

# APPLICATION OF GAME THEORY TO SOME PROBLEMS IN AUTOMOBILE INSURANCE \*)

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## *Introduction*

In this paper we shall study the problem of determining "correct" premium rates for sub-groups of an insurance collective. This problem obviously occurs in all branches of insurance. However, it seems at present to be a really burning issue in automobile insurance. We shall show that the problem can be formulated as a conflict between groups which can gain by co-operating, although their interests are opposed. When formulated in this way, the problem evidently can be analysed and solved by the help of the "Game Theory" of Von Neumann and Morgenstern (5).

### *1. Discussion of a Numerical Example*

**1.1.** We shall first illustrate the problem by a simple example. We consider a group of  $n_1 = 100$  persons, each of whom may suffer a loss of 1, with probability  $p_1 = 0.1$ . We assume that these persons consider forming an insurance company to cover themselves against this risk. We further assume that for some reason, government regulations or prejudices of managers, an insurance company must be organized so that the probability of ruin is less than 0.001

If such a company is formed, expected claim payment will be

$$m = n_1 p_1 = 10$$

and the standard deviation of the claim payments will be

$$\sigma = \sqrt{n_1 p_1 (1 - p_1)} = 3$$

If the government inspection (or the company's actuary) agrees that the ruin probability can be calculated with sufficient approximation by assuming that the claim payments have a normal distribution, the company must have funds amounting to

$$m + 3\sigma = 10 + 9 = 19$$

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This means that the company must collect the following amount from the 100 persons:

$$\begin{array}{r} \text{A net premium} \quad 10 \\ + \text{ a safety loading} \quad \underline{9} \\ = \text{ Total premium} \quad 19 \end{array}$$

Hence each person in this group, which we shall call *Group 1*, must pay a premium of 0.19.

**1.2.** We then consider *Group 2*, which consists of  $n_2 = 100$  persons for whom the probability of a one unit loss is  $p_2 = 0.2$ . If these persons form an insurance company, they will have to pay:

$$\begin{array}{r} \text{Net premium} \quad 20 \\ + \text{ Safety loading} \quad \underline{12} \\ = \text{ Total premium} \quad 32 \end{array}$$

in order to reach the security level required, i.e. each person will have to pay a premium of 0.32.

Assume now that the two groups join, and form one single company. In order to ensure that the ruin probability shall be less than 0.001, this company must have funds amounting to

$$\begin{aligned} & n_1 p_1 + n_2 p_2 + 3 \sqrt{n_1 p_1 (1 - p_1) + n_2 p_2 (1 - p_2)} \\ & = 10 + 20 + 15 = 45 \end{aligned}$$

**1.3.** We see from this example that it is to the advantage of the two groups to form one single company. Total payment of premium will then be 45, whilst it will be  $19 + 32 = 51$  if each group forms its own company.

The open question is how this advantage shall be divided between the two groups. The classical actuarial argument is that each group shall be charged its "fair" premium. However, this principle has meaning only as far as the net premium is concerned, it does not say anything about how the safety loading should be divided between the two groups. The orthodox method would be to divide the safety loading *pro rata* between the two groups, i.e. to let them pay total premium of 15 and 30 respectively. The "fairness" of this rule is certainly open to question, since it gives *Group 1* most of the gain accruing from the formation of one single company. In any case the rule is completely arbitrary.

The Theory of Games has as its purpose just to analyse such situations of conflicting interests. In some cases the theory will enable us to find a solution without resorting to arbitrary rules. In other cases the theory will make it clear that the problem in its very nature is indeterminate, and that some "additional assumption" or "arbitrary rule" is indeed required.

**1.4.** In the example we have analysed, most actuaries will reject as "unfair" the suggestion that both groups should pay the same premium of 22.5, i.e. that each person should pay 0.225. The game theory also rejects this suggestion, but not on the basis of some arbitrary rule of fairness. In game theory one notes that Group 1 by forming its own company will have to pay a premium of 19. If the joint company demands a premium of 22.5, Group 1 will then break out and form its own company. This will increase the premium for Group 2 from 22.5 to 32. Hence it will be to the advantage of this group to offer some concession in order to keep Group 1 in the company. For instance if Group 1 is charged a premium of 18, it will lose if it breaks out and forms its own company. Group 2 will in this case have to pay a premium of 27, which is considerably less than 32, the premium Group 2 will have to pay if it cannot persuade Group 1 to stay in the joint company.

**1.5.** The considerations in the preceding paragraph do not give a determinate solution to our problem.

Let  $P_1$  and  $P_2$  be the amount of premium paid by the two groups. If the groups act "rationally" and form a joint insurance company, we have

$$P_1 + P_2 = 45$$

The groups will stay in this company only if  $P_1 \leq 19$  and  $P_2 \leq 32$ , hence we must have

$$\begin{aligned} 13 &\leq P_1 \leq 19 \\ 26 &\leq P_2 \leq 32 \end{aligned}$$

Any pair of premiums which satisfy the equation and the inequalities in this paragraph, will constitute an acceptable solution to our problem.

**1.6.** We now assume that a Group 3 enters the picture. Let

$n_3 = 120$  and  $p_3 = 0.3$ . It is easy to see that if this group forms its own insurance company, the group will have to pay a total premium of

$$n_3 p_3 + 3 \sqrt{n_3 p_3 (1 - p_3)} = 36 + 15 = 51$$

in order to keep the ruin probability under 0.001. If the three groups join to form one company, the total premium will be

$$10 + 20 + 36 + 21 = 87$$

As in the preceding paragraph we find the indeterminate solution, given by

$$P_1 + P_2 + P_3 = 87$$

$$4 \leq P_1 \leq 19$$

$$17 \leq P_2 \leq 32$$

$$36 \leq P_3 \leq 51$$

It may seem surprising that one of the two first groups actually may be charged an amount less than the net premium. However, this is not complete nonsense. If for instance Group 1 pays only 7, the two other groups together will have to pay 80, which is less than  $32 + 51 = 83$  which they would have to pay if *each* of them had to form its own company.

**1.7.** The rather surprising result in the preceding paragraph cannot materialize if groups 2 and 3 can form an insurance company without Group 1. If they form such a company, the amount of premium to be paid will be

$$n_2 p_2 + n_3 p_3 + 3 \sqrt{n_2 p_2 (1 - p_2) + n_3 p_3 (1 - p_3)} = 20 + 36 + 19.2 = 75.2$$

It is then clear that the two groups will admit Group 1 into their company only if this will reduce their own premium, i.e. lead to a solution where  $P_2 + P_3 < 75.2$ . This means that Group 1 will have to pay a premium  $P_1 > 11.8$ . However, it will be to the advantage of Group 1 to accept this, as long as  $P_1 < 19$ , the premium the group must pay if it forms its own insurance company.

Similar considerations of the companies which can be formed by groups 1 and 2 and by groups 1 and 3 gives

$$P_1 + P_2 < 45$$

$$P_1 + P_3 < 63.4$$

Hence we get the final solution

$$P_1 + P_2 + P_3 = 87$$

where

$$11.8 \leq P_1 \leq 19$$

$$23.6 \leq P_2 \leq 32$$

$$42 \leq P_3 \leq 51$$

**1.8.** This simple example should be sufficient to illustrate the power of game theory when it comes to analysing some of the essential problems in insurance. The basic idea is that a group will have to pay a premium which depends on the alternative actions available, if the group should decide to reject an offer from other groups, i.e. from an insurance company. In other words, the *bargaining strength* of the group will determine the premium. There can be little doubt that this is a more realistic approach to the problem than one based on more orthodox actuarial considerations of "fairness".

During the last decade we have seen that a number of groups, civil servants, physicians, teetotallers etc. have felt strong enough to form their own, usually mutual, automobile insurance companies. A number of authors deplore this development, which they consider a danger to the whole insurance industry. For instance Thépaut (7) states:

Ces groupements ou mutuelles qui bouleverseraient complètement la distribution de l'assurance automobile et partout de l'assurance tout court, paraissent de nature à mettre en question l'existence même des réseaux d'Agents Généraux des Sociétés.

It is possible to find even stronger statements. It seems, however, that these authors, as long as they argue in the terms of more orthodox actuarial concepts, have difficulties, both in explaining the development, and in proposing remedies.

## 2. A More General Case

**2.1.** In this Section we shall try to build a more general theory on the basis of our discussion of the example above.

We shall now consider  $m$  groups. Group  $i$  ( $i = 1, \dots, m$ ) consists of  $n_i$  persons who are exposed to risk of a unit loss with probability

$p_i$ . We shall refer to this set of groups as  $M$ . Let  $S$  be an arbitrary subset of  $M$ .

We assume that the groups in any subset can form an insurance company to protect the members of the groups against the losses, and we assume further that the safety requirements are the same as in the example of the preceding Section (i.e. probability of ruin  $< 0.001$ ).

If the groups in the subset  $S$  form an insurance company, the amount of premium they have to pay will be

$$v(S) = \sum_s n_i p_i + 3 \left( \sum_s n_i p_i (1 - p_i) \right)^{\frac{1}{2}}$$

where summation is over all members of  $S$ .

Our problem can then be formulated as follows:

Which of the  $2^m - 1$  possible subsets will form their own insurance companies, and what premium will be paid by each of the groups which belong to these sets?

**2.2.** Let us consider a set  $S$  consisting of  $s$  groups, and let  $\bar{S}$  be the set consisting of the  $m - s$  groups which are not members of  $S$ .

It is easy to prove by elementary arithmetics that for any  $S$  we have

$$v(S) + v(\bar{S}) > v(M)$$

This inequality states the rather obvious, namely that the total amount of premium will be lowest, if all groups join to form one single insurance company.

Hence, if the groups act rationally, we should expect this company to be formed. We have thus found the answer to the first question in the preceding paragraph. The second question can only be answered in part, all we can conclude so far is that we must have:

$$(1) \quad \sum_{i=1}^m P_i = v(M)$$

where  $P_i$  is the premium to be paid by Group  $i$ .

If Group  $i$  refuses to co-operate with any other group, it will have to pay a premium

$$v(i) = n_i p_i + 3 \sqrt{n_i p_i (1 - p_i)}$$

If the group acts rationally, it will not co-operate with other groups, if such co-operation gives a higher premium than it can

obtain by forming its own insurance company. Hence we must have

$$(2) \quad P_i \leq v(i) \quad \text{for all } i$$

**2.3.** Any set of values  $P_1 \dots P_m$  which satisfy the two conditions (1) and (2) constitute in the terminology of Von Neumann and Morgenstern a *solution* to the  $n$ -person game. The conditions are obviously a generalization of those found in para 1.5.

The solution is indeterminate, in the sense that it gives only an interval in which the premium for each group must lie.

We see this if we write

$$P_i = v(i) - t_i$$

where  $t_i$  is non-negative and satisfies the condition

$$\sum_{i=1}^m t_i = \sum_{i=1}^m v(i) - v(M)$$

$\sum_{i=1}^m t_i$  represents the gain obtained collectively by the groups if they co-operate and form one single insurance company. How this gain should be divided among the groups is left undetermined.

**2.4.** The solution concept of Von Neumann and Morgenstern is obviously not entirely satisfactory. A number of devices or additional assumptions have been proposed in order to make the solution completely, or at least more determinate.

A fairly innocent looking assumption is that for any set  $S$  contained in  $M$  we shall have

$$(3) \quad \sum_s P_j \leq v(S)$$

This is the same assumption which we made in para 1.7. It implies that no set of groups will stay in the joint company, if the total amount of premiums to be paid by these groups will be lower if they form their own company. All sets of values  $P_1 \dots P_m$  which satisfy the conditions (1), (2) and (3) is referred to as the *core* of the game. This term is due to Gillies (see (3), page 194). The core is obviously contained in the *solution* defined by Von Neumann and Morgenstern.

**2.5.** As we did for a special case in para 1.7, we shall use the core to obtain narrower limits for  $P_i$ .

Let  $M - i$  stand for the set consisting of all groups except Group  $i$ . Under our assumptions we have

$$\sum_{j=1}^m P_j = v(M)$$

$$\sum_{j \neq k} P_j \leq v(M - k)$$

By subtracting the inequality from the equation we obtain

$$P_k \geq v(M) - v(M - k)$$

Hence we get the following interval for  $P_i$

$$v(M) - v(M - i) \leq P_i \leq v(i)$$

**2.6.** We now introduce the symbols

$$\pi_j = n_j \phi_j$$

$$\pi = \sum_{j=1}^m \pi_j$$

$$u_j = n_j \phi_j (1 - \phi_j)$$

$$u = \sum_{j=1}^m u_j$$

i.e.  $\pi_j$  and  $u_j$  are the mean and variance of the losses in Group  $j$ . With this notation we have

$$\sum_{j=1}^m P_j = \pi + 3\sqrt{u}$$

It is easy to see that if  $u_j$  is small in relation to  $u$ , the inequality in the preceding paragraph can approximately be written in the following form:

$$\pi_i + 3 \frac{u_i}{2\sqrt{u}} \leq P_i \leq \pi_i + 3\sqrt{u_i}$$

We see from this that a  $P_i$  which belongs to the core cannot be smaller than the net premium  $\pi_i$ . The inequality when written in this form, indicates that it will not be possible to obtain a determinate solution by some limiting process.

If  $n = \sum_{j=1}^m n_j$  increases towards infinity, it is of course trivial that each person will have to pay a premium approximately equal



to the net premium. However, the group to which he belongs will still have to pay a non-zero safety loading.

**2.7.** It is clear that in order to get a determinate solution we need stronger assumptions than the three conditions which define the core. These assumptions must state something about how the groups negotiate their way to a final arrangement, how they make offers and counter-offers, and how they compromise or break off negotiations.

Let us first assume that Group 1 forms its own company, i.e. that

$$P_1 = v(1)$$

Let us then assume that the manager of this company wants his company to grow at all costs, and that he persuades Group 2 to join the company on the condition the group is charged the lowest possible premium, i.e. that Group 1 shall get no reduction in premium owing to Group 2 joining the company. This means that Group 2 will pay

$$P_2 = v(1, 2) - v(1)$$

If similarly Group 3 joins the company on the same conditions, we get

$$P_3 = v(1, 2, 3) - v(1, 2)$$

If Group  $m$  is the last to join the company, it will be charged a premium

$$P_m = v(M) - v(M - (m - 1))$$

**2.8.** The premiums  $P_1 \dots P_m$  which we determined above satisfy the conditions (1), (2) and (3), and hence constitute an acceptable solution. However, we cannot accept this as the final unique solution to our problem, unless we know that the  $m$  groups can join the company only in the particular order we assumed.

Altogether the groups can join the company in  $m!$  orders. If we consider all these orderings as equally acceptable, it is reasonable that Group  $i$  shall pay the average of the premium it will be charged in these orderings. Hence we get

$$P_i = \sum_s \frac{(s-1)!(m-s)!}{m!} \{v(S) - v(S-i)\}$$

where summation is over all subsets  $S$  in  $M$ , and where  $s$  stands for the number of groups in  $S$ .

This solution is due to Shapley (6). It certainly appears reasonable, although one may hesitate in accepting it as the final correct solution to the rating problem in automobile insurance. One may for instance accept that the differences  $v(S) - v(S - i)$  are the essential strategic elements which must determine the premium of Group  $i$ , but one may suggest a different set of weights, for instance a set giving less weights to the extremes  $v(M) - v(M - i)$  and  $v(i)$ .

It is hard to argue against such suggestions from the rather arbitrary way in which we have derived the solution. However, the Shapley solution can be derived in a number of different ways which may be more convincing than the one we have followed.

**2.9.** In his original proof Shapley (6) took a quite different approach. He first proved that the set function  $v(S)$ , usually referred to as the *characteristic function* of the game, can be written as a linear combination

$$v(S) = \sum_R c_R v_R(S)$$

Here summation is over all subsets  $R$  of  $M$ ,  $c_R$  are constants, and  $v_R(S)$  are characteristic functions of symmetric games.

His basic assumptions are, in our symbols:

(i) The premium of each group is determined by the characteristic function, i.e.  $P_i = P_i(v)$

(ii) In a symmetric game, the participants will divide the gain equally among themselves.

(iii)  $P(v)$  is additive, i.e.  $P_i(v + w) = P_i(v) + P_i(w)$ .

From these assumptions it follows that

$$P_i(v) = \sum_R c_R \frac{v_R}{r}$$

where  $r$  are the number of players, or groups in the subset  $R$ . It is then easy to show that this reduces to the expression which we found in para 2.8.

**2.10.** Harsanyi (2) has obtained the Shapley solution as a special case of a far more general game. In the game studied by Harsanyi each player attaches a *utility* to the gain, and this utility may be

different from the monetary value of the gain. The starting point is the Nash (4) solution to the two-person game, according to which two rational players will agree on the solution which maximizes the *product* of the gains in utility. Harsanyi generalizes this to  $n$ -person games, and finds that his solution reduces to the Shapley solution if utility is equal to monetary value.

**2.11.** If the Shapley solution is applied to the two numerical examples in Section 1, we find:

For the two group example:

$$P_1 = 16 \text{ and } P_2 = 29$$

and for the three group example:

$$P_1 = 14.5, P_2 = 26.9 \text{ and } P_3 = 45.6$$

Whether these premiums are more "reasonable" than those found by more intuitive arguments, is of course open to discussion. However, our premiums have been derived from a few simple assumptions about rational behaviour, which seem to have a fairly general validity. This should at least mean that these premiums ought not to be rejected outright in favour of other premiums derived from necessarily arbitrary considerations as to what constitutes actuarial fairness.

**2.12.** In our model we have assumed that each group of persons behaves as one "rational player" in the sense given to this term in game theory. With our present knowledge of group behaviour it is difficult to say much either for or against this assumption.

Our assumption implies, however, that each group attaches the same utility to a given gain, i.e. to a given reduction in the total amount of premium payable by the group. It may be more natural to assume that the utility which the group attaches to a certain reduction in total premium is equal to the reduction obtained for *each member* of the group. Under this assumption the gain  $t_i$  of Group  $i$  will have the utility

$$u_i(t_i) = \frac{t_i}{n_i}$$

If groups in fact behave in this way, the Shapley solution will no longer be valid. We will then have to analyse the problem either

with the more general method of Harsanyi, or use Shapley's approach to a game between  $n$  persons instead of a game between  $m$  groups. This will require some very heavy arithmetics, and we shall not in the present paper pursue the matter any further.

### 3. *Another Numerical Example*

**3.1.** The difference between the traditional approach of fairness and the game theory solution is brought out most clearly if the groups are of very unequal size.

If in the example studied in Section 1, we assume

$$n_1 = n_2 = 10 \quad \text{and} \quad n_3 = 300$$

we find

$$P_1 = 2.20, \quad P_2 = 3.70, \quad P_3 = 111.39$$

Hence the Shapley solution gives the following premiums per person in the three groups:

$$q_1 = 0.220, \quad q_2 = 0.369, \quad q_3 = 0.371$$

The traditional method of making the safety loading proportional to the net premium would give

$$q'_1 = 0.126, \quad q'_2 = 0.252, \quad q'_3 = 0.378$$

**3.2.** Groups 1 and 2 do not get "fair" treatment if we accept the Shapley solution. However, they can do little about this. If the two groups each form their own company, they will have to pay the following premiums

$$q''_1 = 0.385 \quad \text{and} \quad q''_2 = 0.572$$

If the two minority groups join and form one company, they do better. If the gain resulting from this co-operation is divided equally, the premiums per person become

$$q'''_1 = 0.294 \quad \text{and} \quad q'''_2 = 0.481$$

To Group 3 it does not matter much whether the two other groups co-operate or not. If Group 3 has to form a company alone, the premium per member of the group will be

$$q'''_3 = 0.379$$

Hence Group 3 can afford to refuse the demand for actuarial fairness from the other groups.

**3.3.** If all three groups form one company, and if this company charges the same premium to all members, this common premium will be  $q = 0.367$ .

This means in practical terms that if the Shapley solution is accepted, Group 2 will not be able to obtain its own rating, since  $q_2$  and  $q_3$  above are practically equal.

Group 1 will, on the other hand, be recognized as a group of particularly good risks, and will get its own rating. However, the group will have to pay a premium which probably will be considered as "unfair" by any actuary the group may consult.

#### 4. Conclusion

**4.1.** The particular results which we have arrived at in the preceding sections obviously depend on our very arbitrary assumptions about the safety requirements of insurance companies. It is, however, clear that the whole argument could be carried through with safety requirements or equivalent restrictions in a different form.

It might have been more realistic if we had considered administrative costs instead of safety loading. We can for instance assume that these costs in an insurance company depend on the number of policies  $n$ , and on the number of claim payments  $m$ .

If we assume that the cost function is of the form

$$a\sqrt{n} + b\sqrt{m}$$

the expected cost of an insurance company formed by Group 1 will be

$$C_1 = n_1 p_1 + a\sqrt{n_1} + b\sqrt{n_1 p_1}$$

If this group forms a company together with Group 2, expected cost will be

$$C_{12} = n_1 p_1 + n_2 p_2 + a\sqrt{n_1 + n_2} + b\sqrt{n_1 p_1 + n_2 p_2}$$

It is easy to see that

$$C_{12} < C_1 + C_2$$

Hence this model is substantially the same as the one we have studied in the preceding sections. The gain will in this case be a saving in administrative cost.

**4.2.** In a general analysis we would have to consider the *utility* of the different groups. It has been argued in a previous paper (1)

that a utility concept is essential to deeper studies in the theory of insurance. However, the concept is not strictly necessary for our present purpose which is to illustrate how the theory of  $n$ -person games can be applied to some of the central and most controversial problems in insurance.

**4.3.** The problem we have studied seems at present to have particular importance in automobile insurance. However, the problem obviously exists in all branches of insurance.

For instance, a number of fires are caused by careless smokers and children playing with matches. Hence non-smoking and childless home owners could with some right demand lower fire insurance premiums. When they have neither obtained, nor even claimed this, the reason may be that as a group they are not strong enough to form their own insurance company. If they were sufficiently strong, it is likely that the existing companies would offer this group concessions which would balance any advantages the group could gain by forming its own company.

**4.4.** Our problem may have some real importance in life insurance. During the last decades most companies have become more and more "liberal" in accepting at normal premium, risks which previously were considered as "sub-standard". The game theory indicates that there may be limits to how liberal a company can be if it wants to avoid a revolt among the "standard" risks, who in the end pay for the company's liberal policy.

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