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KOROVKIN THEOREMS FOR INTEGRAL OPERATORS WITH KERNELS OF FINITE OSCILLATION

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Introduction. There has been considerable interest recently in the investigation of "Korovkin sets". Briefly, for X a Banach space and $\mathscr{T}$ a family of linear operators on X, a subset $K \subset X$ is a Korovkin set relative to $\mathscr{T}$ if for any bounded sequence $\{T_n\} \subset \mathscr{T}$, $T_n k \to k$ in X for each $k \in K$ implies $T_n x \to x$ for each $x \in X$. A large portion of these investigations have been carried out for X being one of the spaces $C(S)$, S compact Hausdorff, the usual L^p spaces of functions on some finite measure space, or some Banach lattice; while $\mathscr{T}$ is one of the classes $\mathscr{T}_+$-positive operators, $\mathscr{T}_1$-contractions (i.e., $||T|| \leqq 1$), or $\mathscr{T}_+ \cap \mathscr{T}_1$.

In his paper [**8**], P. P. Korovkin showed that $\{1, t, \ldots, t^{m+2}\}$ forms what is now called a Korovkin set for operators $\mathscr{S}_m$ (defined below) and the space $C[a, b]$. We shall consider the class $\mathscr{S}_m$ as operators on the spaces $L^p[a, b]$, $1 \leqq p < \infty$, of Lebesgue measurable functions on the interval $[a, b]$. The operators will have the form

$$A[f : x] = \int_a^b K(x, t) f(t) dt,$$

where $K(x, t)$ is a $[a, b] \times [a, b]$ Lebesgue measurable function.

Definition. An integral operator A is of class $\mathscr{S}_m$ if for almost all x, there exists a partition of $[a, b]$ into at most $m + 1$ intervals, $I_{1,x}, \ldots, I_{r,x}$, $(r = r(x))$, such that the kernel $K(x, t)$ is of one sign ($\geqq 0$ a.e. or $\leqq 0$ a.e.) on each $I_{j,x}$, and alternates in sign on these intervals.

When the operators under consideration are algebraic or trigonometric polynomial valued, then certain quantitative results about the rate of convergence can be obtained. P. P. Korovkin first proved such results for $C[a, b]$ and algebraic polynomial valued operators of class $\mathscr{S}_m$ in [**9**]. For trigonometric polynomial valued convolution operators of class $\mathscr{S}_m$ on thc L^p spaces, corresponding results have been established by Dzjadyk [**6**] for $m = 0$, and by Butzer, Nessel, and Scherer [**7**] for other values of m.

A wealth of literature has developed on operators of class $\mathscr{S}_m$, concerning, for example, existence, relation to classical kernels, and optimal orders of convergence. We refer the interested reader to the survey article by P. L. Butzer [**2**] where a history and much of the literature appears.

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In section 2 of the present work, a simple combination of the techniques of Dzjadyk [**6**] and Korovkin [**9**] yield that $\{1, t, \ldots, t^{m+2}\}$ is also a Korovkin set for $\mathscr{S}_m$ on $L^p[a, b]$. Further, the techniques can be modified to establish the results for other sets of functions; in particular, the periodic case for $L^p[0, 2\pi]$ (sections 3 and 4). In section 5, we derive quantitative results whose form resembles those in [**3**], although their proofs are related to the original method of Korovkin [**9**].

In the sequel, we use the notation

$$A[h(x, t):x] = \int_a^b K(x, t)h(x, t)dt$$

where x and t always refer to the variables in the image space and domain space respectively. Usually, $h(x, t)$ will be of the form $\sum_{j=1}^k g_j(x)f_j(t)$. Also, the phrase "agreeing in sign" allows zero to agree with anything. Further, the constants obtained in our proofs may depend on the length of the interval $[a, b]$, although this will not be explicitly stated.

This problem arose in a seminar on Korovkin sets at the University of Alberta, and we would like to thank the participants of that seminar, particularly, Professors A. Meir, Z. Ditzian, and T. R. Turner.

2. The Korovkin Theorem for powers. We begin with a lemma which is the essential idea used by Dzjadyk [**6**] in his proof for positive operators (i.e., $m = 0$).

LEMMA 2.1. *If $\{A_n\}$ is a sequence of integral operators for which $||A_n[f_k : x] - f_k(x)||_{L^p} \to 0$ for each $f_k(t) = t^k$, $k = 0, 1, \ldots, m$, then $||A_n[p : x] - p(x)||_{L^p} \to 0$ uniformly for all polynomials*

$$p(t) = \sum_{k=0}^m a_k(x, n)f_k(t), \ ||a_k(x, n)||_{L^\infty} \leqq M < +\infty,$$

where M is an absolute constant.

Proof. For such polynomials, we have

$$A_n[p:x] - p(x) = \sum_{k=0}^m a_k(x, n)\{A_n[f_k:x] - f_k(x)\}$$

for almost all x. Thus,

$$||A_n[p:x] - p(x)||_{L^p} \leqq M \sum_{k=0}^m ||A_n[f_k:x] - f_k(x)||_{L^p}$$

which tends to zero by hypothesis so that the lemma is proven.

We shall also require the following remark.

Remark 2.1. Let x be fixed and suppose that either $f(t)$ or $-f(t)$ agrees in sign with $K(x, t)$. If $g(t) \in L^\infty[a, b]$, then

$$\left| \int_a^b K(x, t)g(t)f(t)dt \right| \leqq ||g||_{L^\infty} \left| \int_a^b K(x, t)f(t)dt \right| .$$

THEOREM 2.1. *Suppose that the sequence of integral operators $\{A_n\}$ on $L^p[a, b]$ satisfy:*

(1) $$A_n[f : x] = \int_a^b K_n(x, t) f(t) dt$$

is of class $\mathscr{S}_m$ for each n, $(n = 1, 2, \ldots)$;

(2) $||A_n||_{L^p} \leqq M < +\infty$; *and*

(3) $||A_n[f_k : x] - f_k(x)||_{L^p} \to 0$ *as* $n \to \infty$ *for each function* $f_k(t) = t^k$, $(k = 0, 1, \ldots, m + 2)$. *Then* $||A_n[f : x] - f(x)||_{L^p} \to 0$ *for each* f *in* $L^p[a, b]$.

Proof. Since $||A_n|| \leqq M$ uniformly in n, it suffices to prove the theorem for functions with at least $m + 2$ continuous derivatives. In the sequel, $f \in C^{m+2}[a, b]$, and x will be such that $K_n(x, t)$ has the sign change property for each n (this excludes only a set of measure zero).

For each fixed x and n, we construct an interpolating polynomial $p_{x,n}(t)$ agreeing with $f(t)$ at the endpoints $t_{1,x,n}, \ldots, t_{r,x,n}$ of the intervals $I_{1,x,n}, \ldots, I_{r+1,x,n}$ $(r = r(x, n) \leqq m)$ which are not a or b. Further, if x is not one of the $t_{j,x,n}$, we require $p_{x,n}(x) = f(x)$ and $p_{x,n}'(x) = f'(x)$. The degree, $l = l_{x,n}$, of $p_{x,n}(t)$ is no greater than $m + 1$.

Writing the remainder term of the interpolating polynomial in the Lagrange formula, we obtain

$$\begin{aligned} f(t) &= p_{x,n}(t) + \frac{f^{(l+1)}(c)}{(l+1)!} \prod_{j=1}^{r=r(x,n)} (t - t_{j,x,n})(t - x)^\alpha \\ &= p_{x,n}(t) + g_{x,n}(t) q_{x,n}(t) \end{aligned}$$

where $c = c(t, x, n)$, and $\alpha = 0$ if x is some $t_{j,x,n}$, $\alpha = 2$ otherwise. Here $p_{x,n}(t)$ and $q_{x,n}(t)$ are polynomials of degree at most $m + 2$ whose coefficients are uniformly bounded (the bound, M_1, depends only on f, its derivatives up to order $m + 2$, and m). Further, $|g_{x,n}(t)| \leqq M_2$ where M_2 depends only on f and m, while $q_{x,n}(t)$ agrees in sign with $K_n(x, t)$ or $-K_n(x, t)$ and $q_{x,n}(x) = 0$.

Therefore, since $f(x) = p_{x,n}(x)$,

$$\begin{aligned} ||A_n[f : x] - f(x)||_{L^p} &\leqq ||A_n[p_{x,n} : x] - p_{x,n}(x)||_{L^p} \\ &\quad + ||A_n[g_{x,n} q_{x,n} : x]||_{L^p} \\ &\leqq ||A_n[p_{x,n} : x] - p_{x,n}(x)||_{L^p} + M_2 ||A_n[q_{x,n} : x]||_{L^p} \\ &= ||A_n[p_{x,n} : x] - p_{x,n}(x)||_{L^p} \\ &\quad + M_2 ||A_n[q_{x,n} : x] - q_{x,n}(x)||_{L^p} \end{aligned}$$

by Remark 2.1. Consequently, by Lemma 2.1, $||A_n[f : x] - f(x)||_{L^p}$ converges to zero as n tends to infinity. The theorem is proven.

Remark 2.2. Let $M_j = ||f^{(j)}||_{L^\infty}$, $c = \max\{|a|, |b|\}$, and $\alpha_{ij} = c^{j-i}/i!(j - i)!$. By carefully retracing the steps in the proof of Theorem 2.1, using Newton's

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formula to represent $p_{x,n}(t)$ (see section 5 below), one can show that

$$||A_n[f:x] - f(x)||_{L^p} \leqq \sum_{j=0}^{m+2} M_j \sum_{i=0}^{j} \alpha_{ij}||A_n[t^i:x] - x^i||_{L^p}.$$

This bound, which is useful only for $f \in C^{m+2}[a, b]$, should be compared with a result by Ditzian and Freud [**5**].

3. The Korovkin Theorem for general systems. In this section we modify the arguments of the preceding section so as to obtain a theorem for a general system of functions. Included in this discussion are extended complete Chebyshev systems.

Let

$$L_m y \equiv D^m y + \sum_{i=1}^{m} a_i(t) D^{m-i} y = 0, \quad \left(D = \frac{d}{dt}\right)$$

be a linear differential equation with $a_i \in C(\alpha, \beta)$. Either α or β, or both, may be singular points and we do not exclude the possibility that β is infinite. We do require α to be finite.

We suppose that $L_m y = 0$ is disconjugate on $[\alpha, \beta]$. This means that all solutions have $m - 1$ or fewer zeros on $[\alpha, \beta]$. For a definition of zeros at singularities and other information concerning disconjugate equations see Willett [**11**] and the references therein. When $L_m y = 0$ is disconjugate on $[\alpha, \beta]$, there are functions $\xi_i \in C^{m+1-i}(\alpha, \beta)$, $\xi_i > 0$ on (α, β) such that L_m can be factored as

$$L_m = \omega(t) D_m \ldots D_2 D_1$$

where $D_i y = D(y/\xi_i)$ and $\omega \equiv 1/\xi_{m+1} = \xi_1 \xi_2 \ldots \xi_m$. Moreover, $\xi_2, \ldots, \xi_m$ are integrable only on proper subintervals $[\alpha, c]$ of $[\alpha, \beta]$ and the m-tuple $(u_1, \ldots, u_m)$ of solutions defined by $D_j D_{j-1} \ldots D_1 u_{j+1} = \xi_{j+1}$, $D^k u_{j+1}(\alpha) = 0$ for $k = 0, \ldots, j - 1$, is a fundamental principal system on $[\alpha, \beta]$. (See Willett [**11**, Theorem 2.1].)

As an example, the differential equation

$$(3.1) \quad L_{2r+1} y = D(D^2 + 1)(D^2 + 4) \ldots (D^2 + r^2) y = 0$$

corresponds to

$$(3.2) \quad \xi_1(t) = \sin^{2r}\left(\frac{\beta - t}{2}\right), \quad \xi_{k+1}(t) = k \sin\left(\frac{\beta - \alpha}{2}\right) / 2 \sin^2\left(\frac{\beta - t}{2}\right), \quad k = 1, \ldots, 2r$$

and

$$u_k(t) = \sin^{2r+1-k}\left(\frac{\beta - t}{2}\right) \sin^{k-1}\left(\frac{t - \alpha}{2}\right) \quad \text{for } \alpha \leqq t \leqq \beta$$

provided that $\beta - \alpha < 2\pi$. Observe that, in this example, $u_k(t)$ behaves like

$(t-\alpha)^{k-1}$ for t near α. These $u_k(t)$ are more convenient than the usual set $\{1, \cos t, \sin t, \ldots, \cos rt, \sin rt\}$ because of this fact an the fact that "polynomials" are more easily factored. In particular,

$$\tilde{u}_j(t) = \tilde{u}_j(t; t_1, t_2, \ldots, t_{j-1}) = u_j(t) + \sum_1^{j-1} a_i u_i(t)$$

defined by requiring $\tilde{u}_j(t_1) = \tilde{u}_j(t_2) = \ldots = \tilde{u}_j(t_{j-1}) = 0$ can be represented by

$$(3.3)\quad \tilde{u}_j(t) = \sin^{2r+1-j}\left(\frac{\beta-t}{2}\right)\sin^{j-1}\left(\frac{\beta-\alpha}{2}\right)\prod_{i=1}^{j-1}\left(\sin\left(\frac{t-t_i}{2}\right)\Big/\sin\left(\frac{\beta-t_i}{2}\right)\right).$$

This representation holds for j both even and odd if $\beta - 2\pi < t_i < \beta$.

We need an interpolation formula based on $(u_1, \ldots, u_{m+3})$ whose remainder includes a factor agreeing in sign with $K_n(x, t)$.

For functions $g_1, \ldots, g_k$ and numbers $t_1, \ldots, t_k$, let

$$\begin{bmatrix} g_1 & g_2 \ldots g_k \\ t_1 & t_2 \ldots t_k \end{bmatrix}$$

denote the determinant of the $k \times k$ matrix whose (i, j) entry is $g_i(t_j)$. The notation

$$\begin{bmatrix} g_1 \ldots f \ldots g_k \\ t_1 \ldots t_i \ldots t_k \end{bmatrix}$$

will mean that the function f has replaced the function g_i in defining the ith row. We often suppose that $t_1 \leqq \ldots \leqq t_k$. When ratios of such determinants are encountered with, for example, $t_{j-1} < t_j = t_{j+1} = \ldots = t_{j+k}$ we replace the $j + k$ column in both numerator and denominator by $D_k \ldots D_1 g_i(t_j)$. This is consistent with an evaluation by l'Hospital's rule.

The following statement is the analog of Lemma 2.1.

LEMMA 3.1. *Let $(u_1, \ldots, u_m)$ be a fundamental principal system on $[\alpha, \beta]$ for $L_m y = 0$, and $[a, b] \subset [\alpha, \beta]$. If $\{A_n\}$ is a sequence of integral operators on $L^p[a, b]$ for which $||A_n[u_k : x] - u_k(x)||_{L^p} \to 0$ for $k = 1, \ldots, m$, then $||A_n[p_n : x] - p_n(x)||_{L^p} \to 0$ uniformly for all*

$$p_n(t) = p_{x,n}(t) = \sum_{k=1}^m a_k(x, n) u_k(t), \quad ||a_k(x, n)||_{L^\infty} \leqslant M < +\infty.$$

The proof follows exactly as the proof of Lemma 2.1.

The next lemma will be an induction step in a proof that the coefficients $a_k(x, n)$ in the interpolation formula are bounded independently of x and n.

LEMMA 3.2. *Let $(u_1, \ldots, u_m)$ be a fundamental principal system on $[\alpha, \beta]$ for $L_m y = 0$. Let $f \in C^{r-1}[\alpha, \beta]$ with $1 \leqq r \leqq m$, and set $g = D_1 f$ and $v_i = D_1 u_{i+1}$ for $i = 1, 2, \ldots, m - 1$. Let $\alpha < t_1 \leqq t_2 \leqq \ldots \leqq t_r < \beta$, and i be fixed. Then*

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there exist numbers $\eta_j = \eta_{ij}$ *for* $j = 1, \ldots, r-1$ *such that* $t_j \leqq \eta_{ij} \leqq t_{j+1}$ *and, if* $2 \leqq i \leqq r$,

$$\begin{bmatrix} u_1 \ldots f \ldots u_r \\ t_1 \ldots t_i \ldots t_r \end{bmatrix} \Big/ \begin{bmatrix} u_1 \ldots u_r \\ t_1 \ldots t_r \end{bmatrix} = \begin{bmatrix} v_1 \ldots g \quad \ldots v_{r-1} \\ \eta_1 \ldots \eta_{i-1} \ldots \eta_{r-1} \end{bmatrix} \Big/ \begin{bmatrix} v_1 \ldots v_{r-1} \\ \eta_1 \ldots \eta_{r-1} \end{bmatrix}$$

while, if $i = 1$,

$$\begin{bmatrix} f\, u_2 \ldots u_r \\ t_1\, t_2 \ldots t_r \end{bmatrix} \Big/ \begin{bmatrix} u_1 \ldots u_r \\ t_1 \ldots t_r \end{bmatrix} = (f/u_1)(t_1) - \sum_{k=1}^{r-1} \left[\begin{bmatrix} v_1 \ldots g \ldots v_{r-1} \\ \eta_1 \ldots \eta_k \ldots \eta_{r-1} \end{bmatrix} \Big/ \begin{bmatrix} v_1 \ldots v_{r-1} \\ \eta_1 \ldots \eta_{r-1} \end{bmatrix} \right] (u_k/u_1)(t_1).$$

If α *is not a singular point for* L_m, $t_1 = \alpha$ *is permitted in these statements.*

In the right members above, the relations $\eta_{s-1} < \eta_s = \ldots = \eta_{s+k}$ require that $D_{k+1} \ldots D_2$ be applied to the $s + k$ column in both numerator and denominator. Note that $(v_1, \ldots, v_{m-1})$ is a fundamental principal system on $[\alpha, \beta]$ for $D_m \ldots D_3 D_2$.

Proof. Because of continuity, it suffices to assume that $t_1 < t_2 < \ldots < t_r$. In this case,

$$\begin{bmatrix} u_1 \ldots f \ldots u_r \\ t_1 \ldots t_i \ldots t_r \end{bmatrix} \Big/ \begin{bmatrix} u_1 \ldots u_r \\ t_1 \ldots t_r \end{bmatrix} = \begin{bmatrix} 1 & u_2/u_1 \ldots f/u_1 \ldots u_r/u_1 \\ t_1 & t_2 \quad \ldots t_i \quad \ldots t_r \end{bmatrix} \Big/ \begin{bmatrix} 1 & u_2/u_1 \ldots u_r/u_1 \\ t_1 & t_2 \quad \ldots t_r \end{bmatrix}$$

Upon setting the numerator equal to $(G/u_1)(t_r) = G(t_r)/u_1(t_r)$ and the denominator equal to $(H/u_1)(t_r)$, this ratio becomes

$$\frac{(G/u_1)(t_r)}{(H/u_1)(t_r)} = \frac{(G/u_1)(t_r) - (G/u_1)(t_{r-1})}{(H/u_1)(t_r) - (H/u_1)(t_{r-1})} = \frac{D_1 G(\eta_{r-1})}{D_1 H(\eta_{r-1})}$$

with $t_{r-1} < \eta_{r-1} < t_r$ by Cauchy's extended theorem of the mean.

Since t_r appears only in the last column of G/u_1, D_1G differs from G/u_1 only in the last column, which is differentiated. By continuing in this way, columns 2 through r of both the numerator and denominator may be replaced by derivatives. If $i > 1$, each of these columns is headed by a zero and the determinant orders may be reduced. If $i = 1$, then expansion on the first column of the numerator produces the desired expression.

LEMMA 3.3. *Let* $(u_1, \ldots, u_m)$ *be a fundamental principal system on* $[\alpha, \beta]$ *for* $L_m y = 0$. *Let* $[a, b] \subset (\alpha, \beta)$ *and* $f \in C^{r-1}[a, b]$ *with* $1 \leqq r \leqq m$. *If* $a \leqq t_1 \leqq \ldots \leqq t_r \leqq b$ *and* $1 \leqq i \leqq r \leqq m$, *then*

$$\left| \begin{bmatrix} u_1 \ldots f \ldots u_r \\ t_1 \ldots t_i \ldots t_r \end{bmatrix} \Big/ \begin{bmatrix} u_1 \ldots u_r \\ t_1 \ldots t_r \end{bmatrix} \right| \leqq M < \infty$$

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independent of the t_j. If α is not a singular point for L_m, a may be set equal to α in these statements.

Proof. This follows from the previous lemma by induction on r since $f, u_1, \ldots, u_{m-1}$ are sufficiently smooth on $[a, b]$. Indeed, the induction process gives the quotient as a linear combination of generalized derivatives of the form

$$(3.4) \quad \mathscr{D}^{k-1}f(\eta) \equiv [1/\xi_k(\eta)]D_{k-1}\ldots D_1 f(\eta).$$

Since the ξ_j are strictly positive on $[a, b]$, the coefficients $\alpha_{ik}(\eta)$ in

$$(3.5) \quad \mathscr{D}^{i-1}f(\eta) = \sum_{k=1}^{i} \alpha_{ik}(\eta)D^{k-1}f(\eta)$$

are bounded uniformly and independently of f. These latter facts will be useful in section 5.

For disconjugate equations, the following theorem, similar to Theorem 2.1, holds:

THEOREM 3.1. *Let $L_{m+3}y = 0$ be disconjugate on $[\alpha, \beta]$ and let $[a, b] \subset (\alpha, \beta)$ be a fixed interval. Suppose that the sequence of integral operators $\{A_n\}$ on $L^p[a, b]$, $1 \leqq p < \infty$ satisfy:*

$$(1) \quad A_n[f:x] = \int_a^b K_n(x, t)f(t)dt$$

is of class $\mathscr{S}_m$ for each n;

(2) $||A_n||_{L^p} \leqq M < +\infty$; *and*

(3) $||A_n[u_i : x] - u_i(x)||_{L^p} \to 0$ *as* $n \to \infty$ *for* $i = 1, \ldots, m + 3$, *where* $(u_1, \ldots, u_{m+3})$ *is a fundamental principal system on* $[\alpha, \beta]$ *for* L_{m+3}. *Then, for each* $f \in L^p[a, b]$, *there holds*

$$||A_n[f : x] - f(x)||_{L^p} \to 0, \qquad n \to \infty.$$

Moreover, if α is not a singular point for L_{m+3}, then $a = \alpha$ is permitted in these statements.

Proof. Let $f \in C^{m+2}[a, b]$ and let x be such that $K_n(x, t)$ has the sign change property for each n. For fixed n, let $t_j = t_{j,x,n}$, for $1 \leqq j \leqq r = r(x, n) \leqq m$ be as in the proof of Theorem 2.1. If x equals some $t_{j,x,n}$, set $l = r - 1$. Otherwise, set $l = r + 1$ and $t_{r+1,x,n} = t_{r+2,x,n} = x$.

Consider

$$(3.6) \quad R(t) = \begin{bmatrix} u_1 \ldots u_{l+1} & f \\ t_1 \ldots t_{l+1} & t \end{bmatrix} \Big/ \begin{bmatrix} u_1 \ldots u_{l+1} \\ t_1 \ldots t_{l+1} \end{bmatrix}.$$

When expanded, the right member of (3.6) indicates a formula for interpolating f at $t_1, \ldots, t_{l+1}$ by a "generalized polynomial" of "degree" l.

On the one hand

$$R(t) = \left[\left[\begin{matrix} u_1 \ldots u_{l+1} & f \\ t_1 \ldots t_{l+1} & t \end{matrix}\right] \Big/ \left[\begin{matrix} u_1 \ldots u_{l+1} & u_{l+2} \\ t_1 \ldots t_{l+1} & t \end{matrix}\right]\right]$$
$$\times \left[\left[\begin{matrix} u_1 \ldots u_{l+1} & u_{l+2} \\ t_1 \ldots t_{l+1} & t \end{matrix}\right] \Big/ \left[\begin{matrix} u_1 \ldots u_{l+1} \\ t_1 \ldots t_{l+1} \end{matrix}\right]\right] = g_{x,n}(t) q_{x,n}(t)$$

where $g_{x,n}(t)$ is bounded independently of x, n, t, and the t_j by Lemma 3.3 with m replaced by $m + 3$. Moreover, since $D_{l+2}D_{l+1} \ldots D_1 y = 0$ is disconjugate on $[\alpha, \beta]$, $q_{x,n}(t)$ or $-q_{x,n}(t)$ agrees in sign with $K_n(x, t)$.

On the other hand, $R(t) = f(t) - p_{x,n}(t)$ where

$$\text{(3.7)} \quad p_{x,n}(t) = \sum_{k=1}^{l+1} u_k(t) \left[\begin{matrix} u_1 \ldots f \ldots u_{l+1} \\ t_1 \ldots t_k \ldots t_{l+1} \end{matrix}\right] \Big/ \left[\begin{matrix} u_1 \ldots u_{l+1} \\ t_1 \ldots t_{l+1} \end{matrix}\right]$$
$$= \sum_{k=1}^{l+2} a_k(x, n) u_k(t)$$

where, again by Lemma 3.3, the $a_k(x, n)$ are uniformly bounded.

Finally,

$$q_{x,n}(t) = u_{l+2}(t) - \sum_{k=1}^{l+1} u_k(t) \left[\begin{matrix} u_1 \ldots u_{l+2} \ldots u_{l+1} \\ t_1 \ldots t_k \ldots t_{l+1} \end{matrix}\right] \Big/ \left[\begin{matrix} u_1 \ldots u_{l+1} \\ t_1 \ldots t_{l+1} \end{matrix}\right]$$
$$= \sum_{k=1}^{l+2} b_k(x, n) u_k(t)$$

where, again by Lemma 3.3, the $b_k(x, n)$ are uniformly bounded.

Now, since $f(x) = \sum_{k=1}^{l+1} a_k(x, n) u_k(x)$, we have

$$A_n[f : x] - f(x) = \sum_{k=1}^{l+1} a_k(x, n) \{A_n[u_k : x] - u_k(x)\} + A_n[g_{x,n} q_{x,n} : x].$$

By Remark 2.1, we have

$$||A_n[f : x] - f(x)||_{L^p} \leqq M_1$$
$$\times \sum_{k=1}^{m+2} ||A_n[u_k : x] - u_k(x)||_{L^p} + M_2 ||A_n[q_{x,n} : x]||_{L^p}$$

where M_1 and M_2 are bounds on $|a_k(x, n)|$ and $|g_{x,n}(t)|$ respectively. Since $q_{x,n}(x) = \sum_{k=1}^{l+2} b_k(x, n) u_k(x) = 0$,

$$A_n[q_{x,n} : x] = \sum_{k=1}^{l+2} b_k(x, n) \{A_n[u_k : x] - u_k(x)\}$$

so that

$$||A_n[q_{x,n} : x]||_{L^p} \leqq M_3 \sum_{k=1}^{m+3} ||A_n[u_k : x] - u_k(x)||_{L^p}$$

where M_3 is a bound on $|b_k(x, n)|$. The proof of the theorem is complete.

Remark 3.1. As a corollary of the above proof, we have that if x is one of the $t_{j,x,n}$ for almost all x and all n, then u_{m+2} and u_{m+3} may be deleted from hypothesis (3).

The discussion in this section could have been framed in terms of extended complete Chebyshev systems (see Karlin and Studden [**7**]). We have preferred the language of disconjugacy because it emphasizes the primary role of the operator L_m and because of the applicability to trigonometric polynomials (see the next section).

4. Trigonometric polynomials. In this section, we consider the space $L^p[0, 2\pi]$ of 2π-periodic functions. To account for periodicity, the definition of the class $\mathscr{S}_m$ is modified to include a possible sign change at $t = 2\pi$ $(t = 0)$. The modification is easily implemented by requiring that m be even.

Let $m \geqq 0$ and r be integers such that

(4.1) $\quad 2r \geqq m + 2.$

Consider the set $K = \{1, \sin t, \cos t, \ldots, \sin rt, \cos rt\}$. We now show the following theorem.

THEOREM 4.1. *Let $r, m \geqq 0$ and K be as above. Then K is a Korovkin set for integral operators of class $\mathscr{S}_m$ on $L^p[0, 2\pi]$, $1 \leqq p < \infty$. Further, for those integral operators of class $\mathscr{S}_m$ on $L^p[0, 2\pi]$ whose kernels satisfy: the kernel $K(x, t)$ changes sign at $t = x$* (i.e., x is one of the $t_{j,x,n}$) *for almost all x, then K is still a Korovkin set when* (4.1) *is replaced by*

(4.2) $\quad 2r \geqq m.$

We cannot directly use the results of section 3 because the differential equation (3.1) is not disconjugate on an interval $[\alpha, \beta]$ for which $[0, 2\pi] \subset [\alpha, \beta)$. Since the expressions

(4.3) $\quad (1/\xi_{k+1}(t))D_k \ldots D_1 y(t)$

are non-singular at $t = \beta$ only for $k = 2r$ and $k = 2r + 1$, Lemma 3.3 cannot be extended to the full interval $[\beta - 2\pi, \beta]$. Here the ξ_i, $i = 1, \ldots, 2r + 1$ are given by (3.2).

The device we shall use is to select α, β and $r' \leqq r$ after x and $\{t_{j,x,n} : j = 1, 2, \ldots, 2l'\}$ have been specified. Then $\xi_1(t)$ in (4.3) is replaced by

$$\xi_1(t) = \sin^{2r'}\left(\frac{\beta - t}{2}\right).$$

By selecting $\beta = \beta(x)$ and $\alpha = \beta - 2\pi + \delta$ so that $\alpha + \delta \leqq x \leqq \beta - \delta$ and each of the $t_{j,x,n}$ are more than δ away from $\alpha, \beta, \alpha \pm 2\pi$, and $\beta \pm 2\pi$, and then invoking periodicity to translate to the interval $[\beta - 2\pi, \beta]$, we will obtain the interpolation formula

(4.4) $\quad f(t) = p_{x,n}(t) + g_{x,n}(t)q_{x,n}(t)$

for periodic functions f having $2r'$ continuous derivatives. Here

$$p_{x,n}(t) = \sum_{k=1}^{l+1} a_k(x, n)u_{k,x,n}(t)$$

$$q_{x,n}(t) = \sum_{k=1}^{l+2} b_k(x, n)u_{k,x,n}(t),$$

and

$$\begin{aligned} g_{x,n}(t) &= \begin{bmatrix} u_1 \dots u_{l+1} & f \\ t_1 \dots t_{l+1} & t \end{bmatrix} \Big/ \begin{bmatrix} u_1 \dots u_{l+1} & u_{l+2} \\ t_1 \dots t_{l+1} & t \end{bmatrix} \\ &= (1/\xi_{l+2}(\eta))D_{l+1} \dots D_1 f(\eta) \qquad (\beta - 2\pi < \eta < \beta) \end{aligned}$$

with $\alpha + \delta \leqq t_j = t_{j,x,n}(\pm 2\pi) \leqq \beta - \delta$ and $u_j(t) = u_{j,x,n}(t)$ given by

$$u_j(t) = \xi_1(t) \int_\alpha^t \xi_2(\tau) \dots \int_\alpha^\tau \xi_j(\tau)(d\tau)^{j-1}.$$

Here, also, $l = 2l' - 1$ if x equals some $t_{j,x,n}$ and $l = 2l' + 1$, $t_l = t_{l+1} = x$ if x does not equal some $t_{j,x,n}$. As before, $p_{x,n}(t_j) = f(t_j)$, $q_{x,n}(t_j) = 0$, and, if $l = 2l' + 1$, $Dp_{x,n}(x) = Df(x)$.

With $\delta < 2\pi/3m$, the above construction is possible for almost all x. Since $\alpha + \delta \leqq t_j \leqq \beta - \delta$, Lemma 3.3 assures that $|a_k(x, n)| \leqq M_1$ and $|b_k(x, n)| \leqq M_3$. To insure that $|g_{x,n}(t)| \leqq M_2$, we must choose $2r' = l + 1$ (see (4.3) and the lines that follow it). Since $2l' \leqq m$, we may do so with r satisfying (4.1) or (4.2). Here M_1 and M_2 depend on f and perhaps on its first $2r + 1$ derivatives. This dependence will be explored more deeply in section 5 when we consider constraints on the $t_{j,x,n}$ as $n \to \infty$.

Finally,

$$u_{k,x,n}(t) = \sum_{j=0}^{r'} [c_{j,k}(x, n) \cos jt + d_{j,k}(x, n) \sin jt]$$

with $|c_{j,k}(x, n)| \leqq M_4$ and $|d_{j,k}(x, n)| \leqq M_4$ independently of f and r'. Thus, as in the manner of section 3,

$$\begin{aligned} ||A_n[f : x] - f(x)||_{L^p} &\leqq (M_1 + M_2M_3) \sum_{j=1}^{2r+1} ||A_n[u_{j,x,n} : x] - u_{j,x,n}||_{L^p} \\ &\leqq (M_1 + M_2M_3)(2r + 1)M_4 \\ &\quad \times \sum_{j=0}^{r} \{||A_n[\cos jt : x] - \cos jx||_{L^p} \\ &\qquad + ||A_n[\sin jt : x] - \sin jx||_{L^p}\} \end{aligned}$$

and Theorem 4.1 is proved.

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5. Quantitative results. We now consider the question of rate of convergence to zero of $||A_n[f:x] - f(x)||_{L^p}$ when the operators A_n are of class $\mathscr{S}_m$ and are polynomial valued.

There are two cases. The first case is when A_n is an nth degree algebraic polynomial valued operator on $L^p[a, b]$ and $K_1 = \{1, t, \ldots, t^{m+2}\}$. The second is when A_n is an nth degree trigonometric polynomial valued operator on $L^p[0, 2\pi]$ and $K_2 = \{1, \sin t, \cos t, \ldots \sin rt, \cos rt\}$ $(m + 3 = 2r + 1)$. We shall show that there exist functions $f \in C^{m+2}$ such that

$$\limsup_{n\to\infty} n^{m+2}||A_n[f:x] - f(x)||_{L^p} > 0.$$

This question was studied extensively for convolution kernels by Butzer, Nessel and Scherer [**3**].

If we know more about the structure of the kernel, then we can improve the result. To this end, we make the following definition.

Definition. Let $\{A_n\}$ be a sequence of integral operators of class $\mathscr{S}_m$. The *essential number of sign changes* for the sequence $\{A_n\}$ is the smallest j_0 for which there exist $\delta > 0$ and n_0 such that for each x (except possibly a set of measure zero) and each $n \geqq n_0$ at most j_0 of the $t_{j,x,n}$ lie in any interval of length δ. For the periodic case, we consider intervals of length δ on the circle.

The notion of essential number of sign changes means that at most j_0 of the $t_{j,x,n}$ cluster at some point in the interval. A more natural condition would be that at most j_0 of the $t_{j,x,n}$ cluster at x (see for example, Butzer, Nessel and Scherer [**3**]).

LEMMA 5.1. *Let $L_{m+3}y = 0$ be disconjugate on $[\alpha, \beta]$ with $(u_1, \ldots, u_{m+3})$ as the corresponding fundamental principal system. Let $[a, b] \subset (\alpha, \beta)$ and let $\{A_n\}$ satisfy the conditions of Theorem* 5.1. *Then*

$$||A_n[f:x] - f(x)||_{L^p} \leqq M\{||f||_\infty + ||f^{(j_0+2)}||_\infty\} \times \sum_{k=1}^{m+3} ||A_n[u_k:x] - u_k(x)||_{L^p}$$

for all $f \in C^{j_0+2}$ where M is independent of f. Moreover, if x equals some $t_{j,x,n}$ for almost all x and all n, then

$$||A_n[f:x] - f(x)||_{L^p} \leqq M\{||f||_\infty + ||f^{(j_0)}||\} \sum_{k=1}^{m+3} ||A_n[u:x] - u_k(x)||_{L^p}$$

for $f \in C^{j_0}$.

Lemma 5.1 will apply directly to K_1 and $L^p[a, b]$ if we set $\alpha = 0$ and $\beta = \infty$. With the modifications described in section 4 (see (4.3) and the lines which follow it), the conclusions of Lemma 5.1 will apply to K_2 and $L^p[0, 2\pi]$. In particular, j_0 may be even or odd.

Note that in the case $j_0 = m$, this lemma follows from a careful look at our proofs in section 3 since the coefficients there are bounded by the derivatives of f. For the general case, we need the coefficients given by certain "generalized" divided differences. We begin by giving an alternate form for the interpolating "polynomials" $p_{x,n}(t)$ defined in (3.7). As in section 3, we suppose that $L_{m+3}y = 0$ is disconjugate on $[\alpha, \beta]$ and consider $x, t, t_{j,x,n} \in [a, b] \subset (\alpha, \beta)$.

Let the jth order (generalized) divided difference $f[t_1, \ldots, t_j]$ be specified by

$$f[t_1, \ldots, t_j] \equiv \text{(coefficient of } u_j(t) \text{ in the generalized polynomial } u(t) \text{ of order } j \text{ which interpolates } f(t) \text{ at } t_1, \ldots, t_j).$$

Here $D^i u(t_k) = D^i f(t_k)$ if t_k occurs with multiplicity $i + 1$ or more. The unique existence of $u(t)$ is guaranteed by the disconjugacy of $D_j \ldots D_1 y = 0$. Indeed,

$$u(t) = f(t) - \begin{bmatrix} u_1 \ldots u_j f \\ t_1 \ldots t_j\, t \end{bmatrix} \Big/ \begin{bmatrix} u_1 \ldots u_j \\ t_1 \ldots t_j \end{bmatrix}$$

so that

$$(5.1)\quad f[t_1, \ldots, t_j] = \begin{bmatrix} u_1 \ldots u_{j-1} & f \\ t_1 \ldots t_{j-1} & t_j \end{bmatrix} \Big/ \begin{bmatrix} u_1 \ldots u_j \\ t_1 \ldots t_j \end{bmatrix}.$$

For $f \in C^{k-1}(\alpha, \beta)$, Lemma 3.2 yields

$$(5.2)\quad f[t_1, \ldots, t_k] = \mathscr{D}^{k-1} f(\eta)$$

where $\mathscr{D}^{k-1}$ is as in (3.4) (see also DeVore and Richards [**4**]). For the set K_1, $\mathscr{D}^k y = D^k y / k!$. For K_2, $\mathscr{D}^k y$ is given by (4.3) and (3.2).

With the Newton polynomials $\tilde{u}_i(t) = \tilde{u}_i(t; t_1, \ldots, t_{i-1})$ specified by

$$\tilde{u}_i(t) = \begin{bmatrix} u_1 \ldots u_{i-1} & u_i \\ t_1 \ldots t_{i-1} & t \end{bmatrix} \Big/ \begin{bmatrix} u_1 \ldots u_{i-1} \\ t_1 \ldots t_{i-1} \end{bmatrix},$$

we have

$$(5.3)\quad u(t) = \sum_{i=1}^{j} f[t_1, \ldots, t_i] \tilde{u}_i(t).$$

We shall use (5.3) with $j = l + 1$, $t_j = t_{j,x,n}$, and, possibly $t_l = t_{l+1} = x$ (notation established in the proof of Theorem 3.1) to represent $u(t) = p_{x,n}(t)$ as given by (3.7). Note that the coefficients of the u_j in $\tilde{u}_i(t)$ are bounded independently of f (see Lemma 3.3). From (3.6),

$$(5.4)\quad R(t) = g_{x,n}(t) q_{x,n}(t) \equiv f[t_1, \ldots, t_{l+1}, t] \tilde{u}_{l+2}(t)$$

with $l + 1 \leqq m + 2$. Thus, to establish Lemma 5.1, we must show that

$$(5.5)\quad |f[t_1, \ldots, t_i]| \leqq M[\|f\|_\infty + \|f^{(j_0+2)}\|_\infty]$$

for $i = 1, \ldots, m + 3$.

If $i \leqq j_0 + 3$, then (5.5) follows from (5.2), (3.5) and the well-known relation $||f^{(i-1)}||_\infty \leqq M_i[||f||_\infty + ||f^{(j_0+2)}||_\infty]$ (see, for example, Ditzian and Freud [**5**, Lemma 3.2]).

If $i > j_0 + 3$, we use the recurrence relation

$$(5.6)\quad f[t_1, \ldots, t_j] = [\tilde{u}_{j-1}(t_j)/\tilde{u}_j(t_j)]\{f[t_1, \ldots, t_{j-2}, t_j] - f[t_1, \ldots, t_{j-1}]\}$$

to replace $f[t_1, \ldots, t_j]$ by a linear combination of $(j_0 + 3)$-order divided differences. Relation (5.6) is easily proved by interchanging t_{j-1} and t_j in the discussion preceding (5.3). The coefficients $\tilde{u}_{j-1}(t_j)/\tilde{u}_j(t_j)$ are bounded if and only if $t_{j-1} - t_j$ is bounded away from zero. Since the essential number of sign changes is j_0, the t_j may be ordered so that $|t_{j-1} - t_j| > \delta > 0$. This fact together with the result for $i \leqq j_0 + 3$ establishes (5.5).

By combining the above results, we have Lemma 5.1 proved once we remark that $j_0 + 2$ may be replaced by j_0 in the above discussion if x equals some $t_{j,x,n}$ for almost all x and all n.

We note that for the set K_1, the coefficient in (5.6) is $\tilde{u}_{j-1}(t_j)/\tilde{u}_j(t_j) = 1/(t_j - t_{j-1})$ because of cancellation. For the set K_2, we use the alternative basis given by (3.3). Then we have

$$\tilde{u}_{j-1}(t_j)/\tilde{u}_j(t_j) = \frac{(\sin \frac{1}{2}(\beta - t_{j-1}))(\sin \frac{1}{2}(\beta - t_j))}{(\sin \frac{1}{2}(\beta - \alpha))(\sin \frac{1}{2}(t_j - t_{j-1}))},$$

which shows why we require intervals of length δ on the circle in the periodic case.

THEOREM 5.1. *Let A_n be nth degree algebraic (trigonometric) polynomial valued integral operators of class $\mathscr{S}_m$ on $L^p[a, b]$* (respectively, $L^p[0, 2\pi]$) *with j_0 essential sign changes. Then at least one of the sequences*

$$n^{j_0+2}||A_n[f : x] - f(x)||_{L^p}, \qquad f \in K_1 \text{ (respectively, } f \in K_2)$$

does not converge to zero. In addition, if x equals some $t_{j,x,n}$ for almost all x and all n large, then n^{j_0+2} may be replaced by n^{j_0} in the assertion.

Proof. We first consider K_1 and algebraic polynomial valued operators. If the theorem were not true, then by Lemma 5.1,

$$n^{j_0+2}||A_n[f:x] - f(x)||_{L^p} \leqq M\{||f||_\infty + ||f^{(j_0+2)}||_\infty\}n^{j_0+2} \times \sum_{k=0}^{m+2} ||A_n[t^k : x] - x^k||_{L^p} \to 0$$

as $n \to \infty$. To obtain a contradiction, we obtain $f_n \in C^{j_0+2}$ for which $||f_n||_\infty \leqq 1$, $||f_n^{(j_0+2)}||_\infty \leqq 1$ and $||A_n[f_n : x] - f_n(x)||_{L^1} \geqq C/n^{j_0+2}$ for sufficiently large n, where C is an absolute constant. The following simple construction is due to A. Meir.

Consider the nth degree Chebshev polynomial on $[-1, +1]$ $T_n(x) = x^n + p(x)$ ($p(x)$ lower order terms) (see Achieser [**1**]). Then $||T_n||_\infty = 2^{-n+1}$, so we select a polynomial $Q_n(x)$ so that $Q_n^{(j_0+2)}(x) = 2^{n-1}T_n(x)$ and

$||Q_n(x)|| \leqq 1$. For any algebraic polynomial $P(x)$ of degree at most n, we have

$$\int_{-1}^{1} |Q_n(x) - P(x)| = 2^{n-1}[(n + j_0 + 2) \ldots (n + 1)]^{-1} \times \int_{-1}^{1} |x^{n+j_0+2} - \tilde{P}(x)| dx$$

where $\tilde{P}(x)$ is a polynomial of degree $n + j_0 + 1$. By a well-known result (Achieser [**1**, p. 94]), we obtain

$$\int_{-1}^{1} |Q_n(x) - P(x)| dx \geqslant 2^{-j_0-2}(n + j_0 + 2)^{-j_0-2}.$$

A simple change of variable transforms Q_n to f_n on $[a, b]$.

For the trigonometric case, a similar argument utilizes the fact that zero is a best L^p-approximation $(1 \leqq p < \infty)$ to $\cos(j_0 + n + 2)x$ by trigonometric polynomials of degree $j_0 + n + 1$ (see Shapiro [**10**, p. 57]). The proof is complete.

6. Remarks. The above theorems may be extended to a wider class of operators; namely, those operators belonging to the operator norm closure of the integral operators of class $\mathscr{S}_m$. This would permit certain differences to be considered; e.g., $L^p[0, 2\pi]$, $Af(x) = f(x + \delta) + f(x - \delta) - f(x)$ would be in the extended class $\mathscr{S}_2$.

A careful look at the proofs indicates that most do not depend on the L^p norm per se. In fact, we only require that (i) the C^{m+2} functions be dense in our space, and (ii) that $|f| \leqq |g|$ implies $||f|| \leqq C||g||$. Thus, the theorems are valid for any Banach space of measurable functions which enjoy these properties.

The discussion in section 3 allows infinite intervals to be included. Since functions in C^{m+2} with compact support are dense in the L^p-spaces on an infinite interval, the bounds on the coefficients of the interpolating polynomials depend on the generalized derivatives (3.4) evaluated only on a compact set (depending on f) which avoids the singularities. Thus, the arguments may be carried out. As an example, on $[0, \infty)$, the equation

$$L_{m+3}y \equiv (D + 1) \ldots (D + m + 3)y = 0$$

has the fundamental principal system $(e^{-x}, e^{-2x}, \ldots, e^{-(m+3)x})$. Consequently, the set $K = \{e^{-x}, \ldots, e^{-(m+3)x}\}$ is a Korovkin set for $\mathscr{S}_m$ on $L^p[0, \infty)$ $1 \leqq p < \infty$.

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