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NEW INTERPOLATION THEOREMS RELATED TO THE SPACE BMO_A ON SPACES OF HOMOGENEOUS TYPE

QI-HUI ZHANG AND DA-PING NI

Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type in the sense of Coifman and Weiss, and $\mathrm{BMO}_A(\mathcal{X})$ be the space of BMO type associated with an "approximation to the identity" $\{A_t\}_{t>0}$ and introduced by Duong and Yan. In this paper, we establish new interpolation theorems of operators related to the space $\mathrm{BMO}_A(\mathcal{X})$.

1. INTRODUCTION

We shall work on the space of homogeneous type. Let $\mathcal{X}$ be a set and d be a quasi-metric, that is, d is a function defined from $\mathcal{X} \times \mathcal{X}$ to $[0, \infty)$ satisfying the following conditions:

(i) $d(x, y) \geqslant 0$ for all x, $y \in \mathcal{X}$, and $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in \mathcal{X}$;

(iii) there exists a constant $\kappa \geqslant 1$ such that for any $x, y, z \in \mathcal{X}$,

$$d(x, y) \leqslant \kappa\big[d(x, z) + d(y, z)\big]. \tag{1}$$

Let μ be a positive Borel regular measure on $\mathcal{X}$. We say that $(\mathcal{X}, d, \mu)$ is a space of homogeneous type in the sense of Coifman and Weiss [**2**], if μ satisfies the doubling condition that for all $x \in \mathcal{X}$ and $r > 0$,

$$\mu\big(B(x, 2r)\big) \leqslant C\mu\big(B(x, r)\big) < \infty, \tag{2}$$

where $B(x, r) = \big\{y \in \mathcal{X} : d(x, y) < r\big\}$. Besides, the balls $B(x, r)$ are not necessarily open, but by a theorem of Macias and Segovia [**7**], there is a continuous quasi-metric d', which is equivalent to d in the sense that there exists a constant $C > 0$ such that for all $x, y \in \mathcal{X}$, $C^{-1}d(x, y) \leqslant d'(x, y) \leqslant Cd(x, y)$, and that the balls with respect to d' are open. Thus, throughout this paper, we always assume that the quasi-metric d is continuous and all balls in $\mathcal{X}$ are open. On the other hand, the above doubling property

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implies the strong homogeneity property: there exist positive constants C and n such that for all $\lambda \geqslant 1$, $r > 0$ and $x \in \mathcal{X}$,

$$\mu\big(B(x, \lambda r)\big) \leqslant C\lambda^n \mu\big(B(x, r)\big). \tag{3}$$

There also exist C and N, $0 \leqslant N \leqslant n$, such that for all $x, y \in \mathcal{X}$ and $r > 0$,

$$\mu\big(B(y, r)\big) \leqslant C\Big(1 + \frac{d(x, y)}{r}\Big)^N \mu\big(B(x, r)\big). \tag{4}$$

In [4], to obtain weak $(1,1)$ estimates for certain Riesz transforms, and L^p-boundedness ($p \in (1, \infty)$) of holomorphic functional calculi of linear elliptic operators on irregular domains, Duong and McIntosh introduced singular integral operators with non-smooth kernel on irregular domains via the following generalised approximations to the identity.

Definition 1: A family of operators $\{A_t\}_{t>0}$ is said to be an "approximation to the identity", if for every $t > 0$, A_t can be represented by the kernel $a_t(x, y)$ which is a measurable function defined on $\mathcal{X} \times \mathcal{X}$, in the following sense: for every function $f \in L^p(\mathcal{X})$ with $p \geqslant 1$ and almost everywhere $x \in \mathcal{X}$,

$$A_t f(x) = \int_{\mathcal{X}} a_t(x, y) f(y)\, \mathrm{d}\mu(y),$$

and the kernel $a_t(x, y)$ satisfies that for all x, $y \in \mathcal{X}$ and $t > 0$,

$$\big|a_t(x, y)\big| \leqslant h_t(x, y) = \frac{1}{\mu\big(B(x, t^{1/m})\big)} s\big(d(x, y)^m t^{-1}\big),$$

where m is a positive constant and s is a positive, bounded, decreasing function satisfying

$$\lim_{r \to \infty} r^{n+\delta} s(r^m) = 0$$

for some $\delta > N$ appearing in (4).

During the last several years, considerable attention has been paid to the operators and function spaces associated with the "approximation to the identity". To established the weighted $L^p(\mathcal{X})$ estimate with A_p weights (the weight function class of Muckenhoupt) for singular integral operators with non-smooth kernel, Martell [8] introduced the following sharp maximal operator $M_A^{\#}$.

Definition 2: Let $f \in L^p(\mathcal{X})$ for some $p \in [1, \infty)$. The sharp maximal function $M_A^{\#} f$ associated with the "approximation to the identity" $\{A_t\}_{t>0}$ is defined as

$$M_A^{\#} f(x) = \sup_{x \in B} \frac{1}{\mu(B)} \int_B \big|f(y) - A_{t_B} f(y)\big|\, \mathrm{d}\mu(y),$$

where the supremum is taken over all balls containing x and $t_B = r_B^m$, r_B is the radius of the ball B.

Duong and Yan [5] introduced the following new function space of BMO type, $\mathrm{BMO}_A(\mathcal{X})$, via the sharp maximal operator $M_A^\#$.

DEFINITION 3: Let $\{A_t\}_{t>0}$ be an "approximation to the identity" as in Definition 1. A function $f \in L^p(\mathcal{X})$ with $p \in [1,\infty)$ is said to belong to the space $\mathrm{BMO}_A(\mathcal{X})$, if $M_A^\# f$ is bounded. Moreover, the norm of f in the space $\mathrm{BMO}_A(\mathcal{X})$ is defined by

$$\|f\|_{\mathrm{BMO}_A(\mathcal{X})} = \|M_A^\# f\|_{L^\infty(\mathcal{X})}.$$

REMARK 1. Let M be the standard Hardy-Littlewood maximal operator, and $L^{p,\infty}(\mathcal{X})$ be the weak $L^p(\mathcal{X})$ space with $p \in (0,\infty)$ (recall that $f \in L^{p,\infty}(\mathcal{X})$ if

$$\|f\|_{L^{p,\infty}(\mathcal{X})} = \sup_{\lambda>0} \lambda\mu\Big(\big\{x \in \mathcal{X} : |f(x)| > \lambda\big\}\Big)^{1/p} < \infty,$$

and $\|\cdot\|_{L^{p,\infty}(\mathcal{X})}$ is not a norm). It is well known that M is bounded from $L^1(\mathcal{X})$ to $L^{1,\infty}(\mathcal{X})$, and is bounded from $L^{p,\infty}(\mathcal{X})$ to $L^{p,\infty}(\mathcal{X})$ for any $p \in (1,\infty)$ by a result of [9]. Note that $|A_t f(x)| \leqslant CMf(x)$ (constant C is independent of t) and then if $f \in L^{p,\infty}(\mathcal{X})$ for some $p \in (1,\infty)$, $A_t f$ is meaningful. Thus in Definition 1, Definition 2 and Definition 3, we may replace the condition $f \in L^p(\mathcal{X})$ by $f \in L^{p,\infty}(\mathcal{X})$ with $p \in (1,\infty)$.

As it was pointed out by [8, 5], in some sense, $\mathrm{BMO}_A(\mathcal{X})$ is larger than $\mathrm{BMO}(\mathcal{X})$. Duong and Yan [5] have established an interpolation theorem related to $\mathrm{BMO}_A(\mathcal{X})$. They showed that if a sublinear operator T is bounded on $L^q(\mathcal{X})$ $\big(q \in [1,\infty)\big)$ and is bounded from $L^\infty(\mathcal{X})$ to $\mathrm{BMO}_A(\mathcal{X})$, then T is bounded on $L^p(\mathcal{X})$ for all $p \in (q,\infty)$. The main purpose of this paper is to improve the Duong-Yan's interpolation theorem related to the space $\mathrm{BMO}_A(\mathcal{X})$. To formulate our results, we first recall some definitions and notation.

DEFINITION 4: Let $1 \leqslant q \leqslant \infty$. A function h is called to be a $(1,q)$-atom if

(a) $\sup h \subset B(x,r)$ for some $x \in \mathcal{X}$ and $r > 0$;

(b) $\|h\|_{L^q(\mathcal{X})} \leqslant \Big[\mu\big(B(x,r)\big)\Big]^{1/q-1}$;

(c) $\displaystyle\int_{\mathcal{X}} h(x)\,\mathrm{d}\mu(x) = 0.$

The atomic Hardy space $H^{1,q}(\mathcal{X},d,\mu)$ is defined to be the set of functions $f \in L^1(\mathcal{X})$ satisfying that there exist $\{\lambda_j\}_{j=1}^\infty \subset \mathbb{C}$, $\sum_{j=1}^\infty |\lambda_j| < \infty$ and a sequence of $(1,q)$-atoms $\{h_j\}_{j=1}^\infty$ such that

$$f = \sum_{j=1}^\infty \lambda_j h_j$$

converges in $L^1(\mathcal{X})$. Moreover, the norm of f in $H^{1,q}(\mathcal{X},d,\mu)$ is defined by

$$\|f\|_{H^{1,q}(\mathcal{X},d,\mu)} = \inf\Big\{\sum_{j=1}^\infty |\lambda_j|\Big\},$$

where the infimum is taken over all the possible decompositions of f in $(1,q)$-atoms.

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It was proved by Coifman and Weiss in [3] that $H^{1,q}(\mathcal{X},d,\mu) = H^{1,\infty}(\mathcal{X},d,\mu)$ for $1 \leqslant q \leqslant \infty$. Therefore, in what follows, we denote $H^{1,q}(\mathcal{X},d,\mu)$ simply by $H^1(\mathcal{X},d,\mu)$.

The main results of this paper can be stated as follows.

Theorem 1. *Let $1 < p_0 < \infty$, T and $\widetilde{T}$ be two operators. Suppose that*

(a$_1$) *$|Tf_1(x) - Tf_2(x)| \leqslant |T(f_1 - f_2)(x)|$, for functions f_1, f_2 defined on $\mathcal{X}$;*

(a$_2$) *T is bounded from $H^1(\mathcal{X},d,\mu)$ to $L^1(\mathcal{X})$, and $\widetilde{T}$ is bounded from $L^{p_0}(\mathcal{X})$ to $L^{p_0,\infty}(\mathcal{X})$;*

(a$_3$) *There is a positive constant C_0 such that for any function $f \in L^\infty(\mathcal{X}) \cap L^{p_0}(\mathcal{X})$,*

$$M_A^{\#}(Tf)(x) \leqslant \widetilde{T}f(x) + C_0\|f\|_{L^\infty(\mathcal{X})}.$$

Then T is bounded on $L^p(\mathcal{X})$ for any $1 < p < p_0$.

Theorem 2. *Let $1 < p_1 < p_2 < \infty$, T and $\widetilde{T}$ be two operators. Suppose that*

(b$_1$) *$|Tf_1(x) - Tf_2(x)| \leqslant |T(f_1 - f_2)(x)|$, for functions f_1, f_2 defined on $\mathcal{X}$;*

(b$_2$) *T is bounded from $L^{p_1}(\mathcal{X})$ to $L^{p_1,\infty}(\mathcal{X})$, and $\widetilde{T}$ is bounded from $L^{p_2}(\mathcal{X})$ to $L^{p_2,\infty}(\mathcal{X})$;*

(b$_3$) *There is a positive constant C_1 such that for any function $f \in L^\infty(\mathcal{X}) \cap L^{p_2}(\mathcal{X})$,*

$$M_A^{\#}(Tf)(x) \leqslant \widetilde{T}f(x) + C_1\|f\|_{L^\infty(\mathcal{X})}.$$

Then T is bounded on $L^p(\mathcal{X})$ for any $p_1 < p < p_2$.

Remark 2. If we choose $\widetilde{T} = 0$, Theorem 1 states that an operator satisfying the condition (a$_1$) and bounded from $H^1(\mathcal{X},d,\mu)$ to $L^1(\mathcal{X})$ and from $L^\infty(\mathcal{X})$ to $BMO_A(\mathcal{X})$, is bounded on $L^p(\mathcal{X})$ for any $1 < p < \infty$. Similarly, Theorem 2 implies that an operator satisfying the condition (b$_1$) and bounded from $L^{p_1}(\mathcal{X})$ to $L^{p_1,\infty}(\mathcal{X})$ and from $L^\infty(\mathcal{X})$ to $BMO_A(\mathcal{X})$, is bounded on $L^p(\mathcal{X})$ for any $p_1 < p < \infty$.

Remark 3. One of the main difficulties in the proof of our theorems is that the composite operator $M_A^{\#}T$ may not be a sublinear operator or a quasilinear operator, and the Marcinkiewicz interpolation theorem can not apply directly.

As an application of Theorem 2, we shall consider the $L^p(\mathcal{X})$ boundedness for the following singular integral operator with non-smooth kernel introduced by Duong and McIntosh [4].

Let T be a linear operator with kernel K such that for any bounded function f with bounded support and almost all $x \notin \sup f$,

$$Tf(x) = \int_{\mathcal{X}} K(x,y)f(y)\,\mathrm{d}\mu(y),$$

where K is a measurable function on $\mathcal{X} \times \mathcal{X}\backslash\{(x,y) : x = y\}$. Assume that there exists an "approximation to the identity" $\{A_t\}_{t>0}$ such that the composite operator A_tT with

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$t > 0$ has an associated kernel $K^t(x,y)$, and there are positive constants C_2 and C such that for all $y \in \mathcal{X}$ and $t > 0$,

$$\int_{d(x,y)\geqslant C_2 t^{1/m}} \left|K(x,y) - K^t(x,y)\right| \mathrm{d}\mu(x) \leqslant C. \tag{5}$$

Theorem 3. *Let T be the operator defined as above with kernel K satisfying (5). Suppose that for some fixed q with $1 < q < \infty$, T is bounded from $L^q(\mathcal{X})$ to $L^{q,\infty}(\mathcal{X})$. Then T is bounded on $L^p(\mathcal{X})$ provided that $q < p < \infty$.*

Remark 4. Employing some ideas used in the proof of [**8**, Proposition 5.4], we can show that $M_A^{\#}(Tf)(x) \leqslant C\|f\|_{L^\infty(\mathcal{X})}$ for $f \in L^\infty(\mathcal{X})$. Thus Theorem 3 follows from Theorem 2 directly.

Throughout this paper, C denotes the constants that are independent of the main parameters involved but whose values may differ from line to line. Constants with subscript such as C_1, do not change in different occurrences. For a measurable set E, χ_E denotes the characteristic function of E.

2. Proofs of Theorems

We begin with some preliminary lemmas. We first recall the following Calderón-Zygmund decomposition theorem in [**2**] and a basic covering lemma on spaces of homogeneous type in [**1**].

Lemma 1. *Let $f \in L^1(\mathcal{X})$ and $\lambda > \|f\|_{L^1(\mathcal{X})}\left[\mu(\mathcal{X})\right]^{-1}$. Then there exist a family of balls $\{B_j\}_{j\in\Lambda}$ with almost disjoint interiors (that is, with a bounded overlap) and constants $C > 0$, such that*

(c$_1$) $\dfrac{1}{\mu(B_j)}\displaystyle\int_{B_j}\left|f(y)\right| \mathrm{d}\mu(y) \leqslant C\lambda$;

(c$_2$) $\sum_{j\in\Lambda}\mu(B_j) \leqslant C\lambda^{-1}\|f\|_{L^1(\mathcal{X})}$;

(c$_3$) $f = g + b$ *with* $b = \sum_{j\in\Lambda} b_j$;

(c$_4$) $\left|g(x)\right| \leqslant C\lambda$ *for almost all* $x \in \mathcal{X}$;

(c$_5$) $\sup b_j \subset B_j$, $\displaystyle\int_{B_j} b_j(x)\,\mathrm{d}\mu(x) = 0$, *and* $\|b_j\|_{L^1(\mathcal{X})} \leqslant C\lambda\mu(B_j)$.

Lemma 2. *Let $(\mathcal{X}, d, \mu)$ be a space of homogeneous type, $\mathcal{B} = \{\mathcal{B}_\alpha : \alpha \in \Lambda\}$ be a family of balls in $\mathcal{X}$ such that $E = \bigcup_{\alpha\in\Lambda}\mathcal{B}_\alpha$ is measurable and $\mu(E) < \infty$. Then there exists a disjoint sequence $\left\{B(x_j, r_j)\right\}_{j\in\mathbb{N}} \subset \mathcal{B}$, such that $E \subset \bigcup_{j\in\mathbb{N}} B(x_j, C_3 r_j)$ with C_3 a positive constant depending only on κ (the constant appearing in the inequality (1)). Moreover, for any $\alpha \in \Lambda$, $\mathcal{B}_\alpha$ is contained in some $B(x_j, C_3 r_j)$.*

For the proof of this lemma, see [**1**, Lemma 2.1].

LEMMA 3. *Take $\lambda > 0$, $f \in L^1(\mathcal{X}) \cup L^{p,\infty}(\mathcal{X})$ with $p \in (1,\infty)$ and a ball B_0 such that there exists $x_0 \in B_0$ with $Mf(x_0) \leqslant \lambda$. Then, for every $0 < \eta < 1$, we can find $\gamma > 0$ (independent of λ, B_0, f, x_0) in such a way that*

$$\mu\Big(\big\{x \in B_0 : Mf(x) > \beta\lambda,\, M_A^{\#}f(x) \leqslant \gamma\lambda\big\}\Big) \leqslant \eta\mu(B_0),$$

where $\beta > 1$ is a fixed constant which only depends on the space and the "approximation to the identity" $\{A_t\}_{t>0}$.

PROOF: If $f \in L^p(\mathcal{X})$ for some $p \in [1,\infty)$, this lemma was proved by Martell in [8]. Repeating the proof of [**8**, Proposition 4.1], we see that the conclusion of Lemma 3 is also true for $f \in L^{p,\infty}(\mathcal{X})$ with $p \in (1,\infty)$. ∎

As the composite operator $M_A^{\#}T$ maybe not a quasilinear operator, this gives much trouble to our work. The next lemma is devoted to showing connections between the operators T and $M_A^{\#}T$ (if we replace f by Tf) and is the key point for the proof of our theorems.

LEMMA 4. *Let $p \in (1,\infty)$ and $\{A_t\}_{t>0}$ be an "approximation to the identity" as in Definition 1. There is a positive constant C depending only on p, $\{A_t\}_{t>0}$ and the space $\mathcal{X}$ such that for any function f,*

(i) *if $\mu(\mathcal{X}) = \infty$ and $\sup_{0<\lambda<R} \lambda^p\mu\Big(\big\{x \in \mathcal{X} : Mf(x) > \lambda\big\}\Big) < \infty$ for any $R > 0$, then*

$$\|Mf\|^p_{L^{p,\infty}(\mathcal{X})} \leqslant C\|M_A^{\#}f\|^p_{L^{p,\infty}(\mathcal{X})};$$

(ii) *if $\mu(\mathcal{X}) < \infty$ and $f \in L^1(\mathcal{X})$, then*

$$\|Mf\|^p_{L^{p,\infty}(\mathcal{X})} \leqslant C\|f\|^p_{L^1(\mathcal{X})}\big[\mu(\mathcal{X})\big]^{1-p} + C\|M_A^{\#}f\|^p_{L^{p,\infty}(\mathcal{X})};$$

(iii) *if $\mu(\mathcal{X}) < \infty$ and $f \in L^{q,\infty}(\mathcal{X})$ for some $q \in (1,\infty)$, then*

$$\|Mf\|^p_{L^{p,\infty}(\mathcal{X})} \leqslant C\|f\|^p_{L^{q,\infty}(\mathcal{X})}\big[\mu(\mathcal{X})\big]^{1-p/q} + C\|M_A^{\#}f\|^p_{L^{p,\infty}(\mathcal{X})}.$$

PROOF: For the proof of the cases (i) and (ii), see [**6**, Theorem 2.2]. We only consider the case (iii). Recall that M is bounded from $L^{q,\infty}(\mathcal{X})$ to $L^{q,\infty}(\mathcal{X})$ for any $q \in (1,\infty)$, that is to say, there is a positive constant C_4 such that

$$\|Mf\|_{L^{q,\infty}(\mathcal{X})} \leqslant C_4\|f\|_{L^{q,\infty}(\mathcal{X})}.$$

Let $\lambda_0 = C_4\|f\|_{L^{q,\infty}(\mathcal{X})}\big[\mu(\mathcal{X})\big]^{-1/q}$. For $\lambda > \lambda_0$, Set

$$\mathrm{E}_\lambda = \big\{x \in \mathcal{X} : Mf(x) > \lambda\big\}$$

and

$$\mathrm{F}_\lambda = \big\{x \in \mathcal{X} : Mf(x) > \beta\lambda,\ M_A^{\#}f(x) \leqslant \gamma\lambda\big\},$$

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where β and γ appearing in Lemma 3. It is easy to see that

$$\mu(\mathrm{E}_\lambda) \leqslant C_4^q \lambda^{-q} \|f\|_{L^{q,\infty}(\mathcal{X})}^q < \mu(\mathcal{X}),$$

which in turn implies that

$$\mu(\mathrm{E}_\lambda) < \infty, \text{ and } \mathcal{X} \backslash \mathrm{E}_\lambda \neq \emptyset.$$

For each $x \in \mathrm{E}_\lambda$, denote by r_x the distance of x and the set $\mathcal{X}\backslash \mathrm{E}_\lambda$, namely,

$$r_x = \inf_{y \in \mathcal{X}\backslash \mathrm{E}_\lambda} d(x, y).$$

Then,

$$r_x > 0 \text{ and } \mathrm{E}_\lambda = \bigcup_{x \in \mathrm{E}_\lambda} B(x, C_3^{-1} r_x),$$

where $C_3 > 1$ is the same as in Lemma 2. We can apply Lemma 2 to obtain a disjoint sequence $\{B(x_j, C_3^{-1} r_j)\}_{j \in \mathbf{N}}$ such that

$$\mathrm{E}_\lambda = \bigcup_{j \in \mathbf{N}} B(x_j, r_j), \; B(x_j, C_5 r_j) \cap (\mathcal{X}\backslash \mathrm{E}_\lambda) \neq \emptyset,$$

where $C_5 > 1$ is a constant depending only on the space. Lemma 3 together with the inequality (3) states that

$$\begin{aligned} \mu(\mathrm{F}_\lambda) &\leqslant \sum_{j \in \mathbf{N}} \mu\Big(\big\{x \in B(x_j, C_5 r_j) : Mf(x) > \beta\lambda, \; M_A^{\#} f(x) \leqslant \gamma\lambda\big\}\Big) \\ &\leqslant \eta \sum_{j \in \mathbf{N}} \mu\big(B(x_j, C_5 r_j)\big) \leqslant C\eta\mu(\mathrm{E}_\lambda). \end{aligned}$$

It follows that for any $\lambda > \lambda_0$,

$$\begin{aligned} \mu\Big(\big\{x \in \mathcal{X} : Mf(x) > \beta\lambda\big\}\Big) &\leqslant \mu(\mathrm{F}_\lambda) + \mu\Big(\big\{x \in \mathcal{X} : M_A^{\#} f(x) > \gamma\lambda\big\}\Big) \\ &\leqslant C\eta\mu(\mathrm{E}_\lambda) + \mu\Big(\big\{x \in \mathcal{X} : M_A^{\#} f(x) > \gamma\lambda\big\}\Big). \end{aligned}$$

Then,

$$\begin{aligned} (\beta\lambda)^p \mu\Big(\big\{x \in \mathcal{X} : Mf(x) > \beta\lambda\big\}\Big) &\leqslant C\eta(\beta\lambda)^p \mu(\mathrm{E}_\lambda) \\ &\quad + (\beta\lambda)^p \mu\Big(\big\{x \in \mathcal{X} : M_A^{\#} f(x) > \gamma\lambda\big\}\Big). \end{aligned}$$

Thus, a straightforward computation shows that

$$\begin{aligned} \sup_{0<\lambda<\beta R} \lambda^p \mu(\mathrm{E}_\lambda) &\leqslant \sup_{0<\lambda\leqslant\beta\lambda_0} \lambda^p \mu(\mathrm{E}_\lambda) + \sup_{\beta\lambda_0<\lambda<\beta R} \lambda^p \mu(\mathrm{E}_\lambda) \\ &\leqslant (\beta\lambda_0)^p \mu(\mathcal{X}) + C\eta\beta^p \sup_{\lambda_0<\lambda<R} \lambda^p \mu(\mathrm{E}_\lambda) \\ &\qquad + (\beta\gamma^{-1})^p \sup_{\lambda>0} \lambda^p \mu\Big(\big\{x \in \mathcal{X} : M_A^{\#} f(x) > \lambda\big\}\Big) \\ &\leqslant C\|f\|_{L^{q,\infty}(\mathcal{X})}^p \big[\mu(\mathcal{X})\big]^{1-p/q} + C\eta\beta^p \sup_{0<\lambda<\beta R} \lambda^p \mu(\mathrm{E}_\lambda) \\ &\qquad + (\beta\gamma^{-1})^p \sup_{\lambda>0} \lambda^p \mu\Big(\big\{x \in \mathcal{X} : M_A^{\#} f(x) > \lambda\big\}\Big). \end{aligned}$$

Choose η such that $\eta < (C\beta^p)^{-1}$, we obtain that

$$\sup_{0<\lambda<\beta R} \lambda^p \mu(\mathrm{E}_\lambda) \leqslant C\|f\|^p_{L^{q,\infty}(\mathcal{X})}\big[\mu(\mathcal{X})\big]^{1-p/q} + C\sup_{\lambda>0}\lambda^p\mu\Big(\big\{x\in\mathcal{X} : M^\#_A f(x) > \lambda\big\}\Big)$$

and then completes the proof of the case (iii). ∎

PROOF OF THEOREM 1: We first claim that for any $1 < p < p_0$, $M^\#_A T$ is of weak type (p,p), that is, there is a positive constant C such that for any $f \in L^p(\mathcal{X})$,

$$\sup_{\lambda>0}\lambda^p\mu\Big(\big\{x\in\mathcal{X} : M^\#_A Tf(x) > \lambda\big\}\Big) \leqslant C\|f\|^p_{L^p(\mathcal{X})}. \tag{6}$$

In fact, let $\lambda_1 = 0$ if $\mu(\mathcal{X}) = \infty$ and $\lambda_1 = \|f\|^p_{L^p(\mathcal{X})}\big[\mu(\mathcal{X})\big]^{-1}$ if $\mu(\mathcal{X}) < \infty$. Note that if $\lambda^p \leqslant \lambda_1$, the inequality (6) holds obviously. For each fixed $f \in L^p(\mathcal{X})$ and $\lambda^p > \lambda_1$, by Lemma 1, we can perform the Calderón-Zygmund decomposition to $|f|^p$ at level λ^p and there exist a family of balls $\{B_j\}_{j\in\Lambda}$ and positive constants C, C_6 such that

- (d_1) $\sum_{j\in\Lambda}\mu(B_j) \leqslant C\lambda^{-p}\|f\|^p_{L^p(\mathcal{X})}$;
- (d_2) $f = g + b$ with $b = \sum_{j\in\Lambda} b_j$;
- (d_3) $|g(x)| \leqslant C\lambda$ for almost all $x \in \mathcal{X}$;
- (d_4) for any $j \in \Lambda$, $\operatorname{sup} b_j \subset B_j$, $\int_{B_j} b_j(x)\,\mathrm{d}\mu(x) = 0$ and $\|b_j\|^p_{L^p(\mathcal{X})} \leqslant C_6\lambda^p\mu(B_j)$.

It is obvious that $g \in L^p(\mathcal{X})$ and

$$\|g\|^p_{L^p(\mathcal{X})} \leqslant C\|f\|^p_{L^p(\mathcal{X})} + C\sum_{j\in\Lambda}\|b_j\|^p_{L^p(\mathcal{X})} \leqslant C\|f\|^p_{L^p(\mathcal{X})}.$$

For $1 < p < p_0 < \infty$, it is easy to see that

$$\|g\|^{p_0}_{L^{p_0}(\mathcal{X})} \leqslant C\lambda^{p_0-p}\|g\|^p_{L^p(\mathcal{X})} \leqslant C\lambda^{p_0-p}\|f\|^p_{L^p(\mathcal{X})}.$$

It follows from (d_4) that $\big(C_6^{1/p}\lambda\mu(B_j)\big)^{-1}b_j(x)$ is a $(1,p)$-atom. We know from (d_1) that

$$\sum_{j\in\Lambda} C_6^{1/p}\lambda\mu(B_j) \leqslant C\lambda^{1-p}\|f\|^p_{L^p(\mathcal{X})} < \infty,$$

which in turn implies that

$$b(x) = \sum_{j\in\Lambda} b_j(x) = \sum_{j\in\Lambda} C_6^{1/p}\lambda\mu(B_j)\big(C_6^{1/p}\lambda\mu(B_j)\big)^{-1}b_j(x) \in H^1(\mathcal{X}, d, \mu)$$

and

$$\|b\|_{H^1(\mathcal{X},d,\mu)} \leqslant \sum_{j\in\Lambda} C_6^{1/p}\lambda\mu(B_j) \leqslant C\lambda^{1-p}\|f\|^p_{L^p(\mathcal{X})}.$$

On the other hand, a trivial computation along with the condition (a_1) leads to that

$$\begin{aligned}
M_A^{\#}Tf(x) &= \sup_{x\in B}\frac{1}{\mu(B)}\int_B\Big|T(g+b)(y)-A_{t_B}\big(T(g+b)\big)(y)\Big|\,\mathrm{d}\mu(y)\\
&\leqslant \sup_{x\in B}\frac{1}{\mu(B)}\Big(\int_B|T(g+b)(y)-Tg(y)|\,\mathrm{d}\mu(y)+\int_B|Tg(y)-A_{t_B}(Tg)(y)|\,\mathrm{d}\mu(y)\\
&\qquad +\int_B\Big|A_{t_B}\big(T(g+b)\big)(y)-A_{t_B}(Tg)(y)\Big|\,\mathrm{d}\mu(y)\Big)\\
&\leqslant MTb(x)+M_A^{\#}Tg(x)+\sup_{x\in B}\frac{1}{\mu(B)}\int_B\Big|A_{t_B}\big(T(g+b)\big)(y)-A_{t_B}(Tg)(y)\Big|\,\mathrm{d}\mu(y).
\end{aligned}$$

Since $Tb\in L^1(\mathcal{X})$, an argument similar to that used in the proof of [8, Lemma 3.5] tells us that

$$\begin{aligned}
&\sup_{x\in B}\frac{1}{\mu(B)}\int_B\Big|A_{t_B}\big(T(g+b)\big)(y)-A_{t_B}\big(Tg\big)(y)\Big|\,\mathrm{d}\mu(y)\\
&\qquad\leqslant \sup_{x\in B}\frac{1}{\mu(B)}\int_B\int_{\mathcal{X}}h_{t_B}(y,z)|Tb(z)|\,\mathrm{d}\mu(z)\,\mathrm{d}\mu(y)\\
&\qquad\leqslant CMTb(x).
\end{aligned}$$

Consequently, there exists a positive constant C_7 such that

$$M_A^{\#}Tf(x)\leqslant C_7MTb(x)+M_A^{\#}Tg(x).$$

Applying the condition (a_3) and (d_3), we get

$$\begin{aligned}
\mu\Big(\big\{x\in\mathcal{X}:M_A^{\#}Tf(x)>4C_0C_7\lambda\big\}\Big) &\leqslant \mu\Big(\big\{x\in\mathcal{X}:M_A^{\#}Tg(x)>2C_0C_7\lambda\big\}\Big)\\
&\qquad +\mu\Big(\big\{x\in\mathcal{X}:MTb(x)>2C_0\lambda\big\}\Big)\\
&\leqslant \mu\Big(\big\{x\in\mathcal{X}:\widetilde{T}g(x)>C_0C_7\lambda\big\}\Big)\\
&\qquad +\mu\Big(\big\{x\in\mathcal{X}:MTb(x)>2C_0\lambda\big\}\Big).
\end{aligned}$$

The fact that $\widetilde{T}$ is bounded from $L^{p_0}(\mathcal{X})$ to $L^{p_0,\infty}(\mathcal{X})$ gives that

$$\mu\Big(\big\{x\in\mathcal{X}:\widetilde{T}g(x)>C_0C_7\lambda\big\}\Big)\leqslant C\lambda^{-p_0}\|g\|^{p_0}_{L^{p_0}(\mathcal{X})}\leqslant C\lambda^{-p}\|f\|^{p}_{L^{p}(\mathcal{X})}.$$

Note that M is of weak type (1,1) and T is bounded from $H^1(\mathcal{X},d,\mu)$ to $L^1(\mathcal{X})$. Thus,

$$\mu\Big(\big\{x\in\mathcal{X}:MTb(x)>2C_0\lambda\big\}\Big)\leqslant C\lambda^{-1}\|b\|_{H^1(\mathcal{X},d,\mu)}\leqslant C\lambda^{-p}\|f\|^{p}_{L^{p}(\mathcal{X})},$$

and the inequality (6) follows immediately.

We can now conclude the proof of Theorem 1. If $\mu(\mathcal{X})=\infty$, for any bounded function f with bounded support and $\int_{\mathcal{X}}f(x)\mathrm{d}\mu(x)=0$, we see that $f\in H^1(\mathcal{X},d,\mu)$

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and $f \in L^p(\mathcal{X})$ for $1 < p < p_0$. Then $Tf \in L^1(\mathcal{X})$. It follows from the case (i) of Lemma 4 and the inequality (6) that

$$\sup_{\lambda>0} \lambda^p \mu\Big(\Big\{x \in \mathcal{X} : \big|Tf(x)\big| > \lambda\Big\}\Big) \leqslant \|MTf\|^p_{L^{p,\infty}(\mathcal{X})} \leqslant C\|f\|^p_{L^p(\mathcal{X})}. \tag{7}$$

If $\mu(\mathcal{X}) < \infty$, for each bounded function f with bounded support and $\int_{\mathcal{X}} f(x)\,\mathrm{d}\mu(x) = 0$, it is easy to verify that $\big[\mu(\mathcal{X})\big]^{1/p-1}\|f\|^{-1}_{L^p(\mathcal{X})} f(x)$ is a $(1,p)$-atom and

$$\|f\|_{H^1(\mathcal{X},d,\mu)} \leqslant \big[\mu(\mathcal{X})\big]^{1-1/p}\|f\|_{L^p(\mathcal{X})}.$$

By the case (ii) of Lemma 4 and the inequality (6), we obtain that

$$\begin{aligned}\sup_{\lambda>0} \lambda^p \mu\Big(\Big\{x \in \mathcal{X} : \big|Tf(x)\big| > \lambda\Big\}\Big) &\leqslant C\|f\|^p_{H^1(\mathcal{X},d,\mu)}\big[\mu(\mathcal{X})\big]^{1-p} + C\|M^{\#}_A Tf\|^p_{L^{p,\infty}(\mathcal{X})} \\ &\leqslant C\|f\|^p_{L^p(\mathcal{X})}.\end{aligned} \tag{8}$$

The inequalities (7) and (8) via a standard density argument show that T is bounded from $L^p(\mathcal{X})$ to $L^{p,\infty}(\mathcal{X})$ for $1 < p < p_0$. On the other hand, note that an operator satisfying the condition (a_1) is a subadditive operator. An application of the Marcinkiewicz interpolation theorem gives us the desired result. ∎

Proof of Theorem 2: Given $f \in L^p(\mathcal{X})$ with $1 < p_1 < p < p_2 < \infty$. For each $\lambda > 0$, decompose f as

$$f(x) = f(x)\chi_{\{x\in\mathcal{X}:|f(x)|>C_8\lambda\}}(x) + f(x)\chi_{\{x\in\mathcal{X}:|f(x)|\leqslant C_8\lambda\}}(x) = f_1(x) + f_2(x),$$

where the constant C_8 will be fixed below. A straightforward computation shows that $f_1(x) \in L^{p_1}(\mathcal{X})$, $f_2(x) \in L^{p_2}(\mathcal{X})$ and

$$\|f_1\|^{p_1}_{L^{p_1}(\mathcal{X})} \leqslant (C_8\lambda)^{p_1-p}\|f\|^p_{L^p(\mathcal{X})},\ \|f_2\|^{p_2}_{L^{p_2}(\mathcal{X})} \leqslant (C_8\lambda)^{p_2-p}\|f\|^p_{L^p(\mathcal{X})}.$$

As in the proof of Theorem 1, we can obtain that there is a positive constant C_9 such that

$$M^{\#}_A Tf(x) \leqslant C_9 MTf_1(x) + M^{\#}_A Tf_2(x).$$

We shall choose C_8 such that $C_8 \leqslant C_9$. Then, the conditions (b_2), (b_3) together with the fact that M is bounded from $L^{p_1,\infty}(\mathcal{X})$ to $L^{p_1,\infty}(\mathcal{X})$ yield that

$$\begin{aligned}\mu\Big(\big\{x \in \mathcal{X} : M^{\#}_A Tf(x) > 4C_1C_9\lambda\big\}\Big) &\leqslant \mu\big(\{x \in \mathcal{X} : MTf_1(x) > 2C_1\lambda\}\big) \\ &\quad+\mu\Big(\big\{x \in \mathcal{X} : M^{\#}_A Tf_2(x) > 2C_1C_9\lambda\big\}\Big) \\ &\leqslant \frac{C}{\lambda^{p_1}}\|Tf_1\|^{p_1}_{L^{p_1,\infty}(\mathcal{X})} + \mu\Big(\big\{x \in \mathcal{X} : \widetilde{T}f_2(x) > C_1C_9\lambda\big\}\Big) \\ &\leqslant \frac{C}{\lambda^{p_1}}\|f_1\|^{p_1}_{L^{p_1}(\mathcal{X})} + \frac{C}{\lambda^{p_2}}\|f_2\|^{p_2}_{L^{p_2}(\mathcal{X})} \leqslant \frac{C}{\lambda^{p}}\|f\|^p_{L^p(\mathcal{X})}.\end{aligned}$$

This states that $M_A^{\#}T$ is bounded from $L^p(\mathcal{X})$ to $L^{p,\infty}(\mathcal{X})$. We then consider the following two cases to prove that T is bounded from $L^p(\mathcal{X})$ to $L^{p,\infty}(\mathcal{X})$ for $p_1 < p < p_2$.

For the case $\mu(\mathcal{X}) = \infty$. For any bounded function f with bounded support, it is easy to see that $\sup_{0<\lambda<R} \lambda^p \mu\Big(\{x \in \mathcal{X} : MTf(x) > \lambda\}\Big) < \infty$ for any $R > 0$. Applying the case (i) of Lemma 4 we can get that for $p_1 < p < p_2$,

$$\|Tf\|_{L^{p,\infty}(\mathcal{X})} \leqslant \|MTf\|_{L^{p,\infty}(\mathcal{X})} \leqslant C\|M_A^{\#}Tf\|_{L^{p,\infty}(\mathcal{X})} \leqslant C\|f\|_{L^p(\mathcal{X})}.$$

For the case $\mu(\mathcal{X}) < \infty$. For each $f \in L^p(\mathcal{X})$ with $1 < p_1 < p < p_2$, note that $f \in L^{p_1}(\mathcal{X})$ and

$$\|f\|_{L^{p_1}(\mathcal{X})} \leqslant C\big[\mu(\mathcal{X})\big]^{1/p_1 - 1/p}\|f\|_{L^p(\mathcal{X})}.$$

Thus, $Tf \in L^{p_1,\infty}(\mathcal{X})$. It follows from the case (iii) of Lemma 4 that

$$\begin{aligned}\|Tf\|^p_{L^{p,\infty}(\mathcal{X})} &\leqslant C\|Tf\|^p_{L^{p_1,\infty}(\mathcal{X})}\big[\mu(\mathcal{X})\big]^{1-p/p_1} + C\|M_A^{\#}Tf\|^p_{L^{p,\infty}(\mathcal{X})} \\ &\leqslant C\|f\|^p_{L^p(\mathcal{X})}.\end{aligned}$$

Consequently, T is bounded from $L^p(\mathcal{X})$ to $L^{p,\infty}(\mathcal{X})$ for $p_1 < p < p_2$. Recall that T is a subadditive operator, by the Marcinkiewicz interpolation theorem again, we can obtain that T is bounded on $L^p(\mathcal{X})$ for $1 < p_1 < p < p_2 < \infty$. ∎

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Department of Applied Mathematics
University of Information Engineering
PO Box 1001-747
Zhengzhou 450002
People's Republic of China
e-mail: z_qihui@yahoo.com.cn