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# LOW CODIMENSIONAL EMBEDDINGS OF $Sp(n)$ AND $SU(n)$

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In [4] Elmer Rees proves that the symplectic group $Sp(n)$ can be smoothly embedded in Euclidean space with codimension $3n$, and the unitary group $U(n)$ with codimension $n$. These are special cases of a result he obtains for a compact connected Lie group $G$. The general technique is first to embed $G/T$, where $T$ is a maximal torus, as a maximal orbit of the adjoint representation of $G$, and then to extend to an embedding of $G$ by using a maximal orbit of a faithful representation of $G$. In this note, we observe that in the cases $G=Sp(n)$ or $SU(n)$ an improved result is obtained by using the "symplectic torus" $S^3\times\cdots\times S^3$ in place of $T=S^1\times\cdots\times S^1$. As in Rees's construction, the normal bundle of the embedding of $G$ is trivial.

**Theorem.** (1) *For all $n$, $Sp(n)$ can be embedded with trivial normal bundle in Euclidean space with codimension $n$.*

(2) *For $n\geqq 3$, $SU(n)$ can be embedded with trivial normal bundle in Euclidean space with codimension $\frac{1}{2}n$ if $n$ is even, $[\frac{1}{2}n]+2$ if $n$ is odd. Hence $U(n)$ can be similarly embedded with codimension $\frac{1}{2}n-1$ if $n$ is even, $[\frac{1}{2}n]+1$ if $n$ is odd.*

Since $U(n)$ is diffeomorphic to $S^1\times SU(n)$, the result for $U(n)$ follows immediately from that for $SU(n)$. The situation for low values of $n$ may be summarised as follows. Recall that $Sp(1)\cong SU(2)\cong S^3$, so that $Sp(1)$, $SU(2)$ and $U(2)$ are obviously hypersurfaces. For $SU(3)$ the result is also elementary: by identifying $SU(3)$ with the space of unitary 2-frames in $\mathbb{C}^3$ we have an embedding

$$SU(3)\to S^5\times S^5\to\mathbb{R}^{11}$$

whose normal bundle is easily seen to be trivial. We shall prove below that $SU(3)$ (and *a fortiori* $U(3)$) does not embed in $\mathbb{R}^{10}$, so that the result is best possible in this case. This is also the case for $Sp(2)$ and for $SU(4)$, since Theorem 5 of [3] asserts that $Sp(n)$ and $SU(n)$ are never hypersurfaces except in the trivial cases mentioned above. (Apart from this, there seem to be no *non*-embedding results known for compact Lie groups.)

## Proof of the theorem for $Sp(n)$

The pattern of the proof of the theorem is the same in both cases, but we begin with the symplectic case since the details are simpler. Let $H(n)$ denote the set of all $n\times n$ symplectic Hermitian matrices defined by the condition $Q^*=Q$ where $Q$ is a matrix with quaternion entries and $Q^*$ denotes its conjugate transpose. As a real vector space $H(n)$ has dimension $2n^2-n$. Let $\Lambda$ denote a diagonal matrix in $H(n)$ with distinct real entries arranged in increasing order down the diagonal. The formula $AQA^*$ for $A$ in $Sp(n)$ and $Q$ in $H(n)$ defines a representation of $Sp(n)$ and the stabilizer of $\Lambda$ is the symplectic torus

$$\Delta(n)=Sp(1)\times\cdots\times Sp(1)$$

consisting of diagonal elements of $Sp(n)$. It follows that the quotient space $Sp(n)/\Delta(n)$ of cosets $A\,\Delta(n)$ embeds in $H(n)$ as a maximal orbit of the representation, and by a general result cited in [1] the normal bundle of the embedding is trivial. In our special case this is easy to see directly. Let $x_r$ vary in a small interval $I_r=(\lambda_r-\varepsilon,\lambda_r+\varepsilon)$ around the entry $\lambda_r$ of $\Lambda$. Let $X$ denote the diagonal matrix with entries $x_1,\dots,x_n$ which, for $\varepsilon$ small enough, are distinct numbers. It is then easy to check that for two such matrices $X_1$, $X_2$ the equation $X_1C=CX_2$ for $C$ in $Sp(n)$ implies $X_1=X_2$ and $C$ diagonal. Consequently the formula $AXA^*$ defines a smooth embedding

$$Sp(n)/\Delta(n)\times I_1\times\cdots\times I_n\to H(n)$$

which, on counting dimensions, is seen to trivialise a tubular neighbourhood of the maximal orbit containing $\Lambda$. Identifying each interval $I_r$ diffeomorphically with the real line and $H(n)$ with $\mathbb{R}^{2n^2-n}$ we have an embedding

$$f:Sp(n)/\Delta(n)\times\mathbb{R}^n\to\mathbb{R}^{2n^2-n}.$$

Now we consider the standard action of $Sp(n)$ on quaternionic $n$-space $\mathbb{H}^n$. Let $v$ in $\mathbb{H}^n$ have all entries equal to 1. If $y_1$, $y_2$ are real vectors sufficiently near to $v$ and $A$ is in $\Delta(n)$ then the equation $Ay_1=y_2$ implies that $y_1=y_2$ and $A$ is the identity matrix. A simple calculation then shows that the function $Sp(n)\to Sp(n)/\Delta(n)\times\mathbb{H}^n$ defined by $A\mapsto([A],Av)$ is an embedding which, as in the case of $f$ above, extends to an embedding

$$g:Sp(n)\times\mathbb{R}^n\to Sp(n)/\Delta(n)\times\mathbb{R}^{4n}.$$

Finally, identifying $\mathbb{R}^{4n}$ with $\mathbb{R}^n\times\mathbb{R}^{3n}$, we obtain the composite embedding

$$Sp(n)\times\mathbb{R}^n\xrightarrow{g}Sp(n)/\Delta(n)\times\mathbb{R}^n\times\mathbb{R}^{3n}\xrightarrow{f\times id}\mathbb{R}^{2n^2-n}\times\mathbb{R}^{3n}=\mathbb{R}^{2n^2+2n}.$$

Since the dimension of $Sp(n)$ is $2n^2+n$, this concludes the proof that $Sp(n)$ embeds in codimension $n$.

## Proof of the theorem for $SU(n)$

We now indicate how the argument above may be varied to apply to $SU(n)$. The representation space is the space $K(n)$ of all skew-symmetric complex $n\times n$ matrices: as

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a real vector space, $K(n)$ has dimension $n^2-n$. The formula $UKU^t$ for $U$ in $SU(n)$ and $K$ in $K(n)$ defines a representation of $SU(n)$ which is well known to be equivalent to the second exterior power representation. Let $\Lambda$ denote a "skew-diagonal" matrix in $K(n)$, i.e. one which is the direct sum of $2\times 2$ matrices

$$\begin{pmatrix} 0 & \lambda_r \\ -\lambda_r & 0 \end{pmatrix}, \qquad 1 \leqq r \leqq [\tfrac{1}{2}n].$$

If we take the $\lambda_r$ to be real, distinct and positive then the stabilizer of $\Lambda$ is the "symplectic torus"

$$\Delta(n) = SU(2)\times\cdots\times SU(2)$$

formed by matrices which are the direct sum of $[\tfrac{1}{2}n]$ $2\times 2$ blocks

$$\begin{pmatrix} a_r & b_r \\ -\bar{b}_r & \bar{a}_r \end{pmatrix}, \qquad |a_r|^2+|b_r|^2=1.$$

(If $n$ is odd, $\Delta(n)$ may be identified with the subgroup $\Delta(n-1)$ and $K(n)$ with the subspace $K(n-1)$.) Hence the quotient space $SU(n)/\Delta(n)$ of cosets $U\Delta(n)$ is embedded in $K(n)$ as an orbit of the representation.

We may again check directly that the normal bundle is trivial. Suppose first that $n$ is odd: then the normal bundle has dimension $m=[\tfrac{1}{2}n]$. Let $x_r$ vary in a small interval $I_r=(\lambda_r-\varepsilon, \lambda_r+\varepsilon)$ around the entry $\lambda_r$ of $\Lambda$. Let $X$ denote the skew-diagonal matrix with entries $\pm x_1, \ldots, \pm x_m$, which, for $\varepsilon$ small enough, are distinct real numbers. It is then easy to check that for two such matrices $X, Y$ the equation $XU=UY$ for $U$ in $SU(n)$ implies $X=Y$ and $U\in\Delta(n)$. Consequently the formula $UXU^t$ defines a smooth embedding

$$SU(n)/\Delta(n)\times I_1\times\cdots\times I_m \to K(n)$$

which trivialises a tubular neighbourhood of the orbit containing $\Lambda$. Identifying each interval $I_r$ diffeomorphically with $\mathbb{R}$ and $K(n)$ with $\mathbb{R}^{n^2-n}$, we have an embedding

$$SU(n)/\Delta(n)\times\mathbb{R}^m \to \mathbb{R}^{n^2-n} \quad (n \text{ odd}, m=[\tfrac{1}{2}n]).$$

Now consider the case where $n$ is even. The normal bundle has dimension $\tfrac{1}{2}n+1$ in this case, so we require one extra degree of freedom in choosing the $x_r$. By taking determinants, the equation $XU=UY$ implies that

$$x_1x_2\ldots x_{\frac{1}{2}n} = y_1y_2\ldots y_{\frac{1}{2}n}.$$

Hence we may replace one of the intervals, $I_1$ say, by the disc $|z-\lambda_1|<\varepsilon$ in $\mathbb{C}$, and argue as before to obtain an embedding

$$SU(n)/\Delta(n)\times\mathbb{C}\times\mathbb{R}^{\frac{1}{2}n-1} \to \mathbb{R}^{n^2-n} \quad (n \text{ even})$$

We now consider the standard action of $SU(n)$ on $\mathbb{C}^n$. Let $v=(1,1,\ldots,1)\in\mathbb{C}^n$. The equation $Uv=v$ for $U\in\Delta(n)$ implies that $U$ is the identity, so the formula $U\to(U\Delta(n), Uv)$ defines an embedding

$$SU(n)\to SU(n)/\Delta(n)\times\mathbb{C}^n.$$

Again we show that the normal bundle is trivial. If $n$ is even, the normal bundle has dimension $n/2$, and we consider a real vector

$$t=(t_1,t_1,t_2,t_2,\ldots,t_{\frac{1}{2}n},t_{\frac{1}{2}n})$$

sufficiently close to $v$. If $t,u$ are two such vectors, then the equation $Ut=u$ for $U\in\Delta(n)$ implies that $t=u$ and $U$ is the identity. Hence we have an embedding

$$SU(n)\times\mathbb{R}^{\frac{1}{2}n}\to SU(n)/\Delta(n)\times\mathbb{C}^n \quad (n \text{ even}).$$

If $n$ is odd, the normal bundle has dimension $m+2$ where $m=[\frac{1}{2}n]$. Since $\Delta(n)=\Delta(n-1)$, the above argument can be modified by adding arbitrary complex numbers as the $n$th components of $t$ and $u$. In this way we obtain an embedding

$$SU(n)\times\mathbb{R}^m\times\mathbb{C}\to SU(n)/\Delta(n)\times\mathbb{C}^n \quad (n \text{ odd}).$$

Finally, we identify $\mathbb{C}^n$ with appropriate products of copies of $\mathbb{R}$ and $\mathbb{C}$ to obtain the composite embeddings

$$SU(n)\times\mathbb{R}^{\frac{1}{2}n}\to\mathbb{R}^{n^2+\frac{1}{2}n-1} \quad (n \text{ even})$$

$$SU(n)\times\mathbb{R}^m\times\mathbb{C}\to\mathbb{R}^{n^2+m+1} \quad (n \text{ odd}, m=[\tfrac{1}{2}n])$$

which we set out to construct. This completes the proof of the theorem for $SU(n)$.

In conclusion, we show that $SU(3)$ cannot be smoothly embedded in $\mathbb{R}^{10}$. By the Pontrjagin–Thom construction, such an embedding would yield a map from $S^{10}$ to the double suspension $\Sigma^2 SU(3)$ of degree one on the top cell. This would imply that $\Sigma^2 SU(3)$ is reducible; we shall show that, on the contrary, the attaching map of the top cell in $\Sigma^2 SU(3)$ is essential.

Recall that $SU(3)$ is a principal $S^3$-bundle over $S^5$ whose characteristic element is the generator $\eta_3$ of $\pi_4(S^3)\cong\mathbb{Z}_2$. (We shall follow the notation of [5] for homotopy elements.) Hence by [2] we have a cell decomposition

$$\Sigma SU(3)\cong S^4\cup_\eta e^6\cup_{i\circ\phi} e^9$$

where $i: S^4\to S^4\cup_\eta e^6$ denotes the inclusion map, and $\phi\in\pi_8(S^4)$ is obtained by applying the Hopf construction to $\eta_3$. Since the Hopf construction applied to the identity class $\iota_3$ on $S^3$ yields the Hopf invariant one element $\nu_4\in\pi_7(S^4)$, $\phi=\nu_4\circ\eta_7$ by naturality. From [5, p. 43] we know that $\Sigma\phi=\nu_5\circ\eta_8$ generates $\pi_9(S^5)\cong\mathbb{Z}_2$. By the homotopy excision theorem $\pi_{10}(S^5 \cup_\eta e^7)$ projects isomorphically on to $\pi_{10}(S^7)$, so that we have an exact sequence

$$\pi_{10}(S^7) \xrightarrow{\Delta} \pi_9(S^5) \xrightarrow{i_*} \pi_9(S^5 \cup_\eta e^7),$$

where $\Delta\iota_7 = \eta_5$. By naturality, $\Delta\eta_7 = \eta_5 \circ \nu_6 = 0$ [5, p. 44]. Hence $i_*$ is injective, so that the top cell of $\Sigma^2 SU(3)$ is attached essentially by $i_*(\Sigma\phi)$.

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