &= \gamma(t) \int_{K_v} \psi_m \left(-\frac{1}{2} \operatorname{Tr}(\kappa t) z z^\sigma + \operatorname{Tr}(\theta^\sigma(1 - t^\sigma) z^\sigma w) - x_{(1-t)z} \right) \varphi(-w + z(1 - t)) dz, \end{aligned}$$

where

$$\begin{aligned} \gamma(t) &= |\mathbf{N}(m\kappa(1 - t))|_v^{1/2} \lambda_{K_v}(\psi_v)^{-1} \chi_0 \left(\frac{y}{x + y\theta} m\kappa^2 \right) \\ &= |\mathbf{N}(m\kappa(1 - t))|_v^{1/2} \overline{\lambda_{K_v}(\psi_{m,v})^{-1}} \chi_0 \left(\frac{1 - t}{\kappa} \right). \end{aligned}$$

It follows that the left-hand side of (*) is equal to

$$\begin{aligned} & \gamma(t) \int_{K_v} \int_{K_v} \psi_m \left(-\frac{1}{2} \operatorname{Tr}(\kappa t) z z^\sigma + \operatorname{Tr}(\theta^\sigma (1 - t^\sigma) z^\sigma w) - x_{(1-t)z} - x_w \right) \\ & \quad \times \varphi(-w + z(1-t)) \overline{\mathcal{M}_\chi^{\kappa,m}(r(w, 0))} \Phi \, dz \, dw \\ & = \chi_0 \left(\frac{1-t}{\kappa} \right) |\mathbf{N}(m\kappa(1-t))|_v^{1/2} \int_{K_v} \varphi(-w) \\ & \quad \times \lambda_{K_v}(\psi_{m,v}) \int_{K_v} \psi_m \left(\frac{1}{2} \langle z, tz \rangle_\kappa \right) \overline{\mathcal{M}_\chi^{\kappa,m}(r((1-t)z, 0)(w, x_w))} \Phi \, dz \, dw \\ & = (\chi\chi_0) \left(\frac{1-t}{\kappa} \right) \int_{K_v} \varphi(-w) \overline{\mathcal{M}_\chi^{\kappa,m}(rt(w, x_w))} \Phi \, dw. \end{aligned}$$

Since $(\chi\chi_0)((1-t)/\kappa) = (\chi\chi_0)^1(t^{-1})(t \in K_{\mathbf{A}}^1)$, we are done. □

PROPOSITION 8.4. Let $m \in \mathbb{Q}^\times, \beta \in K_{\mathbf{A}}^\times, \chi \in \mathcal{X}$ and $\Theta \in \mathbf{T}_\chi^m$.

- (i) We have $\Lambda_{\beta,\Theta}^m(\iota_m(t)h) = (\chi/\chi_0)^1(t^{-1})\Lambda_{\beta,\Theta}^m(h)$ ($t \in K_{\mathbf{A}}^1, h \in H_{\mathbf{A}}$).
- (ii) For $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$, we have

$$\langle\langle (\mathcal{L}f)_\beta^m, \Theta \rangle\rangle_R = \int_{H_{\eta_m, \mathbf{A}} \setminus H_{\mathbf{A}}} \Lambda_{\beta,\Theta}^m(h) W_{\chi,f}^m(h) \, dh,$$

where $W_{\chi,f}^m$ is defined by (7.7) and the Haar measure dx on $H_{\eta_m, \mathbf{A}}$ is normalized by

$$\int_{H_{\eta_m, \mathbf{A}}} \varphi(x) \, dx = \int_{K_{\mathbf{A}}^1} \varphi(\iota_m(t)) \, d^\times t \quad (\varphi \in L^1(H_{\eta_m, \mathbf{A}})).$$

Proof. The proposition is a direct consequence of Lemmas 8.2 and 8.3. □

9. Holomorphy and cuspidality

9.1 Real spherical functions

For $m \in \mathbb{Q}^\times$ and $k \in \mathbb{Z}$, let $\mathcal{W}_\infty^m(k)$ be the space of smooth functions W on H_∞ satisfying:

- (i) $W(\iota_m(t)hu) = t^k j'(u, z_0)^{1-l} W(h)$ ($t \in K_\infty^1, h \in H_\infty, u \in \mathcal{U}_\infty$);
- (ii) $\widetilde{W}(h\langle z_0 \rangle) := j'(h, z_0)^{l-1} W(h)$ is holomorphic on \mathfrak{H} .

9.2 Inverse Cayley transform

Let $\mathbf{D} = \{w \in \mathbb{C} \mid |w| < 1\}$ be the unit disc in \mathbb{C} . We define the inverse Cayley transform $C_m : \mathbf{D} \rightarrow \mathfrak{H}$ by

$$C_m(w) = \begin{cases} \frac{m(\theta - \theta^\sigma w)}{1 - w} & \text{if } m > 0, \\ \frac{m(\theta^\sigma - \theta w)}{1 - w} & \text{if } m < 0 \end{cases}$$

($w \in \mathbf{D}$). The following fact is easily verified.

LEMMA 9.1. Let $t \in K_\infty^1, w \in \mathbf{D}$ and $W \in \mathcal{W}_\infty^m(k)$. Then we have

$$\begin{aligned} \iota_m(t)\langle C_m(w) \rangle &= \begin{cases} C_m(t^{-1}w) & \text{if } m > 0, \\ C_m(tw) & \text{if } m < 0, \end{cases} \\ j'(\iota_m(t), C_m(w)) &= \begin{cases} \frac{1-t^{-1}w}{1-w} & \text{if } m > 0, \\ \frac{t^{-1}-w}{1-w} & \text{if } m < 0, \end{cases} \\ \widetilde{W}(\iota_m(t)\langle C_m(w) \rangle) &= \begin{cases} t^k \left(\frac{1-t^{-1}w}{1-w} \right)^{l-1} \widetilde{W}(C_m(w)) & \text{if } m > 0, \\ t^{k-l+1} \left(\frac{1-tw}{1-w} \right)^{l-1} \widetilde{W}(C_m(w)) & \text{if } m < 0. \end{cases} \end{aligned}$$

LEMMA 9.2. Let $W \in \mathcal{W}_\infty^m(k)$ and $z \in \mathfrak{H}$.

(i) If $m > 0$, we have

$$\widetilde{W}(z) = \begin{cases} c(z - m\theta)^{-k}(z - m\theta^\sigma)^{k-l+1} & \text{if } k \leq 0, \\ 0 & \text{if } k > 0 \end{cases}$$

with $c \in \mathbb{C}$.

(ii) If $m < 0$, we have

$$\widetilde{W}(z) = \begin{cases} c(z - m\theta)^{-k}(z - m\theta^\sigma)^{k-l+1} & \text{if } k \geq l - 1, \\ 0 & \text{if } k < l - 1 \end{cases}$$

with $c \in \mathbb{C}$.

Proof. First assume that $m > 0$. Put $W^*(w) = (1-w)^{1-l}\widetilde{W}(C_m(w))$ for $w \in \mathbf{D}$. Then W^* is holomorphic on \mathbf{D} and satisfies $W^*(t^{-1}w) = t^k W^*(w)$ for $t \in K_\infty^1$ by Lemma 9.1. It follows that $W^*(w) = c'\delta(k \leq 0)w^{-k}$ with $c' \in \mathbb{C}$. Since $w = (z-m\theta)/(z-m\theta^\sigma)$ and $1-w = m(\theta-\theta^\sigma)/(z-m\theta^\sigma)$, we have proved assertion (i). The assertion (ii) can be proved similarly. \square

9.3 Global spherical functions

In this subsection, we let $f \in S_{l-1}(\mathcal{U}'_f, \widetilde{\chi}_0)$ and $m \in \mathbb{Q}^\times$. Recall that, for $\chi \in \mathcal{X}$, $W_{\chi,f}^m$ is defined by (7.7).

PROPOSITION 9.3. We have the following.

- (i) If $m > 0$ and $w_\infty(\chi) < -1$, we have $W_{\chi,f}^m = 0$.
- (ii) If $m < 0$ and $w_\infty(\chi) > 1 - 2l$, we have $W_{\chi,f}^m = 0$.

Proof. Observe that $h_\infty \mapsto W_{\chi,f}^m(h_f h_\infty)$ belongs to $\mathcal{W}_\infty^m(-(w_\infty(\chi) + 1)/2)$ for any $h_f \in H_{\mathbf{A},f}$. Then the proposition directly follows from Lemma 9.2. \square

PROPOSITION 9.4. We have the following.

- (i) If $m > 0$, we have $(\mathcal{L}f)_\beta^m \in \mathbf{T}_{\text{hol}}^m$ ($\beta \in K_{\mathbf{A}}^\times$).
- (ii) If $m < 0$, we have $(\mathcal{L}f)^m = 0$.

Proof. By (7.2), we have $(\mathcal{L}f)_\beta^m \in \mathbf{T}^{m,0}$. Suppose that $m > 0$. By Lemma 4.2(i), we have $\mathbf{T}^{m,0} = \bigoplus_{\chi \in \mathcal{X}, w_\infty(\chi) \leq -1} \mathbf{T}_\chi^{m,0}$. On the other hand, Propositions 8.4 and 9.3 imply that $(\mathcal{L}f)_\beta^m \in \bigoplus_{\chi \in \mathcal{X}, w_\infty(\chi) \geq -1} \mathbf{T}_\chi^{m,0}$. We thus have $(\mathcal{L}f)_\beta^m \in \bigoplus_{\chi \in \mathcal{X}, w_\infty(\chi) = -1} \mathbf{T}_\chi^{m,0} = \mathbf{T}_{\text{hol}}^m$ (see Lemma 4.2(ii)). The assertion (ii) can be proved similarly. \square

9.4 Calculation of local integrals at ∞

From now on, we assume that $m > 0$. Let $\Theta \in \mathbf{T}_{\text{hol}}^m$ and fix an $r_f \in R_{\mathbf{A},f}$. We put $\tau(r_\infty) = I_\Theta(r_f r_\infty)$ for $r_\infty \in R_\infty$.

LEMMA 9.5. For $z \in \mathfrak{H}$, we have

$$\int_{\mathbb{C}} \mathbf{e}[z \mathbf{N}(w)] \tau(w, x_w) dw = \frac{\sqrt{-1}}{z + m\theta} \tau(1).$$

Proof. Since

$$w \mapsto \tilde{\tau}(w) := \mathbf{e}\left[-\frac{m\kappa}{2} \mathbf{N}(w)\right] \tau(w, 0)$$

is holomorphic on \mathbb{C} , the integral of the lemma is equal to

$$\int_{\mathbb{C}} \mathbf{e}[(z + m\theta) \mathbf{N}(w)] \tilde{\tau}(w) dw = \tilde{\tau}(0) \int_{\mathbb{C}} \mathbf{e}[(z + m\theta) \mathbf{N}(w)] dw = \frac{\sqrt{-1}}{z + m\theta} \tau(1)$$

and we are done. □

For $\beta_\infty \in K_\infty^\times$ and $W \in \mathcal{W}_\infty^m(0)$, set

$$\begin{aligned} \mathcal{I}_{\beta_\infty}(W, \tau) &= \mathbf{N}(\beta_\infty) \int_{H_{\eta_m, \infty} \setminus H_\infty} \int_{\mathbb{C}} \chi_0^{-1}(\det h) \Phi_{0, \infty}(\beta_\infty h^{-1} \eta_m) (\mathcal{M}_{\chi_0}^T(h) \varphi_{0, \infty})(-w) W(h) \overline{\tau(w, x_w)} dw dh. \end{aligned}$$

Here we note that

$$h \mapsto \chi_0^{-1}(\det h) \Phi_{0, \infty}(\beta_\infty h^{-1} \eta_m) W(h) \int_{\mathbb{C}} (\mathcal{M}_{\chi_0}^T(h) \varphi_{0, \infty})(-w) \overline{\tau(w, x_w)} dw$$

is left $H_{\eta_m, \infty}$ -invariant. This fact is verified by using Lemma 8.3 and the fact that τ is left K_∞^1 -invariant.

PROPOSITION 9.6. We have

$$\mathcal{I}_{\beta_\infty}(W, \tau) = c_\infty \cdot \beta_\infty^l \mathbf{e}_m \left[\frac{\kappa}{2} (\mathbf{N}(\beta_\infty) - 1) \right] \overline{\tau(1)} W(h_{m\theta, \infty})$$

(for the definitions of c_∞ and $h_{m\theta, \infty}$, see § 7.6).

Proof. To simplify the notation, we suppress ∞ from the notation. Since H_{η_m} is compact and $\text{vol}(H_{\eta_m}) = 1$, $\mathcal{I}_\beta(W, \tau)$ is equal to

$$\mathbf{N}(\beta) \int_H \int_{\mathbb{C}} \chi_0^{-1}(\det h) \Phi_{0, \infty}(\beta h^{-1} \eta_m) (\mathcal{M}_{\chi_0}^T(h) \varphi_{0, \infty})(-w) W(h) \overline{\tau(w, x_w)} dw dh.$$

Observe that the integrand of the above integral is, as a function of h , right \mathcal{U}_∞ -invariant, since $\Phi_{0, \infty}(u^{-1}Z) = j^l(u, z_0) \Phi_{0, \infty}(Z)$ and $\mathcal{M}_{\chi_0}^T(u) \varphi_{0, \infty} = j(u, z_0)^{-1} \varphi_{0, \infty}$ for $u \in \mathcal{U}_\infty$. Thus, $\mathcal{I}_\beta(W, \tau)$ is equal to

$$\begin{aligned} &2\pi \mathbf{N}(\beta) \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{C}} y^{-1} \Phi_{0, \infty}(\beta \mathbf{d}_H(\sqrt{y}^{-1}) \mathbf{n}_H(-x) \eta_m) \\ &\quad \times (\mathcal{M}_{\chi_0}^T(\mathbf{n}_H(x) \mathbf{d}_H(\sqrt{y})) \varphi_0)(-w) W(\mathbf{n}_H(x) \mathbf{d}_H(\sqrt{y})) \overline{\tau(w, x_w)} dw d^\times y dx \\ &= 2\pi \mathbf{N}(\beta) \beta^l \int_{\mathbb{R}} \int_0^\infty \int_{\mathbb{C}} y^{-1} \{\sqrt{y}^{-1}(-x + m\theta^\sigma) - \sqrt{-1}\sqrt{y}\}^l \\ &\quad \times \mathbf{e}\left[\frac{\sqrt{-1}}{2} \mathbf{N}(\beta) \{y + y^{-1} \mathbf{N}(-x + m\theta^\sigma)\}\right] \cdot y^{1/2} \mathbf{e}[(x + \sqrt{-1}y) \mathbf{N}(w)] \\ &\quad \times y^{(l-1)/2} \widetilde{W}(x + \sqrt{-1}y) \overline{\tau(w, x_w)} dw d^\times y dx. \end{aligned}$$

By Lemma 9.2, we have $\widetilde{W}(x + \sqrt{-1}y) = (m\kappa)^{l-1}\widetilde{W}(m\theta) \cdot (x + \sqrt{-1}y - m\theta^\sigma)^{-l+1}$ and, hence,

$$\begin{aligned} \mathcal{I}_\beta(W, \tau) &= 2\pi N(\beta)\beta^l(m\kappa)^{l-1}\widetilde{W}(m\theta) \int_{\mathbb{R}} \int_0^\infty y^{-1}(x + \sqrt{-1}y - m\theta^\sigma) \\ &\quad \times e\left[\frac{\sqrt{-1}}{2}N(\beta)\{y + y^{-1}N(-x + m\theta^\sigma)\}\right] \int_{\mathbb{C}} e[(x + \sqrt{-1}y)N(w)]\overline{\tau(w, x_w)} dw d^\times y dx. \end{aligned}$$

Using Lemma 9.5, we obtain

$$\mathcal{I}_\beta(W, \tau) = 2\pi\sqrt{-1}(m\kappa)^{l-1}N(\beta)\beta^l \cdot \overline{\tau(1)}\widetilde{W}(m\theta) \cdot I,$$

where

$$I = \int_{\mathbb{R}} \int_0^\infty y^{-1}e\left[\frac{\sqrt{-1}}{2}N(\beta)(y + y^{-1}N(-x + m\theta^\sigma))\right] d^\times y dx = \left(\frac{mN(\beta)|\kappa|}{2}\right)^{-1} e_m\left[\frac{\kappa}{2}N(\beta)\right].$$

Since $\widetilde{W}(m\theta) = j'(h_{m\theta, \infty}, z_0)^{l-1}W(h_{m\theta, \infty})$, we have

$$\mathcal{I}_\beta(W, \tau) = 2\pi\sqrt{-1}(m\kappa)^{l-1}(m|\kappa|/2)^{-1}j'(h_{m\theta, \infty}, z_0)^{l-1}e[m\kappa/2]\beta^l e_m\left[\frac{\kappa}{2}(N(\beta) - 1)\right]\overline{\tau(1)}W(h_{m\theta, \infty}),$$

which proves the proposition. □

By Propositions 8.4 and 9.6, we have proved the following result, from which Proposition 7.3 follows.

PROPOSITION 9.7. For $m > 0, \beta = \beta_f\beta_\infty \in K_{\mathbf{A}}^\times$ and $\Theta \in \mathbf{T}_{\text{hol}}^m$, we have

$$\begin{aligned} &\langle\langle (\mathcal{L}f)_\beta^m, \Theta \rangle\rangle_R \\ &= c_\infty \cdot \beta_\infty^l e_m\left[\frac{\kappa}{2}(N(\beta_\infty) - 1)\right] |N(\beta_f)|_{\mathbf{A}} \int_{(H_{\eta_m})_{\mathbf{A}, f} \setminus H_{\mathbf{A}, f}} \chi_0^{-1}(\det h_f)\Phi_{0, f}(\beta_f h_f^{-1}\eta_m)W_{\chi, f}^m(h_f h_{m\theta, \infty}) \\ &\quad \times \int_{K_{\mathbf{A}, f}} (\mathcal{M}_{\chi_0}^T(h_f)\varphi_{0, f})(-w)\overline{I_\Theta(w, x_w)} dw dh_f. \end{aligned}$$

10. Local integrals at finite primes

10.1 Local integrals

In this section, we reduce the calculation of the Fourier–Jacobi components of $\mathcal{L}f$ to that of certain local integrals. Until the end of the paper, we fix a finite prime p and often suppress the subscript p . We write F and K for \mathbb{Q}_p and K_p , respectively. Let $\text{ord}_F: F^\times \rightarrow \mathbb{Z}$ be the additive valuation of F normalized by $\text{ord}_F(\pi) = 1$. Let $\mathfrak{p}_F = \pi\mathcal{O}_F$ be the maximal ideal of \mathcal{O}_F and put $\delta = \text{ord}_F D$. We denote by \mathcal{X} the set of characters of K^\times with $\chi|_{F^\times} = \omega_{K/F}$. Let \mathcal{E} be the set of (m, \mathfrak{a}, χ) , where $m \in F^\times$, \mathfrak{a} is an ideal of K with $\mu := \text{ord}_F(mN(\mathfrak{a})) \geq 0$ and $\chi \in \mathcal{X}$ satisfying the following two conditions:

$$(i) \quad A(\chi) = A_p(\chi) = \begin{cases} \mu & \text{if } \delta = 0, \\ 2(\mu + \delta) & \text{if } \delta > 0 \text{ and } \mu > 0, \\ 2\delta \text{ or } 2\delta - 1 & \text{if } \delta > 0 \text{ and } \mu = 0, \end{cases}$$

where $A_p(\chi)$ is defined in § 4.7;

$$(ii) \quad \epsilon(1/2, \chi, \psi_{m, K}) = \chi(\kappa^{-1}).$$

Let $X_{\text{unr}}(K^1)$ be the group of unitary characters Ω of $K^1 = \{t \in K^\times \mid t t^\sigma = 1\}$ with $\Omega|_{\mathcal{O}_K^1} = 1$.

For $\chi \in \mathcal{X}$, let $\mathcal{W}^m(\chi)$ be the space of \mathbb{C} -valued smooth functions W on H satisfying

$$W(\iota_m(t)hu) = (\chi/\chi_0)^1(t)\tilde{\chi}_0(u)W(h) \tag{10.1}$$

for $t \in K^1, h \in H, u \in \mathcal{U}' = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{U} \mid c \in D\mathcal{O}_K \}$ (for the definition of ι_m , see (7.6)). For $\Omega \in X_{\text{unr}}(K^1)$, let $\mathcal{W}^m(\chi; \chi_0\Omega)$ be the space of $W \in \mathcal{W}^m(\chi)$ satisfying $W(t1_2 \cdot h) = (\chi_0\Omega)(t)W(h)$ for $t \in K^1$.

For $(m, \mathfrak{a}, \chi) \in \mathcal{E}$, let $\mathcal{T}^m(\mathfrak{a}, \chi)$ be the space consisting of \mathbb{C} -valued functions τ on R satisfying the following four conditions.

(10.2) We have $\tau(trr_0(0, x)) = \psi_m(x)\tau(r)$ ($t \in K^1, r \in R, r_0 \in R(\mathfrak{a}), x \in F$).

(10.3) If \mathfrak{a}' is an ideal of K with $\mathfrak{a}' \not\supseteq \mathfrak{a}$, then

$$\int_{N(\mathfrak{a}')} \tau(rn) \, dn = 0.$$

(Recall that $N(\mathfrak{a}') = \{ (w, x + x_w) \in N \mid w \in \mathfrak{a}', x \in \mathfrak{a}'\mathfrak{a}'^\sigma \}$.)

(10.4) Suppose that K/F is ramified. Then

$$\int_{\Pi^{-1}\mathfrak{a}} \tau(r(w, x_w)) \, d_{\Pi^{-1}\mathfrak{a}}w = \begin{cases} 0 & \text{if } \chi \text{ is ordinary,} \\ \tau(r) & \text{if } \chi \text{ is exceptional.} \end{cases}$$

(For the definition of $d_{\Pi^{-1}\mathfrak{a}}w$, see § 2.4.)

(10.5) There exists a lattice L_0 of K such that, for any lattice L of K with $L \supset L_0$, we have

$$\tau(rt) = \lambda_K(\psi_m)^{-1} \chi\left(\frac{1-t}{\kappa}\right) |N(1-t)|_F^{1/2} \int_L \psi_m\left(\frac{1}{2}\langle w, tw \rangle_\kappa\right) \tau(r((1-t)w, 0)) \, d^m w$$

for $t \in K^1 - \{1\}$.

Let $(m, \mathfrak{a}, \chi) \in \mathcal{E}$ and \mathfrak{b} be an ideal of K . For $W \in \mathcal{W}^m(\chi)$ and $\tau \in \mathcal{T}^m(\mathfrak{a}, \chi)$, we set

$$\mathcal{I}_{\mathfrak{b}}(W, \tau) = |N(\beta)|_F \int_{H_{\eta_m} \backslash H} \int_K \chi_0^{-1}(\det h) \Phi_0(\beta h^{-1} \eta_m) (\mathcal{M}_{\chi_0}^T(h) \varphi_0)(-w) W(h) \overline{\tau(w, x_w)} \, dw \, dh,$$

where β is an element of K^\times such that $\mathfrak{b} = \beta\mathcal{O}_K$. Note that the right-hand side of $\mathcal{I}_{\mathfrak{b}}(W, \tau)$ does not depend on the choice of β .

Remark. Going back to the global situation, let $(m, \mathfrak{a}, \chi) \in \mathcal{E}$, $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0)$ and $\Theta \in \mathbf{T}_{\text{hol,prim}}^m(\mathfrak{a}, \chi)$ (cf. § 7.6). Then, for every p , the restriction of $W_{\chi, f}^m$ to H_p belongs to $\mathcal{W}^m(\chi_p)$. If $f \in S_{l-1}(\mathcal{U}'_f, \tilde{\chi}_0; \chi_0\Omega)$, we have $W_{\chi, f}^m|_{H_p} \in \mathcal{W}^m(\chi_p; (\chi_0\Omega)_p)$. We also note that the restriction of I_Θ to R_p belongs to $\mathcal{T}^m(\mathfrak{a}_p, \chi_p)$.

10.2 Local main results

The proofs of Theorems 7.5 and 7.6 are reduced to those of the following results.

THEOREM 10.1. *Let $(m, \mathfrak{a}, \chi) \in \mathcal{E}$. Then, for $W \in \mathcal{W}^m(\chi)$ and $\tau \in \mathcal{T}^m(\mathfrak{a}, \chi)$, we have*

$$\begin{aligned} \mathcal{I}_{\mathfrak{a}}(W, \tau) &= \chi_0^{-1}(\alpha) |N(\alpha)|_F^{3/2} |D|_F^{1/2} v_0 \cdot \overline{\tau(1)} \\ &\quad \times \{W(\mathfrak{d}_H(\alpha^{-1})) + \delta_e(\chi) \cdot p^{-1/2} \chi^{-1}(m\theta) W(\mathfrak{d}_H(\alpha^{-1})w_0)\}. \end{aligned}$$

Here α is an element of K^\times with $\mathfrak{a} = \alpha\mathcal{O}_K$,

$$v_0 = p^\mu \times \begin{cases} 1 & \text{if } \delta = 0 \text{ and } \mu = 0, \\ 1 - \omega_{K/F}(p)p^{-1} & \text{if } \delta = 0 \text{ and } \mu > 0, \\ \frac{2p}{1+p} & \text{if } \delta > 0 \end{cases}$$

and

$$\delta_e(\chi) = \begin{cases} 1 & \text{if } p|D \text{ and } \chi \text{ is exceptional,} \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 10.2. *Let $(m, \mathfrak{a}, \chi) \in \mathcal{E}$ and $\Omega \in X_{\text{unr}}(K^1)$. For $W \in \mathcal{W}^m(\chi; \chi_0\Omega)$ and $\tau \in \mathcal{T}^m(\mathfrak{a}, \chi)$, we have the following.*

(i) *If K/F is an unramified quadratic extension, we have*

$$p^2\mathcal{I}_{p^{k+1}\mathfrak{a}}(W, \tau) - p^{-1}\mathcal{I}_{p^k\mathfrak{a}}(W * \phi_1, \tau) + \{\delta(k + \mu > 0)p^{-1} - 1\}\mathcal{I}_{p^k\mathfrak{a}}(W, \tau) + p^{-2}\mathcal{I}_{p^{k-1}\mathfrak{a}}(W, \tau) = 0$$

for $k \geq 0$.

(ii) *If K/F is a ramified quadratic extension, we have*

$$p\mathcal{I}_{\Pi^{k+1}\mathfrak{a}}(W, \tau) - p^{-1/2}\mathcal{I}_{\Pi^k\mathfrak{a}}(W * (\phi^+ + \phi^-), \tau) - \delta(k = 0 \text{ and } \chi \text{ is ordinary})\mathcal{I}_{\Pi^k\mathfrak{a}}(W, \tau) + p^{-1}\mathcal{I}_{\Pi^{k-1}\mathfrak{a}}(W, \tau) = 0$$

for $k \geq 0$.

(iii) *If $K = F \oplus F$, we have*

$$p\mathcal{I}_{\Pi_1^{k_1+1}\Pi_2^{k_2}\mathfrak{a}}(W, \tau) - p^{-1/2}\mathcal{I}_{\Pi_1^{k_1}\Pi_2^{k_2}\mathfrak{a}}(W * \phi_1, \tau) - \mathcal{I}_{\Pi_1^{k_1}\Pi_2^{k_2}\mathfrak{a}}(W, \tau) + \mathcal{I}_{\Pi_1^{k_1-1}\Pi_2^{k_2+1}\mathfrak{a}}(W * \phi_0^+, \tau) + p^{-1}\mathcal{I}_{\Pi_1^{k_1}\Pi_2^{k_2-1}\mathfrak{a}}(W, \tau) = 0$$

and

$$p\mathcal{I}_{\Pi_1^{k_1}\Pi_2^{k_2+1}\mathfrak{a}}(W, \tau) - p^{-1/2}\mathcal{I}_{\Pi_1^{k_1}\Pi_2^{k_2}\mathfrak{a}}(W * \phi_2, \tau) - \mathcal{I}_{\Pi_1^{k_1}\Pi_2^{k_2}\mathfrak{a}}(W, \tau) + \mathcal{I}_{\Pi_1^{k_1+1}\Pi_2^{k_2-1}\mathfrak{a}}(W * \phi_0^-, \tau) + p^{-1}\mathcal{I}_{\Pi_1^{k_1-1}\Pi_2^{k_2}\mathfrak{a}}(W, \tau) = 0$$

for $k_1, k_2 \geq 0$.

10.3 We close this section with an elementary fact, which is needed in the proof of Theorem 10.2. Let $C^\infty(H/\mathcal{U}', \tilde{\chi}_0)$ be the space of smooth functions f on H satisfying $f(hu) = \tilde{\chi}_0(u)f(h)$ ($h \in H, u \in \mathcal{U}'$). Then the Hecke algebra \mathcal{H}^H (cf. §3.4) acts on $C^\infty(H/\mathcal{U}', \tilde{\chi}_0)$ on the right in a natural manner. For $\phi \in \mathcal{H}^H$, put $\phi^\vee(h) = \chi_0(\det h)\phi(h^{-1})$. Note that ϕ^\vee is also in \mathcal{H}^H .

LEMMA 10.3. *For $f, f' \in C^\infty(H/\mathcal{U}', \tilde{\chi}_0)$ and $\phi \in \mathcal{H}^H$, we have*

$$\int_H \chi_0^{-1}(\det h)(f * \phi)(h)f'(h) dh = \int_H \chi_0^{-1}(\det h)f(h)(f' * \phi^\vee)(h) dh.$$

11. Calculation of local integrals at finite primes: primitive case

11.1 Until the end of the paper, we keep the notation of §10, and let $(m, \mathfrak{a}, \chi) \in \mathcal{E}, W \in \mathcal{W}^m(\chi)$ and $\tau \in \mathcal{T}^m(\mathfrak{a}, \chi)$. Take and fix an $\alpha \in K^\times$ such that $\mathfrak{a} = \alpha\mathcal{O}_K$. To simplify the notation, we write η and ι for η_m and ι_m , respectively. We also write $\mathbf{d}(a), \mathbf{n}(x)$ and $\bar{\mathbf{n}}(x)$ for $\mathbf{d}_H(a), \mathbf{n}_H(x)$ and $\bar{\mathbf{n}}_H(x)$ respectively, if there is no fear of confusion. Let $H_\eta = \{h \in H \mid h\eta = \eta\}$ and put $L_k = \mathcal{O}_F + \theta\mathfrak{p}_F^k$ for $k \in \mathbb{Z}, k \geq 0$. In this section, we prove Theorem 10.1.

11.2 Inert case

In this subsection, we consider the case where K/F is inert. Note that $\mathcal{U} = \mathcal{U}'$ in this case. Set $h_l = \mathbf{d}(\pi^{-l}\alpha^{-1})$ ($l \in \mathbb{Z}$). We need the following Cartan-type decomposition of H .

LEMMA 11.1. *We have the following.*

- (i) *We have $H = \bigcup_{l \geq -[\mu/2]} H_\eta h_l \mathcal{U}$ (disjoint union).*
- (ii) *For $l \in \mathbb{Z}$ with $l \geq -[\mu/2]$, we have*

$$v_l := \text{vol}(H_\eta h_l \mathcal{U}) = p^{2l+\mu} \times \begin{cases} 1 & \text{if } 2l + \mu = 0, \\ 0 & \text{if } 2l + \mu > 0. \end{cases}$$

Proof. The first assertion is easily verified. To prove the second, observe that

$$H_\eta h_l \mathcal{U} = \bigcup_{z \in \mathcal{O}_K^\times / (\mathcal{O}_K^\times \cap L_{2l+\mu})} \iota(z^\sigma/z) h_l \mathcal{U} \quad (\text{disjoint union}).$$

Since $[\mathcal{O}_K^\times : (\mathcal{O}_K^\times \cap L_k)]$ is equal to $p^k(1 + p^{-1})$ if $k \geq 1$ and 1 if $k = 0$, we are done. □

LEMMA 11.2. *For $W \in \mathcal{W}^m(\chi)$, we have $\text{Supp}(W) \subset \bigcup_{l \geq 0} H_\eta h_l \mathcal{U}$.*

Proof. There is nothing to prove if $\mu \leq 1$. Assume that $\mu \geq 2$ and $-[\mu/2] \leq l < 0$. It is sufficient to show that $W(h_l) = 0$. Since $A(\chi) = \mu$, there exists an $x \in \mathcal{O}_F$ such that $\chi(1 + \pi^{\mu-1}x\theta) \neq 1$. Put

$$t_x = \frac{1 + \pi^{\mu-1}x\theta^\sigma}{1 + \pi^{\mu-1}x\theta} \in K^1.$$

Then $\iota(t_x)h_l = h_l u$, where

$$u = \frac{1}{1 + \pi^{\mu-1}x\theta} \begin{pmatrix} 1 & m \mathbf{N}(\alpha) \pi^{\mu+2l-1} \mathbf{N}(\theta)x \\ -(m \mathbf{N}(\alpha))^{-1} \pi^{\mu-2l-1} x & 1 + \text{Tr}(\theta) \pi^{\mu-1} x \end{pmatrix}.$$

Since $u = (u_{ij}) \in \mathcal{U}$ and $u_{21} \in \pi \mathcal{O}_K$, we have

$$(\chi/\chi_0)(1 + \pi^{\mu-1}x\theta)W(h_l) = W(\iota(t_x)h_l) = W(h_l u) = \tilde{\chi}_0(u)W(h_l) = \chi_0^{-1}(1 + \pi^{\mu-1}x\theta)W(h_l).$$

This implies that $W(h_l) = 0$. □

In view of Lemmas 11.1 and 11.2, $\mathcal{I}_a(W, \tau)$ is equal to

$$\begin{aligned} & |\mathbf{N}(\alpha)|_F \sum_{l=0}^{\infty} v_l W(h_l) \chi_0(\alpha^\sigma/\alpha) \Phi_0 \left(\frac{\pi^l m \mathbf{N}(\alpha) \theta^\sigma}{\pi^{-l}} \right) \chi_0(\pi^l \alpha) |\mathbf{N}(\pi^{-l} \alpha^{-1})|_F^{1/2} \int_K \varphi_0(\pi^{-l} \alpha^{-1} w) \overline{\tau(w, x_w)} dw \\ &= v_0 \chi_0^{-1}(\alpha) |\mathbf{N}(\alpha)|_F^{1/2} W(h_0) \int_a \overline{\tau(w, x_w)} dw \\ &= v_0 \chi_0^{-1}(\alpha) |\mathbf{N}(\alpha)|_F^{3/2} W(\mathbf{d}(\alpha^{-1})) \overline{\tau(1)}. \end{aligned}$$

This completes the proof of Theorem 10.1 in the inert case.

11.3 Ramified case

In this subsection, we consider the case where K/F is a ramified quadratic extension. Recall that θ is a prime element of K . Set $h_l = \mathbf{d}(\Pi^{-l} \alpha^{-1})$ for $l \in \mathbb{Z}$.

LEMMA 11.3. *We have the following.*

- (i) *We have*

$$H = \bigcup_{l \geq -\mu} H_\eta h_l \mathcal{U}' \cup \bigcup_{l \geq -\mu} \bigcup_{a \in \mathfrak{p}_F / D\mathcal{O}_F} H_\eta h_l \mathbf{n}(a) w_0 \mathcal{U}',$$

where $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

- (ii) *We have $H_\eta h_{-\mu} \mathbf{n}(a) w_0 \mathcal{U}' = H_\eta h_{-\mu} w_0 \mathcal{U}'$ for $a \in \mathfrak{p}_F$.*

Proof. For $h, h' \in H$, we write $h \sim h'$ if $h' \in H_\eta h \mathcal{U}'$. It is easily verified that $H = \bigcup_{l \geq -\mu} H_\eta h_l \mathcal{U}$ and $H_\eta h_l \mathcal{U} = \{h \in H \mid \Pi^l \alpha \cdot h^{-1} \eta \in \mathcal{O}_K^\times - \Pi \mathcal{O}_K^\times\}$. Let $h = h_l u (l \geq -\mu, u = (u_{ij}) \in \mathcal{U})$. To prove part (i), it suffices to show that

$$h \sim \begin{cases} h_l & \text{if } u_{11} \in \mathcal{O}_K^\times, \\ h_l \mathbf{n}(a) w_0 & \text{for some } a \in \mathfrak{p}_F \text{ otherwise.} \end{cases} \tag{11.1}$$

First suppose that $u_{11} \in \mathcal{O}_K^\times$. Then $h \sim h_l \bar{\mathbf{n}}(c)$ with $c = u_{11}^{-1} u_{21} \in \mathcal{O}_F$. If we put

$$z = 1 + y\theta \quad \text{with } y = \frac{cm \mathbf{N}(\alpha) \pi^l}{1 - c \operatorname{Tr}(\theta) m \mathbf{N}(\alpha) \pi^l} \in \mathcal{O}_F,$$

we have

$$h \sim \iota(z^\sigma/z) h_l \bar{\mathbf{n}}(c) h_l \frac{1}{z} \begin{pmatrix} 1 + cm \mathbf{N}(\alpha \theta) \pi^l y & m \mathbf{N}(\alpha \theta) \pi^l y \\ 0 & 1 + \operatorname{Tr}(\theta) y \end{pmatrix} \sim h_l.$$

Next suppose that $u_{11} \in \Pi \mathcal{O}_K$ (and, hence, $u_{21} \in \mathcal{O}_K^\times$). Then $h \mathbf{n}(-u_{21}^{-1} u_{22}) \mathbf{d}(-(u_{21}^\sigma)^{-1}) = h_l \mathbf{n}(a) w_0$ with $a = u_{11} u_{21}^{-1} \in \mathfrak{p}_F$, which completes the proof of (11.1). To prove part (ii), let $a \in \mathfrak{p}_F$ and put

$$z = 1 + y\theta \quad \text{with } y = -\frac{a \pi^\mu}{m \mathbf{N}(\alpha \theta)}.$$

We then have $y \in \mathcal{O}_F$ and

$$\iota(z^\sigma/z) h_{-\mu} \mathbf{n}(a) w_0 = h_{-\mu} w_0 \begin{pmatrix} z^\sigma & m^{-1} \mathbf{N}(\alpha)^{-1} \pi^\mu y z^{-1} \\ 0 & z^{-1} \end{pmatrix},$$

which implies $h_{-\mu} \mathbf{n}(a) w_0 \sim h_{-\mu} w_0$. □

LEMMA 11.4. *We have the following.*

(i) *For $l \geq -\mu$, we have*

$$\operatorname{vol}(H_\eta h_l \mathcal{U}') = \frac{2p}{1+p} p^{l+\mu}.$$

(ii) *We have*

$$\operatorname{vol}(H_\eta h_{-\mu} w_0 \mathcal{U}') = \frac{2}{1+p}.$$

Proof. It is easily verified that

$$H_\eta h_l \mathcal{U}' = \bigcup_{z \in \mathcal{O}_K^\times / (\mathcal{O}_K^\times \cap L_{\mu+l+\delta})} (\iota(z^\sigma/z) h_l \mathcal{U}' \cup \iota((\Pi z)^\sigma / (\Pi z)) h_l \mathcal{U}') \quad (\text{disjoint union})$$

for $l \geq -\mu$, and

$$H_\eta h_{-\mu} w_0 \mathcal{U}' = \bigcup_{z \in \mathcal{O}_K^\times / (\mathcal{O}_K^\times \cap L_{\delta-1})} (\iota(z^\sigma/z) h_{-\mu} w_0 \mathcal{U}' \cup \iota((\Pi z)^\sigma / (\Pi z)) h_{-\mu} w_0 \mathcal{U}') \quad (\text{disjoint union}).$$

The lemma now follows from $[\mathcal{O}_K^\times : (\mathcal{O}_K^\times \cap L_k)] = p^k$ ($k \geq 0$) and $[\mathcal{U} : \mathcal{U}'] = p^\delta (1 + p^{-1})$. □

LEMMA 11.5. *Let $W \in \mathcal{W}^m(\chi)$.*

(i) *If χ is ordinary, we have*

$$\operatorname{Supp}(W) \subset \bigcup_{l \geq 0} H_\eta h_l \mathcal{U}' \cup \bigcup_{l \geq 1} \bigcup_{a \in \mathfrak{p}_F / D\mathcal{O}_F} H_\eta h_l \mathbf{n}(a) w_0 \mathcal{U}'.$$

(ii) *If χ is exceptional, we have*

$$\operatorname{Supp}(W) \subset \bigcup_{l \geq 0} H_\eta h_l \mathcal{U}' \cup \bigcup_{l \geq 1} \bigcup_{a \in \mathfrak{p}_F / D\mathcal{O}_F} H_\eta h_l \mathbf{n}(a) w_0 \mathcal{U}' \cup H_\eta h_0 w_0 \mathcal{U}'.$$

Proof. There is nothing to prove when χ is exceptional. Assume that χ is ordinary. We first show that $W(h_l) = 0$ if $-\mu \leq l < 0$. Take a $y \in \mathcal{O}_F$ such that $\text{ord}_F y = \mu + \delta - 1$ and $\chi(z) \neq 1$ with $z = 1 + \theta y$. Then $\iota(z^\sigma/z)h_l = h_l u$, where

$$u = \frac{1}{z} \begin{pmatrix} 1 & \pi^l m \mathbf{N}(\alpha \theta) y \\ -y/\pi^l m \mathbf{N}(\alpha) & 1 + \text{Tr}(\theta) y \end{pmatrix}.$$

If $-\mu \leq l < 0$, we have $u \in \mathcal{U}'$ and, hence, $(\chi/\chi_0)(z)W(h_l) = \chi_0^{-1}(z)W(h_l)$. This implies that $W(h_l) = 0$. A similar argument shows that $W(h_l \mathbf{n}(a)w_0) = 0$ if $-\mu \leq l \leq 0$ and $a \in \mathfrak{p}_F$. \square

To prove Theorem 10.1, we first observe that $\Phi_0(\alpha h^{-1} \eta) = 0$ if $h = h_l$ ($l \geq 1$) or $h = h_l \mathbf{n}(a)w_0$ ($l \geq 1, a \in \pi \mathcal{O}_F$). In view of Lemmas 11.4 and 11.5, we obtain $\mathcal{I}_a(W, \tau) = I_1 + \delta_e(\chi)I_2$, where

$$I_1 = |\mathbf{N}(\alpha)|_F \chi_0(\alpha^\sigma/\alpha) \frac{2}{1+p} p^{\mu+1} W(h_0) \int_K \mathcal{M}_{\chi_0}^T(h_0) \varphi_0(-w) \overline{\tau(w, x_w)} dw,$$

$$I_2 = |\mathbf{N}(\alpha)|_F \chi_0(\alpha^\sigma/\alpha) \frac{2}{1+p} W(h_0 w_0) \int_K \mathcal{M}_{\chi_0}^T(h_0 w_0) \varphi_0(-w) \overline{\tau(w, x_w)} dw.$$

Since $\mathcal{M}_{\chi_0}^T(h_0) \varphi_0(-w) = \chi_0(\alpha) |\mathbf{N}(\alpha)|_F^{-1/2} \varphi_0(\alpha^{-1} w)$ and $\text{vol}(\mathfrak{a}) = |\mathbf{N}(\alpha)|_F p^{-\delta/2}$, we have

$$I_1 = \frac{2}{1+p} p^{\mu+1} \chi_0(\alpha^\sigma/\alpha) W(h_0) \chi_0(\alpha) |\mathbf{N}(\alpha)|_F^{1/2} \int_{\mathfrak{a}} \overline{\tau(w, x_w)} dw$$

$$= \frac{2}{1+p} p^{\mu+1-\delta/2} \chi_0^{-1}(\alpha) |\mathbf{N}(\alpha)|_F^{3/2} W(h_0) \overline{\tau(1)}.$$

We next have $\mathcal{M}_{\chi_0}^T(h_0 w_0) \varphi_0(-w) = \chi_0(\alpha) |\mathbf{N}(\alpha)|_F^{-1/2} p^{-\delta/2} \lambda_K(\psi) \varphi_0(\Pi^\delta \alpha^{-1} w)$ and, hence,

$$I_2 = \frac{2}{1+p} \chi_0^{-1}(\alpha) |\mathbf{N}(\alpha)|_F^{1/2} p^{-\delta/2} \lambda_K(\psi) W(h_0 w_0) \int_{\Pi^{-\delta} \mathfrak{a}} \overline{\tau(w, x_w)} dw.$$

To complete the proof of Theorem 10.1, it now remains to show the following.

LEMMA 11.6. *Suppose that $\chi \in \mathcal{X}_{\text{prim}}^+(m, \mathfrak{a})$ is exceptional. For $\tau \in \mathcal{T}^m(\mathfrak{a}, \chi)$, we have*

$$\int_{\Pi^{-\delta} \mathfrak{a}} \tau(w, x_w) dw = \lambda_K(\psi_m) \chi(\theta) |\mathbf{N}(\alpha)|_F p^{1/2} \tau(1). \tag{11.2}$$

Proof. Put $t_\theta = \theta^\sigma/\theta \in K^1$. By (10.5), taking a sufficiently large lattice L' of K , we have

$$\begin{aligned} \tau(rt_\theta) &= \lambda_K(\psi_m)^{-1} \chi\left(\frac{1-t_\theta}{\kappa}\right) |\mathbf{N}(1-t_\theta)|_F^{1/2} \int_{L'} \psi_m\left(\frac{1}{2}(w, t_\theta w)_\kappa\right) \tau(r(w(1-t_\theta), 0)) d^m w \\ &= \lambda_K(\psi_m)^{-1} \chi^{-1}(\theta) |\mathbf{N}(m\theta)|_F^{1/2} \int_L \psi\left(\frac{m}{2} \text{Tr}\left(\frac{\kappa}{1-t_\theta}\right) w w^\sigma\right) \tau(r(w, 0)) dw \\ &= \lambda_K(\psi_m)^{-1} \chi^{-1}(\theta) |m|_F p^{-1/2} \int_L \tau(r(w, x_w)) dw, \end{aligned}$$

where $L = (1-t_\theta)^{-1} L'$. We may (and do) suppose that $L \supset \Pi^{-\delta} \mathfrak{a}$. Since χ is exceptional, the last integral is equal to

$$\begin{aligned} &\int_L \int_{\Pi^{-1} \mathfrak{a}} \tau(r(w, x_w)(w', x_{w'})) d_{\Pi^{-1} \mathfrak{a}} w' dw \\ &= \int_L \tau(r(w, x_w)) \int_{\Pi^{-1} \mathfrak{a}} \psi_m(-\text{Tr}(\theta^\sigma w^\sigma w')) d_{\Pi^{-1} \mathfrak{a}} w' dw = \int_{\Pi^{-\delta} \mathfrak{a}} \tau(r(w, x_w)) dw. \end{aligned}$$

We thus have

$$\int_{\Pi^{-\delta} \mathfrak{a}} \tau(w, x_w) dw = \lambda_K(\psi_m) \chi(\theta) |m|_F^{-1} p^{1/2} \tau(t_\theta) = \lambda_K(\psi_m) \chi(\theta) |\mathbf{N}(\alpha)|_F p^{1/2} \tau(1),$$

which proves the lemma. \square

Remark. When $\delta = 1$, the left-hand side of (11.2) is independent of the choice of θ and equal to $|\mathbf{N}(\alpha)|_F p^{1/2} \tau(1)$ by the assumption that χ is exceptional. This implies that $\lambda_K(\psi_m)\chi(\theta) = 1$ (we can show this fact directly). Note that, if $\delta \geq 2$, the left-hand side of (11.2) depends on the choice of θ .

11.4 Split case

In this subsection, we consider the case where $K = F \oplus F$, and let $\Pi_1 = (\pi, 1), \Pi_2 = (1, \pi)$. We may (and do) take $\theta = (1, 0)$. Then $x_w = \frac{1}{2} \mathbf{N}(w)$ for $w \in K$. Let $Z = \{t1_2 \mid t \in K^1\}$ be the center of H . For $l \in \mathbb{Z}$, we put $h_l = \mathbf{d}(\Pi_1^{-l} \alpha^{-1})$.

LEMMA 11.7. *We have $H = \bigcup_{l \geq -\mu} ZH_\eta h_l \mathcal{U}$.*

Proof. For $h, h' \in H$, we write $h \sim h'$ if $h' \in ZH_\eta h \mathcal{U}$. We first show that, for any $h \in H$, there exists an $l \in \mathbb{Z}$ such that $h \sim h_l$. By Iwasawa decomposition of H , we have $h = \bar{\mathbf{n}}(x)\mathbf{d}(a)u$ for some $x \in F, a \in K^\times, u \in \mathcal{U}$. We may (and do) assume that $x \neq m^{-1}$. Putting $y = (1 - mx)^{-1} \in F^\times$, we have

$$\bar{\mathbf{n}}(x) = \iota((y, y^{-1}))\mathbf{d}((1, y^{-1})) \tag{11.3}$$

and, hence, $h \sim \mathbf{d}(a')$ for some $a' \in K^\times$. Our claim immediately follows from this. It is now sufficient to show that $h_l \sim h_{-l}$ if $l < -\mu$. Put $y' = (1 - \pi^l m \mathbf{N}(\alpha))^{-1} \in F^\times$. Note that $\text{ord}_F y' = -(l + \mu)$. By (11.3), we have $h_l \sim h_l \bar{\mathbf{n}}(1) = \bar{\mathbf{n}}(\pi^l \mathbf{N}(\alpha))h_l \sim \mathbf{d}((1, 1/y'))h_l \sim \mathbf{d}((1/y', 1))h_l \sim \mathbf{d}(\Pi_1^{l+\mu})h_l = h_{-l}$. \square

LEMMA 11.8. *For $l \geq -\mu$, we have*

$$v_l := \text{vol}(ZH_\eta \backslash ZH_\eta h_l \mathcal{U}) = \begin{cases} 1 & \text{if } l = -\mu, \\ p^{l+\mu}(1 - p^{-1}) & \text{if } l > -\mu. \end{cases}$$

Proof. Let $l \geq -\mu$ and put $X = Z_0 H_{\eta,0} h_l \mathcal{U}$, where $Z_0 = \{t1_2 \mid t \in \mathcal{O}_K^1\}$ and $H_{\eta,0} = \{\iota(t) \mid t \in \mathcal{O}_K^1\}$. Then $v_l = \text{vol}(X)$. Since $\iota(t)\eta^\sigma = t \cdot \eta^\sigma$ ($t \in K^1$), we have

$$X = \{h \in H \mid \Pi_1^l \alpha \cdot h^{-1} \eta, \Pi_1^l \alpha \cdot h^{-1} \eta^\sigma \in \mathcal{O}_K^2 - (\Pi_1 \mathcal{O}_K^2 \cup \Pi_2 \mathcal{O}_K^2)\}.$$

It follows that $X = Y \cup \bigcup_{k=1}^{l+\mu} Y_k$ (disjoint union), where

$$Y = \{\mathbf{n}(x)\mathbf{d}(y)u \mid x \in \pi^{-l} \mathbf{N}(\alpha)^{-1} \mathcal{O}_F, y \in \Pi_1^{-l} \alpha^{-1} \mathcal{O}_K^\times, u \in \mathcal{U}\},$$

$$Y_k = \{\mathbf{n}(x)\mathbf{d}(y)u \mid x, -x + m \in \pi^{k-l} \mathbf{N}(\alpha)^{-1} \mathcal{O}_F^\times, y \in \Pi_1^{k-l} \Pi_2^k \alpha^{-1} \mathcal{O}_K^\times, u \in \mathcal{U}\}.$$

The lemma now follows from

$$\text{vol}(Y) = 1, \quad \text{vol}(Y_k) = p^k \times \begin{cases} 1 - p^{-1} & \text{if } 1 \leq k \leq l + \mu - 1, \\ 1 - 2p^{-1} & \text{if } k = l + \mu. \end{cases} \tag{\square}$$

LEMMA 11.9. *For $W \in \mathcal{W}^m(\chi)$, we have $\text{Supp}(W) \subset \bigcup_{l \geq 0} ZH_\eta h_l \mathcal{U}$.*

Proof. This is proved by an argument similar to the proof of Lemma 11.2. \square

By Lemma 11.9, we have

$$\begin{aligned} \mathcal{I}_a(W, \tau) &= |\mathbf{N}(\alpha)|_F \sum_{k \in \mathbb{Z}} \sum_{l=0}^{\infty} v_l \chi_0^{-1}(\det \mathbf{d}(\Pi_1^{-k-l} \Pi_2^k \alpha^{-1})) \Phi_0(\alpha \mathbf{d}(\Pi_1^{k+l} \Pi_2^{-k} \alpha) \eta) \\ &\quad \times W(\mathbf{d}(\Pi_1^{-k-l} \Pi_2^k \alpha^{-1})) \int_K (\mathcal{M}_{\chi_0}^T(\mathbf{d}(\Pi_1^{-k-l} \Pi_2^k \alpha^{-1})) \varphi_0)(-w) \overline{\tau(w, x_w)} dw. \end{aligned}$$

For $k \in \mathbb{Z}$ and $l \geq 0$, we have $\Phi_0(\alpha \mathbf{d}(\Pi_1^{k+l} \Pi_2^{-k} \alpha) \eta) = \delta(k = l = 0)$. By this and Lemma 11.8, we obtain

$$\begin{aligned} \mathcal{I}_a(W, \tau) &= \chi_0^{-1}(\alpha) |\mathbf{N}(\alpha)|_F^{1/2} v_0 \cdot W(\mathbf{d}(\alpha^{-1})) \int_K \varphi_0(\alpha^{-1} w) \overline{\tau(w, x_w)} dw \\ &= \chi_0^{-1}(\alpha) |\mathbf{N}(\alpha)|_F^{3/2} \overline{\tau(1)} W(\mathbf{d}(\alpha^{-1})) \times p^\mu \begin{cases} 1 - p^{-1} & \text{if } \mu > 0, \\ 1 & \text{if } \mu = 0. \end{cases} \end{aligned}$$

This completes the proof of Theorem 10.1 in the split case.

12. Calculation of local integrals at finite primes: non-primitive case

12.1 In this section, we let $\Omega \in X_{\text{unr}}(K^1)$ and $W \in \mathcal{W}^m(\chi; \chi_0 \Omega)$ (cf. § 10.1), and prove Theorem 10.2. We keep the notation of § 11.

12.2 Inert case

In this subsection, we consider the case where K/F is inert. For $k \in \mathbb{Z}$, we have

$$\mathcal{I}_{\pi^k \mathbf{a}}(W, \tau) = |\mathbf{N}(\pi^k \alpha)|_F \int_{H_\eta \setminus H} \int_K \chi_0^{-1}(\det h) W(h) A_k(w, h) \overline{\tau(w, x_w)} dw dh,$$

where $A_k(w, h) = \Phi_0(\pi^k \alpha h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h) \varphi_0)(-w)$. In view of Lemma 10.3 and the fact that $\phi_1^\vee = \phi_1$, we have

$$\mathcal{I}_{\pi^k \mathbf{a}}(W * \phi_1, \tau) = |\mathbf{N}(\pi^k \alpha)|_F \int_{H_\eta \setminus H} \int_K \chi_0^{-1}(\det h) W(h) A'_k(w, h) \overline{\tau(w, x_w)} dw dh,$$

where

$$\begin{aligned} A'_k(w, h) &= -\Phi_0(\pi^k \alpha \mathbf{d}(\pi) h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h \mathbf{d}(\pi^{-1})) \varphi_0)(-w) \\ &\quad - \sum_{a \in (\mathcal{O}_F - \pi \mathcal{O}_F) / \pi \mathcal{O}_F} \Phi_0(\pi^k \alpha \mathbf{n}(-\pi^{-1} a) h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h \mathbf{n}(\pi^{-1} a)) \varphi_0)(-w) \\ &\quad - \sum_{a \in \mathcal{O}_F / \pi^2 \mathcal{O}_F} \Phi_0(\pi^k \alpha \mathbf{d}(\pi^{-1}) \mathbf{n}(-a) h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h \mathbf{n}(a) \mathbf{d}(\pi)) \varphi_0)(-w). \end{aligned}$$

It follows that

$$\begin{aligned} p^2 \mathcal{I}_{\pi^{k+1} \mathbf{a}}(W, \tau) - p^{-1} \mathcal{I}_{\pi^k \mathbf{a}}(W * \phi_1, \tau) + \{\delta(k + \mu > 0) p^{-1} - 1\} \mathcal{I}_{\pi^k \mathbf{a}}(W, \tau) + p^{-2} \mathcal{I}_{\pi^{k-1} \mathbf{a}}(W, \tau) \\ = |\mathbf{N}(\pi^k \alpha)|_F \int_{H_\eta \setminus H} \int_K \chi_0^{-1}(\det h) W(h) B_k(w, h) \overline{\tau(w, x_w)} dw dh, \end{aligned}$$

where

$$B_k(w, h) = A_{k+1}(w, h) - p^{-1} A'_k(w, h) + \{\delta(k + \mu > 0) p^{-1} - 1\} A_k(w, h) + A_{k-1}(w, h).$$

Set

$$\mathcal{B}_{k,l} = \int_K B_k(w, h_l) \overline{\tau(w, x_w)} dw.$$

To prove Theorem 10.2, it suffices to show the following result in view of Lemma 11.2.

PROPOSITION 12.1. For $k, l \geq 0$, we have $\mathcal{B}_{k,l} = 0$.

Proof. For $l \in \mathbb{Z}$, set $\phi_l(w) = \varphi_0(\pi^{-l} \alpha^{-1} w)$. Let $l \geq 0$ and $k \in \mathbb{Z}$. Observe that

$$A_k(w, h_l) = \delta(k \geq l) (-p)^l \chi_0(\alpha) |\mathbf{N}(\alpha)|_F^{-1/2} \phi_l(w)$$

and that

$$A'_k(w, h_l) = (-1)^l p^{l+1} \chi_0(\alpha) |\mathbf{N}(\alpha)|_F^{-1/2} \{ \delta(k \geq l + 1) \phi_{l+1}(w) - p^{-1} \delta(k \geq l + 1) I_1(w) + p^{-2} I_2(w) \},$$

where

$$I_1(w) = \sum_{a \in (\mathcal{O}_F - \pi \mathcal{O}_F) / \pi \mathcal{O}_F} \psi(-\pi^{-2l-1} \mathbf{N}(\alpha)^{-1} \mathbf{N}(w)a) \phi_l(w),$$

$$I_2(w) = \sum_{a \in \mathcal{O}_F / \pi^2 \mathcal{O}_F} \Phi_0 \left(\frac{\pi^{k+l-1} m \mathbf{N}(\alpha) \theta^\sigma - \pi^{k-l-1} a}{\pi^{k+1-l}} \right) \psi(-\pi^{-2l} \mathbf{N}(\alpha)^{-1} \mathbf{N}(w)a) \phi_{l-1}(w).$$

A straightforward calculation shows that

$$I_1(w) = p \phi_{l+1}(w) - \phi_l(w),$$

$$I_2(w) = \delta(k + l + \mu > 0) [\delta(k \geq l + 1) p^2 \phi_l(w) + \delta(k = l) p \phi_l(w) + \delta(k = l - 1) \phi_{l-1}(w)]$$

and, hence, that

$$A'_k(w, h_l) = (-1)^l p^{l+1} \chi_0(\alpha) |\mathbf{N}(\alpha)|_F^{-1/2} \times \delta(k + l + \mu > 0) [\delta(k = l) p^{-1} \phi_l(w) + \delta(k \geq l + 1) (1 + p^{-1}) \phi_l(w) + \delta(k = l - 1) p^{-2} \phi_{l-1}(w)].$$

This implies that, for $k, l \geq 0$,

$$B_k(w, h_l) = (-p)^l \chi_0(\alpha) |\mathbf{N}(\alpha)|_F^{-1/2} \delta(l = k + 1) \{ \phi_l(w) - p^{-2} \phi_{l-1}(w) \}$$

and, hence,

$$\mathcal{B}_{k,l} = (-p)^l \chi_0(\alpha) |\mathbf{N}(\alpha)|_F^{-1/2} \delta(k = l - 1) (J_l - p^{-2} J_{l-1}),$$

where

$$J_l = \int_K \phi_l(w) \overline{\tau(w, x_w)} dw.$$

Since $J_l = p^{-2l} |\mathbf{N}(\alpha)|_F \overline{\tau(1)}$ for $l \geq 0$, we have proved that $\mathcal{B}_{k,l} = 0$. □

12.3 Ramified case

In this subsection, we consider the case where K/F is a ramified quadratic extension. For $k \in \mathbb{Z}$, we have

$$\mathcal{I}_{\Pi^k \mathfrak{a}}(W, \tau) = p^{-k} |\mathbf{N}(\alpha)|_F \int_{H_\eta \backslash H} \int_K \chi_0^{-1}(\det h) A_k(w, h) W(h) \overline{\tau(w, x_w)} dw dh,$$

where $A_k(w, h) = \Phi_0(\Pi^k \alpha h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h) \varphi_0)(-w)$ ($w \in K, h \in H$). By Lemma 10.3 and the fact that $(\phi^\pm)^\vee = \phi^\mp$, we obtain

$$\mathcal{I}_{\Pi^k \mathfrak{a}}(W * (\phi^+ + \phi^-), \tau) = p^{-k} |\mathbf{N}(\alpha)|_F \int_{H_\eta \backslash H} \int_K \chi_0^{-1}(\det h) A'_k(w, h) W(h) \overline{\tau(w, x_w)} dw dh,$$

where

$$A'_k(w, h) = \chi_0(\Pi) \sum_{c \in \mathcal{O}_F / \pi \mathcal{O}_F} A_k(w, h \mathbf{n}(c) \mathbf{d}(\Pi)) + \chi_0^{-1}(\Pi) \sum_{c \in \mathcal{O}_F / \pi \mathcal{O}_F} A_k(w, h \overline{\mathbf{n}}(Dc) \mathbf{d}(\Pi^{-1})).$$

It follows that

$$p \mathcal{I}_{\Pi^{k+1} \mathfrak{a}}(W, \tau) - p^{-1/2} \mathcal{I}_{\Pi^k \mathfrak{a}}(W * (\phi^+ + \phi^-), \tau) - \delta(k = 0 \text{ and } \chi \text{ is ordinary}) \mathcal{I}_{\Pi^k \mathfrak{a}}(W, \tau) + p^{-1} \mathcal{I}_{\Pi^{k-1} \mathfrak{a}}(W, \tau) = p^{-k} |\mathbf{N}(\alpha)|_F \int_{H_\eta \backslash H} dh \int_K dw \chi_0^{-1}(\det h) W(h) B_k(w, h) \overline{\tau(w, x_w)},$$

where

$$B_k(w, h) = A_{k+1}(w, h) - p^{-1/2}A'_k(w, h) - \delta(k = 0 \text{ and } \chi \text{ is ordinary})A_k(w, h) + A_{k-1}(w, h).$$

Set $\phi_l(w) = \varphi_0(\Pi^{-l}\alpha^{-1}w)$ for $l \in \mathbb{Z}$.

LEMMA 12.2. *Let $k \geq 0$.*

(i) *For $l \geq 0$, we have*

$$B_k(w, h_l) = \chi_0(\Pi^l\alpha)p^{l/2}|N(\alpha)|_F^{-1/2}[\delta(k = l \text{ or } l - 1)\{\phi_l(w) - p^{-1}\phi_{l-1}(w)\} - \delta(l = k = 0 \text{ and } \chi \text{ is ordinary})\phi_0(w)].$$

(ii) *For $l \geq 0$ and $a \in \mathfrak{p}_F$, we have*

$$B_k(w, h_l\mathbf{n}(a)w_0) = \begin{cases} -\beta_l \cdot \psi(a\mathbf{N}(\Pi^{-l}\alpha^{-1}w))\phi_{l-\delta}(w) & \text{if } k = l = 0 \text{ and } \chi \text{ is ordinary,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\beta_l = p^{(l-\delta)/2}|N(\alpha)|_F^{-1/2}\chi_0(\Pi^l\alpha)\lambda_K(\psi)$.

Proof. First observe that $A_k(w, h_l) = \delta(l \leq k)\chi_0(\Pi^l\alpha)p^{l/2}|N(\alpha)|_F^{-1/2}\phi_l(w)$. Using the formulas

$$\sum_{c \in \mathcal{O}_F/\pi\mathcal{O}_F} \mathcal{M}_{\chi_0}^T(\mathbf{n}(\pi^{-1}c))\varphi_0(w) = p\varphi_0(\Pi^{-1}w)$$

and

$$\sum_{c \in \mathcal{O}_F/\pi\mathcal{O}_F} \mathcal{M}_{\chi_0}^T(\bar{\mathbf{n}}(\pi^{-1}Dc))\varphi_0(w) = \varphi_0(\Pi w),$$

we obtain

$$\begin{aligned} A'_k(w, h_l) &= \chi_0(\Pi) \sum_{c \in \mathcal{O}_F/\pi\mathcal{O}_F} \Phi_0 \left(\frac{\Pi^{k+(l-1)\sigma}m\mathbf{N}(\alpha)\theta^\sigma - \Pi^{k-l-\sigma}c}{\Pi^{k+1-l}} \right) \mathcal{M}_{\chi_0}^T(h_{l-1}\mathbf{n}(\pi^{-1}c))\varphi_0(-w) \\ &\quad + \chi_0^{-1}(\Pi) \sum_{c \in \mathcal{O}_F/\pi\mathcal{O}_F} \Phi_0 \left(\frac{\Pi^{k+(l+1)\sigma}m\mathbf{N}(\alpha)\theta^\sigma}{-\Pi^{k-1+l\sigma}m\mathbf{N}(\alpha)Dc\theta^\sigma + \Pi^{k-l-1}} \right) \mathcal{M}_{\chi_0}^T(h_{l+1}\bar{\mathbf{n}}(\pi^{-1}Dc))\varphi_0(-w) \\ &= \chi_0(\Pi^l\alpha) \left\{ \delta(k \geq l-1)|N(\Pi^{-l+1}\alpha^{-1})|_F^{1/2}\phi_{l-1}(w) \right. \\ &\quad + \delta(k \geq l+1)|N(\Pi^{-l+1}\alpha^{-1})|_F^{1/2} \sum_{c \in \mathcal{O}_F^\times/\pi\mathcal{O}_F} \mathcal{M}_{\chi_0}^T(\mathbf{n}(\pi^{-1}c))\varphi_0(-\Pi^{-l+1}\alpha^{-1}w) \\ &\quad \left. + \delta(k \geq l+1)|N(\Pi^{-l-1}\alpha^{-1})|_F^{1/2} \sum_{c \in \mathcal{O}_F/\pi\mathcal{O}_F} \mathcal{M}_{\chi_0}^T(\bar{\mathbf{n}}(\pi^{-1}Dc))\varphi_0(-\Pi^{-l-1}\alpha^{-1}w) \right\} \\ &= \chi_0(\Pi^l\alpha)p^{(l+1)/2}|N(\alpha)|_F^{-1/2}\{\delta(k = l \text{ or } l - 1)p^{-1}\phi_{l-1}(w) + 2 \cdot \delta(k \geq l + 1)\phi_l(w)\}. \end{aligned}$$

The assertion (i) is now proved by a straightforward calculation. Next let $h = h_l\mathbf{n}(a)w_0$ ($l \geq 0$, $a \in \mathfrak{p}_F$). A similar argument as above shows that $A_k(w, h) = \delta(k \geq l)\beta_l \cdot \psi(a\mathbf{N}(\Pi^{-l}\alpha^{-1}w))\phi_{l-\delta}(w)$ and $A'_k(w, h) = \{\delta(k \geq l+1) + \delta(k \geq l-1)\}p^{1/2}\beta_l \cdot \psi(a\mathbf{N}(\Pi^{-l}\alpha^{-1}w))\phi_{l-\delta}(w)$. The second assertion immediately follows from these. \square

PROPOSITION 12.3. *Let $\tau \in \mathcal{T}^m(\mathfrak{a}, \chi)$.*

(i) *For $k, l \geq 0$, we have*

$$\int_K B_k(w, h_l)\overline{\tau(w, x_w)} dw = 0.$$

(ii) For $k \geq 0, l \geq 1$ and $a \in \mathfrak{p}_F$, we have

$$\int_K B_k(w, h_l \mathbf{n}(a) w_0) \overline{\tau(w, x_w)} dw = 0.$$

(iii) Assume that χ is exceptional. Then, for $k \geq 0$, we have

$$\int_K B_k(w, h_0 w_0) \overline{\tau(w, x_w)} dw = 0.$$

Proof. For $l \in \mathbb{Z}$, set

$$J_l = \int_K \phi_l(w) \overline{\tau(w, x_w)} dw.$$

In view of (10.2) and (10.4), we have

$$J_l = \begin{cases} p^{-l-\delta/2} |\mathbf{N}(\alpha)|_F \overline{\tau(1)} & \text{if } l \geq 0 \text{ or } \mu = 0, \chi \text{ is exceptional and } l = -1, \\ 0 & \text{otherwise} \end{cases}$$

for $l \geq -1$. By Lemma 12.2, for $k, l \geq 0$, we have

$$\begin{aligned} & \int_K B_k(w, h_l) \overline{\tau(w, x_w)} dw \\ &= \chi_0(\Pi^l \alpha) p^{l/2} |\mathbf{N}(\alpha)|_F^{-1/2} \{ \delta(k = l \text{ or } l - 1) (J_l - p^{-1} J_{l-1}) - \delta(l = k = 0 \text{ and } \chi \text{ is ordinary}) J_0 \} \\ &= 0, \end{aligned}$$

which shows assertion (i). The assertions (ii) and (iii) directly follow from Lemma 12.2(ii). □

In view of Lemma 11.5 and Proposition 12.3, we have completed the proof of Theorem 10.2 in the ramified case.

12.4 Split case

In this subsection, we consider the case where $K = F \oplus F$ and use the notation of § 11.4.

LEMMA 12.4. Let $k_1, k_2 \in \mathbb{Z}$.

(i) We have

$$\mathcal{I}_{\Pi_1^{k_1} \Pi_2^{k_2} \mathfrak{a}}(W, \tau) = p^{-k_1-k_2} |\mathbf{N}(\alpha)|_F \int_{H_\eta \backslash H} \int_K \chi_0^{-1}(\det h) W(h) A_{k_1, k_2}(w, h) \overline{\tau(w, x_w)} dw dh,$$

where $A_{k_1, k_2}(w, h) = \Phi_0(\Pi_1^{k_1} \Pi_2^{k_2} \alpha h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h) \varphi_0)(-w)$.

(ii) We have

$$\mathcal{I}_{\Pi_1^{k_1} \Pi_2^{k_2} \mathfrak{a}}(W * \phi_1, \tau) = p^{-k_1-k_2} |\mathbf{N}(\alpha)|_F \int_{H_\eta \backslash H} \int_K \chi_0^{-1}(\det h) W(h) A_{k_1, k_2}^1(w, h) \overline{\tau(w, x_w)} dw dh,$$

where

$$\begin{aligned} A_{k_1, k_2}^1(w, h) &= \chi_0^{-1}(\Pi_2) \left\{ \Phi_0(\Pi_1^{k_1} \Pi_2^{k_2} \alpha \mathbf{d}(\Pi_2) h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h \mathbf{d}(\Pi_2^{-1})) \varphi_0)(-w) \right. \\ &\quad \left. + \sum_{c \in \mathcal{O}_F / \pi \mathcal{O}_F} \Phi_0(\Pi_1^{k_1} \Pi_2^{k_2} \alpha \mathbf{d}(\Pi_1^{-1}) \mathbf{n}(-c) h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h \mathbf{n}(c) \mathbf{d}(\Pi_1)) \varphi_0)(-w) \right\}. \end{aligned}$$

(iii) We have

$$\mathcal{I}_{\Pi_1^{k_1} \Pi_2^{k_2} \mathfrak{a}}(W * \phi_2, \tau) = p^{-k_1-k_2} |\mathbf{N}(\alpha)|_F \int_{H_\eta \backslash H} \int_K \chi_0^{-1}(\det h) W(h) A_{k_1, k_2}^2(w, h) \overline{\tau(w, x_w)} dw dh,$$

where

$$A_{k_1, k_2}^2(w, h) = \chi_0^{-1}(\Pi_1) \left\{ \Phi_0(\Pi_1^{k_1} \Pi_2^{k_2} \alpha \mathbf{d}(\Pi_1) h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h \mathbf{d}(\Pi_1^{-1}))) \varphi_0(-w) \right. \\ \left. + \sum_{c \in \mathcal{O}_F / \pi \mathcal{O}_F} \Phi_0(\Pi_1^{k_1} \Pi_2^{k_2} \alpha \mathbf{d}(\Pi_2^{-1}) \mathbf{n}(-c) h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h \mathbf{n}(c) \mathbf{d}(\Pi_2))) \varphi_0(-w) \right\}.$$

(iv) We have

$$\mathcal{I}_{\Pi_1^{k_1} \Pi_2^{k_2} \mathfrak{a}}(W * \phi_0^\pm, \tau) = p^{-k_1 - k_2} |\mathbf{N}(\alpha)|_F \int_{H_\eta \backslash H} \int_K \chi_0^{-1}(\det h) W(h) A_{k_1, k_2}^\pm(w, h) \overline{\tau(w, x_w)} dw dh,$$

where

$$A_{k_1, k_2}^\pm(w, h) = \chi_0(\Pi_1 / \Pi_2)^{\pm 1} \Phi_0(\Pi_1^{k_1} \Pi_2^{k_2} (\Pi_1 / \Pi_2)^{\pm 1} \alpha h^{-1} \eta) (\mathcal{M}_{\chi_0}^T(h \mathbf{d}((\Pi_1 / \Pi_2)^{\pm 1}))) \varphi_0(-w).$$

(v) We have

$$p \mathcal{I}_{\Pi_1^{k_1+1} \Pi_2^{k_2} \mathfrak{a}}(W, \tau) - p^{-1/2} \mathcal{I}_{\Pi_1^{k_1} \Pi_2^{k_2} \mathfrak{a}}(W * \phi_1, \tau) - \mathcal{I}_{\Pi_1^{k_1} \Pi_2^{k_2} \mathfrak{a}}(W, \tau) \\ + \mathcal{I}_{\Pi_1^{k_1-1} \Pi_2^{k_2+1} \mathfrak{a}}(W * \phi_0^+, \tau) + p^{-1} \mathcal{I}_{\Pi_1^{k_1} \Pi_2^{k_2-1} \mathfrak{a}}(W, \tau) \\ = -p^{-k_1 - k_2} |\mathbf{N}(\alpha)|_F \int_{H_\eta \backslash H} \int_K \chi_0^{-1}(\det h) W(h) B_{k_1, k_2}(w, h) \overline{\tau(w, x_w)} dw dh,$$

where

$$B_{k_1, k_2}(w, h) = p^{-1/2} A_{k_1, k_2}^1(w, h) + A_{k_1, k_2}(w, h) - A_{k_1+1, k_2}(w, h) - A_{k_1, k_2-1}(w, h) - A_{k_1-1, k_2+1}^+(w, h).$$

(vi) We have

$$p \mathcal{I}_{\Pi_1^{k_1} \Pi_2^{k_2+1} \mathfrak{a}}(W, \tau) - p^{-1/2} \mathcal{I}_{\Pi_1^{k_1} \Pi_2^{k_2} \mathfrak{a}}(W * \phi_2, \tau) - \mathcal{I}_{\Pi_1^{k_1} \Pi_2^{k_2} \mathfrak{a}}(W, \tau) \\ + \mathcal{I}_{\Pi_1^{k_1+1} \Pi_2^{k_2-1} \mathfrak{a}}(W * \phi_0^-, \tau) + p^{-1} \mathcal{I}_{\Pi_1^{k_1-1} \Pi_2^{k_2} \mathfrak{a}}(W, \tau) \\ = -p^{-k_1 - k_2} |\mathbf{N}(\alpha)|_F \int_{H_\eta \backslash H} \int_K \chi_0^{-1}(\det h) W(h) B'_{k_1, k_2}(w, h) \overline{\tau(w, x_w)} dw dh,$$

where

$$B'_{k_1, k_2}(w, h) = p^{-1/2} A_{k_1, k_2}^2(w, h) + A_{k_1, k_2}(w, h) - A_{k_1, k_2+1}(w, h) - A_{k_1-1, k_2}(w, h) - A_{k_1+1, k_2-1}^-(w, h).$$

Proof. The first assertion is obvious from the definition. The assertions (ii)–(iv) follow from Lemma 10.3 and the fact that $\phi_1^\vee = \phi_2, \phi_2^\vee = \phi_1$ and $(\phi_0^\pm)^\vee = \phi_0^\mp$. The last two assertions are direct consequences of assertions (i)–(iv). \square

By Lemma 11.9, we have

$$\text{Supp}(W) \subset \bigcup_{l_1, l_2 \in \mathbb{Z}, l_1 + l_2 \geq 0} H_\eta h_{l_1, l_2} \mathcal{U},$$

where $h_{l_1, l_2} = \mathbf{d}(\Pi_1^{-l_1} \Pi_2^{-l_2} \alpha^{-1})$. Set

$$\mathcal{B}_{k_1, k_2, l_1, l_2} = \int_K B_{k_1, k_2}(w, h_{l_1, l_2}) \overline{\tau(w, x_w)} dw, \\ \mathcal{B}'_{k_1, k_2, l_1, l_2} = \int_K B'_{k_1, k_2}(w, h_{l_1, l_2}) \overline{\tau(w, x_w)} dw.$$

To prove Theorem 10.2, it suffices to show the following.

PROPOSITION 12.5. For $k_1, k_2, l_1, l_2 \in \mathbb{Z}$ with $k_1 \geq 0, k_2 \geq 0, l_1 + l_2 \geq 0$, we have $\mathcal{B}_{k_1, k_2, l_1, l_2} = 0$ and $\mathcal{B}'_{k_1, k_2, l_1, l_2} = 0$.

12.5 Proof of Proposition 12.5

To simplify the notation, we set

$$\begin{aligned} \phi_{l_1, l_2}(w) &= \varphi_0(\Pi_1^{-l_1} \Pi_2^{-l_2} \alpha^{-1} w), \\ J(l_1, l_2) &= \int_K \phi_{l_1, l_2}(w) \overline{\tau(w, x_w)} dw, \\ J'(l_1, l_2) &= \int_K \phi_{l_1, l_2}(w) \psi_m(N(w)) \overline{\tau(w, x_w)} dw = \int_K \phi_{l_1, l_2}(w) \overline{\tau(w, -x_w)} dw \end{aligned}$$

for $l_1, l_2 \in \mathbb{Z}$. Note that $J(l_1, l_2) = J'(l_1, l_2)$ if $l_1 + l_2 + \mu \geq 0$.

LEMMA 12.6. For $k_1, k_2, l_1, l_2 \in \mathbb{Z}$ with $k_1 \geq 0, k_2 \geq 0, l_1 + l_2 \geq 0$, we have

$$\begin{aligned} \mathcal{B}_{k_1, k_2, l_1, l_2} &= p^{(l_1+l_2)/2} \chi_0(\Pi_1^{l_1} \Pi_2^{l_2} \alpha) |N(\alpha)|_F^{-1/2} \\ &\times \begin{cases} -J(l_1, l_2) + p^{-1} J(l_1 - 1, l_2) & \text{if } k_1 = l_1 - 1, k_2 \geq l_2, \\ -J(l_1 - 1, l_2 + 1) + p^{-1} J'(l_1 - 1, l_2) & \text{if } k_1 \geq l_1, k_2 = l_2, \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \mathcal{B}'_{k_1, k_2, l_1, l_2} &= p^{(l_1+l_2)/2} \chi_0(\Pi_1^{l_1} \Pi_2^{l_2} \alpha) |N(\alpha)|_F^{-1/2} \\ &\times \begin{cases} -J(l_1, l_2) + p^{-1} J'(l_1, l_2 - 1) & \text{if } k_1 \geq l_1, k_2 = l_2 - 1, \\ -J(l_1 + 1, l_2 - 1) + p^{-1} J(l_1, l_2 - 1) & \text{if } k_1 = l_1, k_2 \geq l_2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. By the definitions of A_{k_1, k_2} and A_{k_1, k_2}^\pm , we have

$$\begin{aligned} A_{k_1, k_2}(w, h_{l_1, l_2}) &= \delta(k_1 \geq l_1, k_2 \geq l_2) p^{(l_1+l_2)/2} \chi_0(\Pi_1^{l_1} \Pi_2^{l_2} \alpha) |N(\alpha)|_F^{-1/2} \phi_{l_1, l_2}(w), \\ A_{k_1, k_2}^+(w, h_{l_1, l_2}) &= \delta(k_1 \geq l_1 - 1, k_2 \geq l_2 + 1) p^{(l_1+l_2)/2} \chi_0(\Pi_1^{l_1} \Pi_2^{l_2} \alpha) |N(\alpha)|_F^{-1/2} \phi_{l_1-1, l_2+1}(w), \\ A_{k_1, k_2}^-(w, h_{l_1, l_2}) &= \delta(k_1 \geq l_1 + 1, k_2 \geq l_2 - 1) p^{(l_1+l_2)/2} \chi_0(\Pi_1^{l_1} \Pi_2^{l_2} \alpha) |N(\alpha)|_F^{-1/2} \phi_{l_1+1, l_2-1}(w) \end{aligned}$$

for $k_1, k_2, l_1, l_2 \in \mathbb{Z}$. Next suppose that $k_1, k_2, l_1 + l_2 \geq 0$. By using the formula

$$\sum_{c \in \mathcal{O}_F / \pi \mathcal{O}_F} \mathcal{M}_{\chi_0}^T(\mathbf{n}(\pi^{-1}c)) \varphi_0(w) = p \{ \varphi_0(\Pi_1^{-1}w) + \varphi_0(\Pi_2^{-1}w) - \varphi_0(\pi^{-1}w) \},$$

we obtain

$$\begin{aligned} A_{k_1, k_2}^1(w, h_{l_1, l_2}) &= p^{(l_1+l_2+1)/2} \chi_0(\Pi_1^{l_1} \Pi_2^{l_2} \alpha) |N(\alpha)|_F^{-1/2} \times [\delta(k_1 \geq l_1, k_2 \geq l_2 + 1) \{ \phi_{l_1, l_2}(w) + \phi_{l_1-1, l_2+1}(w) \} \\ &\quad + p^{-1} \delta(k_1 = l_1 - 1, k_2 \geq l_2) \phi_{l_1-1, l_2}(w) + p^{-1} \delta(k_1 \geq l_1, k_2 = l_2) \psi_m(N(w)) \phi_{l_1-1, l_2}(w)] \end{aligned}$$

and

$$\begin{aligned} A_{k_1, k_2}^2(w, h_{l_1, l_2}) &= p^{(l_1+l_2+1)/2} \chi_0(\Pi_1^{l_1} \Pi_2^{l_2} \alpha) |N(\alpha)|_F^{-1/2} \times [\delta(k_1 \geq l_1 + 1, k_2 \geq l_2) \{ \phi_{l_1, l_2}(w) + \phi_{l_1+1, l_2-1}(w) \} \\ &\quad + p^{-1} \delta(k_1 = l_1, k_2 \geq l_2 - 1) \phi_{l_1, l_2-1}(w) + p^{-1} \delta(k_1 \geq l_1 + 1, k_2 = l_2 - 1) \psi_m(N(w)) \phi_{l_1, l_2-1}(w)]. \end{aligned}$$

A straightforward calculation shows that

$$\begin{aligned}
 B_{k_1, k_2}(w, h_{l_1, l_2}) &= p^{(l_1+l_2)/2} \chi_0(\Pi_1^{l_1} \Pi_2^{l_2} \alpha) |N(\alpha)|^{-1/2} \\
 &\quad \times [\delta(k_1 = l_1 - 1, k_2 \geq l_2) \{-\phi_{l_1, l_2}(w) + p^{-1} \phi_{l_1-1, l_2}(w)\} \\
 &\quad + \delta(k_1 \geq l_1, k_2 = l_2) \{-\phi_{l_1-1, l_2+1}(w) + p^{-1} \psi_m(N(w)) \phi_{l_1-1, l_2}(w)\}]
 \end{aligned}$$

and

$$\begin{aligned}
 B'_{k_1, k_2}(w, h_{l_1, l_2}) &= p^{(l_1+l_2)/2} \chi_0(\Pi_1^{l_1} \Pi_2^{l_2} \alpha) |N(\alpha)|^{-1/2} \\
 &\quad \times [\delta(k_1 \geq l_1, k_2 = l_2 - 1) \{-\phi_{l_1, l_2}(w) + p^{-1} \psi_m(N(w)) \phi_{l_1, l_2-1}(w)\} \\
 &\quad + \delta(k_1 = l_1, k_2 \geq l_2) \{-\phi_{l_1+1, l_2-1}(w) + p^{-1} \phi_{l_1, l_2-1}(w)\}].
 \end{aligned}$$

These immediately show the lemma. □

To complete the proof of Proposition 12.5, it is sufficient to show the following.

LEMMA 12.7. *Let $l_1, l_2 \in \mathbb{Z}$ with $l_1 + l_2 \geq 0$.*

- (i) *If $l_1 \geq 1$, we have $J(l_1, l_2) = p^{-1} J(l_1 - 1, l_2)$.*
- (ii) *If $l_2 \geq 1$, we have $J(l_1, l_2) = p^{-1} J'(l_1, l_2 - 1)$.*

Proof. Suppose that $l_1 + l_2 \geq 0$ and $l_2 \geq 1$. Take $\alpha = (\alpha_1, \alpha_2) \in K^\times$ such that $\mathfrak{a} = \alpha \mathcal{O}_K$. We then have

$$\begin{aligned}
 J(l_1, l_2) &= \int_{\Pi_1^{l_1} \Pi_2^{l_2} \mathfrak{a}} \overline{\tau(w, x_w)} dw \\
 &= |\pi^{l_2-1} \alpha_2|_F^{-1} \int_{\Pi_1^{l_1} \Pi_2^{l_2} \mathfrak{a}} \int_{\pi^{l_2-1} \alpha_2 \mathcal{O}_F} \overline{\tau((w, x_w)((0, u), 0))} du dw \\
 &= |\pi^{l_2-1} \alpha_2|_F^{-1} \int_{\Pi_1^{l_1} \Pi_2^{l_2} \mathfrak{a}} \int_{\pi^{l_2-1} \alpha_2 \mathcal{O}_F} \overline{\tau\left(w + (0, u), \frac{1}{2} w_1 w_2 - \frac{1}{2} w_1 u\right)} du dw \\
 &= |\pi^{l_2-1} \alpha_2|_F^{-1} \int_{\Pi_1^{l_1} \Pi_2^{l_2} \mathfrak{a}} \int_{\pi^{l_2-1} \alpha_2 \mathcal{O}_F} \overline{\tau\left((w_1, w_2 + u), -\frac{1}{2} w_1 (w_2 + u)\right)} du dw \\
 &= p^{-1} \int_{\Pi_1^{l_1} \Pi_2^{l_2-1} \mathfrak{a}} \overline{\tau(w, -x_w)} dw \\
 &= p^{-1} J'(l_1, l_2 - 1),
 \end{aligned}$$

which proves assertion (ii). The proof of assertion (i) is similar and we omit it. □

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