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A SHORT PROOF OF VLADIMIRSKII'S THEOREM ON PRECOMPACT PERTURBATIONS IN LOCALLY CONVEX SPACES

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1. **Introduction and notations.** Let T, P denote two continuous operators from E into F, where E and F are locally convex spaces. It is proved by L. Schwartz [8] and G. Köthe [6] that if E and F are Fréchet spaces, T is a Φ_{-}-operator and P a compact operator, then $T+P$ is a Φ_{-}-operator.

In [9], Ju. N. Vladimirskii shows that for arbitrary locally convex spaces, this result is no longer true, but the following holds: if T is an almost open operator, with a closed graph, such that the closure of its range has a finite codimension in F, and P is a precompact operator, then $T+P$ is almost open and the closure of the range of $T+P$ has a finite codimension in F.

Ju. N. Vladimirskii's proof of this latter result is based upon an involved technique to reduce it to that of L. Schwartz and G. Köthe, and is rather long.

The aim of this paper is to present a much shorter proof of Ju. N. Vladimirskii's theorem, using standard techniques of duality and a simple method developed in [2]. These considerations also lead to a result on the dimension of the kernel (null space) and the codimension of the closure of the range of $T+P$ when T is open and P is bounded (i.e. P maps a neighbourhood into a bounded set) and "small" enough.

We adopt the following notations. Unless otherwise specified, E and F always denote two general locally convex Hausdorff spaces and T, P two linear operators from E into F such that $D(T) \subset D(P)$, where $D(T)$ is the domain of definition of T. We denote by $N(T)$, $R(T)$, and $G(T)$ respectively the kernel, range and graph (in $E \times F$) of T.

If A is a subset of E, then $\langle A \rangle$, $\rangle A \langle$ and $[A]^{-}$ denote the absolutely convex hull, the linear hull and the closure of A in E respectively. An absolutely convex set is also called a *disk*. A *finite disk* is the absolutely convex hull of a finite set of points. If A is a disk, $\rangle A \langle$ equipped with the topology defined by the Minkowski's gauge of A is referred to as the space generated by A. A set B is *A-compact* (resp. *A-precompact*) if $B \subset \rangle A \langle$ and B is compact (resp. precompact) in the space generated by A.

By neighbourhood, we always mean an absolutely convex open neighbourhood of the origin. An operator T is *open* (resp. *almost open*), if for any neighbourhood

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U in E, there is a neighbourhood V in F such that $TU \supset V \cap R(T)$ (resp. $[TU]^{-} \supset V \cap R(T)$).

An operator T is called a Φ_{+} (resp. Φ_{-}) *-operator* if T is open, $G(T)$ and $R(T)$ are closed and dim $N(T) < \infty$ (resp. codim $R(T) < \infty$), where dim (resp. codim) is an abbreviation of dimension (resp. codimension).

If A is a set then we write

$$T^{-1}(A) = \{x \in D(T) : Tx \in A\}.$$

Finally we denote by E^* the space of continuous linear functionals defined on E (dual of E), T^* the adjoint operator of T, acting from F^* into E^* when $D(T)$ is dense in E, and A^0 the polar of A in the duality.

2. **Ju. N. Vladimirskii's theorem.** We recall first two important results due to I. C. Gokhberg, M. G. Krein [3] and T. Kato [5].

THEOREM 2.1. *Let T and P be two operators from a Banach space E into a Banach space F, such that $D(T) \subset D(P)$.*

If T is a Φ_{+}-operator and P a compact operator, then $T+P$ is a Φ_{+}-operator.

Moreover, dim $N(T+\lambda P) = n$ *is constant for all $\lambda \in \mathbb{C}$, except for at most a countable set of isolated points λ_i where* dim $N(T+\lambda_i P) > n$.

THEOREM 2.2. *Let T and P be two operators from a Banach space E into a Banach space F such that $D(T) \subset D(P)$.*

If T is a Φ_{+} (resp. Φ_{-})-operator and if there are C and C' such that $0 < C < C'$ and $TU \supset C'V \cap R(T)$, $PU \subset CV$, where U and V are the unit balls of E and F, then $T+P$ is a Φ_{+} (resp. Φ_{-})-operator and

$$\dim N(T+P) \leq \dim N(T),$$

$$\operatorname{codim} R(T+P) \leq \operatorname{codim} R(T).$$

In Theorem 2.1, it could be seen that the set of the exceptional isolated points has no accumulation point at finite distance because n is the minimum of dim $N(T+\lambda P)$. See [7] and [4, Theorems V. 1.8 and V. 2.1].

We also need the following lemma.

LEMMA 2.3. *If L is a closed subspace of E, then* dim $L < \infty$ *if and only if* codim $L^0 < \infty$ *in E^*, and in this case,* dim $L =$ codim L^0.

In particular, if T is an operator from E into F with $D(T) = E$ and $G(T)$ closed, then

$$\dim N(T) = \operatorname{codim}[R(T^*)]^{-},$$

$$\dim N(T^*) = \operatorname{codim}[R(T)]^{-},$$

when either side is finite, and where the closure is relative to the weak (pointwise) topologies.

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Proof. The first part is an easy application of Hahn-Banach Theorem. The second follows from the fact that

$$N(T)^0 = [R(T^*)]^-, \qquad R(T)^0 = N(T^*).$$

The second relation is trivial. For the first, we need to prove that

$$N(T) \supset R(T^*)^0,$$

which will imply that $N(T)=R(T^*)^0$, the inverse inclusion being trivial. If $x \notin N(T)$ then $Tx\neq 0$. Since T has a closed graph, it is well known that $D(T^*)$ is (weak) dense in F^*, hence there is $\Gamma \in D(T^*)$ such that $\Gamma(Tx)\neq 0$. Therefore $x \notin R(T^*)^0$.

THEOREM 2.4. (*Ju. N. Vladimirskii* [9]). *Let E, F be locally convex spaces and T, P operators from E into F such that $D(T)\subset D(P)$.*

If T is almost open, $G(T)$ closed, $\operatorname{codim}[R(T)]^-<\infty$, *and P is precompact, then $T+P$ is almost open, $G(T+P)$ closed and* $\operatorname{codim}[R(T+P)]^-<\infty$.

Moreover, $\operatorname{codim}[R(T+\lambda P)]^-=n$ *is constant for all $\lambda \in \mathbb{C}$ except for at most a countable set of exceptional points λ_i with no accumulation point at finite distance. At these points,* $\operatorname{codim}[R(T+\lambda P)]^->n$.

Proof. That $G(T+P)$ is closed is trivial, since P is continuous.

Let U be an arbitrary neighbourhood in F. We may suppose without loss of generality that $D(T)=E$ and $PU\subset K$, where K is precompact in F.

There is a neighbourhood V in E such that $V \cap R(T)\subset [TU]^-$. Since $V \cap R(T)$ is dense in $V \cap [R(T)]^-$, we have $V \cap [R(T)]^-\subset [TU]^-$. This relation may be improved as

$$[V]^- \cap [R(T)]^- \subset [V \cap [R(T)]^-]^- \subset [TU]^-,$$

because V is a disk and $[R(T)]^-$ is linear.

Taking the polars, we obtain successively

$$(TU)^0 \subset [\langle V^0 \cup R(T)^0\rangle]^-,$$

$$T^{*-1}U^0 \subset V^0+N(T^*),$$

$$U^0 \cap R(T^*) \subset T^*V^0,$$

where the closure is taken with respect to the weak topology and V^0 is weakly compact.

Since PU is precompact, V^0 is $(PU)^0$-precompact. Hence P^*V^0 is U^0-precompact.

On the other hand, $\dim N(T^*)=\operatorname{codim}[R(T)]^-<\infty$, by Lemma 2.3. Let D be a finite disk generating $N(T^*)$.

Consider

$$B = V^0 \cap T^{*-1}U^0.$$

It is a Banach disk (i.e. generating a Banach space) since U^0 and V^0 are weakly compact and $G(T^*)$ is weakly closed in $F^*\times E^*$.

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Let L (resp. M) denote the Banach space generated by $B+D$ (resp. U^0). We have $T^*L\subset M$, $P^*L\subset M$.

As a matter of fact, $R(P^*)\subset M$, because $P^{*-1}U^0\supset K^0$, hence $P^*K^0\subset U^0$, where K^0 is absorbent in F^*.

Let T' and P' denote the restrictions of T^* and P^* to L with range space M. We have

$$T'(B+D) \supset U^0 \cap R(T').$$

Moreover, $N(T')$ is finite dimensional, $G(T')$ closed in $L\times M$, $R(T')$ closed in M (as T' is open), $D(T')\subset D(P')$ (as $D(P^*)=F^*$), and $P'(B+D)$ is U^0-compact ($P'(D)$ is a finite disk).

By Theorem 2.1, $T'+P'$ is open and $\dim N(T'+P')<\infty$. For all $\lambda\in\mathbb{C}$ except for at most a countable set with no finite accumulation value, $\dim N(T'+\lambda P')$ is constant and minimum. From this we may derive the desired conclusions of the statement by duality. This is possible due to the following remarks.

We have $N(T^*+P^*)\subset L$, and thus

$$N(T^*+P^*) = N(T'+P').$$

Indeed, if $x\in N(T^*+P^*)$, then $T^*x=-P^*x$ is an element of $R(T^*)\cap M=R(T')$. Thus $x\in L+N(T^*)$. But $L+N(T^*)=L$.

Similarly, $R(T^*+P^*)\cap M=R(T'+P')$. For if $y\in R(T^*+P^*)\cap M$, then $y=(T^*+P^*)x$, and $T^*x=y-P^*x\in R(T^*)\cap M$. We conclude as above.

Notice that

$$\dim N(T^*+P^*) = \operatorname{codim}[R(T+P)]^- < \infty,$$
$$\dim N(T^*+\lambda P^*) = \operatorname{codim}[R(T+\lambda P)]^-.$$

On the other hand, there is $\mu>0$ such that

$$(T'+P')(B+D) \supset \mu U^0 \cap R(T'+P')$$
$$\supset \mu U^0 \cap R(T^*+P^*).$$

Therefore,

$$\mu(T^*+P^*)^{-1}U^0 \subset B+D+N(T^*+P^*)$$
$$\subset 3\langle B \cup D \cup N(T^*+P^*)\rangle,$$

and

$$[(T+P)U]^- \supset \mu/3\, B^0 \cap D^0 \cap R(T+P)$$
$$\supset \mu/3\, V \cap D^0 \cap R(T+P).$$

Since D^0 is a (weak) closed neighbourhood in F, it follows that $T+P$ is almost open.

3. **Bounded perturbation**. When P is only bounded, similar considerations yield the following.

Suppose that there exist two neighbourhoods U and V in E and F such that

$$[TU]^- \supset V\cap R(T), \qquad PU \subset K,$$

where K is a bounded set in F, and that T has a closed graph.

Then there is $\rho>0$ such that, for all $\lambda\in\mathbb{C}$ satisfying $|\lambda|<\rho$,

(a) *if* $\operatorname{codim}[R(T)]^-<\infty$ *then*

$$\operatorname{codim}[R(T+\lambda P)]^- \le \operatorname{codim}[R(T)]^-,$$

(b) *if* $\dim N(T)<\infty$ *and T is weakly open, then*

$$\dim N(T+\lambda P) \le \dim N(T).$$

In general, we may take

$$\rho = \sup\{\lambda > 0 : \lambda K \subset V\}.$$

If $R(P)\subset R(T)$ then we may take

$$\rho = \sup\{\lambda > 0 : \lambda K \cap R(T) \subset [TU]^-\}.$$

In particular, (a) *and* (b) *hold if T is open.*

Proof. We keep the same notations as in §2. We suppose that $D(T)=E$, and we have

$$T^*B \supset U^0 \cap R(T^*), \qquad P^*K^0 \subset U^0,$$

where $B=V^0 \cap T^{*-1}U^0$ is a Banach disk, bounded for the weak topology, and K^0 is absorbent. Therefore K^0 absorbs B. Write $\rho=\sup\{\lambda>0:\lambda B\subset K^0\}$. For any $\lambda\in\mathbb{C}$, with $|\lambda|<\rho$, there is ξ such that $|\lambda|<\xi<\rho$.

In case (a), $\dim N(T^*)=\operatorname{codim}[R(T)]^-<\infty$. Let D be a finite disk generating $N(T^*)$ such that $\xi(B+D)\subset K^0$.

As in §2, let L and M denote the Banach spaces generated by $B+D$ and U^0 respectively and T', P', the corresponding restrictions of T^* and P^*. Then,

$$\begin{aligned} T'(B+D) &\supset U^0 \cap R(T'),\\ \lambda P'(B+D) &\subset \lambda/\xi\, P'[\xi(B+D)],\\ &\subset \lambda/\xi\, U^0. \end{aligned}$$

Since $\dim N(T')<\infty$ and $|\lambda/\xi|<1$, Theorem 2.2 applies to yield

$$\begin{aligned} \dim N(T^*+\lambda P^*) = \dim N(T'+\lambda P') &\le \dim N(T')\\ &\le \dim N(T^*). \end{aligned}$$

By duality,

$$\operatorname{codim}[R(T+\lambda P)]^- \le \operatorname{codim}[R(T)]^-.$$

In case (b), $R(T^*)$ is closed (for the weak topology) because T is weakly open $(R(T^*)=N(T)^0)$, and $\operatorname{codim} R(T^*)<\infty$.

Now let L denote the Banach space generated by B. The other notations are unchanged. We have

$$T'B \supset U^0 \cap R(T'), \qquad \lambda P'B \subset \lambda/\xi\, U^0,$$

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with codim $R(T')<\infty$ (in M) and $|\lambda/\xi|<1$. By Theorem 2.2, in M,

$$\operatorname{codim} R(T'+\lambda P') \leq \operatorname{codim} R(T'),$$

and a fortiori

$$\operatorname{codim} R(T^*+\lambda P^*) \cap M \leq \operatorname{codim} R(T^*) \cap M.$$

Notice that $R(T^*+\lambda P^*)+M=R(T^*)+M$. It can be seen therefore that, in E^*,

$$\operatorname{codim} R(T^*+\lambda P^*) \leq \operatorname{codim} R(T^*),$$

hence (weak topology)

$$\begin{aligned}\operatorname{codim}[R(T^*+\lambda P^*)]^- &\leq \operatorname{codim} R(T^*)\\ &\leq \operatorname{codim}[R(T^*)]^-.\end{aligned}$$

By duality,

$$\dim N(T+\lambda P) \leq \dim N(T).$$

Both (a) and (b) are direct applications of Proposition 4.2 in [2]. This proposition in fact gives the further relationship that $\kappa(T^*+\lambda P^*)=\kappa(T^*)$, where $\kappa(T)=\dim N(T)-\operatorname{codim} R(T)$ when at least one term is finite ($\kappa(T)$ is the *index* of T) Hence in case (b), if $\operatorname{codim}[R(T)]^-=\infty$ then

$$\kappa(T^*+\lambda P^*) = \kappa(T^*) = \infty,$$

thus

$$\operatorname{codim}[R(T+\lambda P)]^- = \infty.$$

If we have the assumptions of both (a) and (b) then

$$\dim N(T+\lambda P)-\operatorname{codim}[R(T+\lambda P)]^- \leq \dim N(T)-\operatorname{codim}[R(T)]^-.$$

The equality holds if and only if $R(T^*+\lambda P^*)$ is weakly closed in E^*, i.e. if and only if $T+\lambda P$ is weakly open ($T+\lambda P$ has a closed graph). Some conditions to ensure that $T+\lambda P$ is almost open or open are discussed in a paper to appear in Bull. Soc. Royale Sc. Liège.

We wish to point out that M. De Wilde has communicated to us still another short proof of the first part of Theorem 2.4 (that $T+P$ is almost open and $\operatorname{codim}[R(T+P)]^-<\infty$), using only his results on "perturbation of disks" [1,2]. We wish also to thank him for many helpful discussions.

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