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HOMOMORPHISMS OF NEAR-RINGS OF CONTINUOUS REAL-VALUED FUNCTIONS

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For any topological near-ring (which is not a ring) whose additive group is the additive group of real numbers, we investigate the near-ring of all continuous functions, under the pointwise operations, from a compact Hausdorff space into that near-ring. Specifically, we determine all the homomorphisms from one such near-ring of functions to another and we show that within a rather extensive class of spaces, the endomorphism semigroup of the near-ring of functions completely determines the topological structure of the space.

1. INTRODUCTION

For information about near-rings and in particular, for any algebraic terms not defined here, one may consult [**1**, **6**, **7**]. Let $(R,+)$ denote the additive topological group of real numbers. In [4], we determined all the binary operations $*$ on R such that $(R,+,*)$ is a topological near-ring. Specifically, we showed that if $a \leqslant 0$, $b \geqslant 0$ and $a^2+b^2 \neq 0$ and we define

$$x * y = \begin{cases} axy & \text{for } y \leqslant 0 \\ bxy & \text{for } y > 0 \end{cases} \tag{1.1}$$

then $(R,+,*)$ is a topological near-ring which is not a ring and every binary operation for which $(R,+,*)$ is a topological near-ring which is not a ring is of this form. We shall subsequently denote the near-ring with multiplication defined as in (1.1) by $R_{a,b}$. Now let X be a compact Hausdorff space and denote by $N_{a,b}(X)$ the near-ring of all continuous maps from X to R where $f+g$ and fg are defined by $(f+g)(x) = f(x)+g(x)$ and $(fg)(x) = (f(x)) * (g(x))$ where the operation $*$ is defined as in (1.1). In Section 2, we investigate homomorphisms of $N_{a,b}(X)$ into $N_{a,b}(Y)$ and we determine the automorphism group of the near-ring $N_{a,b}(X)$. We also determine the endomorphism semigroup of $N_{a,b}(X)$ when X is a continuum. Here, a continuum is a compact connected Hausdorff space. Furthermore, we show that there is an extensive class of continua such that within this class, the endomorphism semigroup of the near-ring $N_{a,b}(X)$ completely determines the topological structure of the space X. Specifically, we show that if X and Y are continua which contain arcs, then the endomorphism semigroups of the near-rings $N_{a,b}(X)$ and $N_{a,b}(Y)$ are isomorphic if and only if the spaces X and Y are homeomorphic.

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2. Homomorphisms from $N_{a,b}(X)$ into $N_{a,b}(Y)$

A few of the results in this section are somewhat analogous to some of the results in [**2**, Chapter 10]. As is customary, a homomorphism from one topological near-ring to another will be assumed to be continuous as well.

Theorem 2.1. *Let φ be a nonzero endomorphism of $R_{a,b}$. Then $\varphi(r) = r$ for all $r \in R_{a,b}$ if $a \neq -b$. If $a = -b$, then either $\varphi(r) = r$ for all $r \in R_{a,b}$ or $\varphi(r) = -r$ for all $r \in R_{a,b}$.*

Proof: Let φ be a nonzero endomorphism of $R_{a,b}$. According to Theorem (3.14) of [**4**], φ is a linear map from $R_{a,b}$ onto $R_{a,b}$ with the property that $f = f \circ \varphi$ where

$$f(x) = \begin{cases} ax & \text{for } x \leqslant 0 \\ bx & \text{for } x > 0. \end{cases}$$

Since φ is a nonzero endomorphism, there exists a real number $c \neq 0$ such that $\varphi(x) = cx$. Suppose $c > 0$. Then

$$-a = f(-1) = f \circ \varphi(-1) = f(-c) = -ac$$

and

$$b = f(1) = f \circ \varphi(1) = f(c) = bc.$$

Since not both a and b can be zero, it readily follows that $c = 1$ so that, in this case, φ must be the identity map. Now suppose $c < 0$. In this case, we have

$$-a = f(-1) = f \circ \varphi(-1) = f(-c) = -bc$$

and

$$b = f(1) = f \circ \varphi(1) = f(c) = ac.$$

Thus, we have $a = bc$ and $b = ac$ which imply $a = ac^2$ and $b = bc^2$. Since not both a and b can be zero, we conclude that $c^2 = 1$ and since $c < 0$, we must have $c = -1$. Finally, if $c = -1$, it follows that $a = -b$. Consequently, if $a \neq -b$, then the only nonzero endomorphism of $R_{a,b}$ is the identity map. ∎

Throughout the remainder of this paper, it will be assumed that the spaces under consideration will all be compact Hausdorff spaces and the terms *space* and *topological space* will mean *compact Hausdorff space*. The symbol $\langle x \rangle$ will be used to denote the constant function which maps everything into the point x. The domain of the function will be clear from the context.

Lemma 2.2. *Suppose $a \neq -b$ and let φ be be a nonzero homomorphism from $N_{a,b}(X)$ into $R_{a,b}$. Then there exists a unique point $x \in X$ such that $\varphi(f) = f(x)$ for all $f \in N_{a,b}(X)$.*

Proof: Either $a \neq 0$ or $b \neq 0$ and there is no loss in generality in assuming the former. Suppose further that $\varphi\langle 1/a\rangle = 0$. Then for any $f \in N_{a,b}(X)$, we have $\varphi(f) = \varphi(f\langle 1/a\rangle) = \varphi(f) * \varphi\langle 1/a\rangle = 0$ which is a contradiction. Consequently $\varphi\langle 1/a\rangle \neq 0$. Next, define a homomorphism ψ from $R_{a,b}$ into $N_{a,b}(X)$ by $\psi(r) = \langle r\rangle$. Then $\varphi \circ \psi$ is an endomorphism of $R_{a,b}$ which, in view of our previous observation, is nonzero. Theorem 2.1 tells us that $\varphi \circ \psi(r) = r$ for all $x \in R_{a,b}$, which implies that

$$\varphi\langle r\rangle = r \text{ for all } r \in R_{a,b}. \tag{2.2.1}$$

It follows from (2.2.1) that φ maps $N_{a,b}(X)$ onto $R_{a,b}$ and since $R_{a,b}$ is simple, Ker φ is a maximal ideal of $N_{a,b}(X)$. According to [**5**, Theorem 3.10], there exists an $x \in X$ such that Ker $\varphi = M_x = \{f \in N_{a,b}(X) : f(x) = 0\}$. For each $f \in N_{a,b}(X)$, we see that $f - \langle f(x)\rangle \in M_x$. This and (2.2.1) together imply that $0 = \varphi(f - \langle f(x)\rangle) = \varphi(f) - f(x)$ and the lemma has been verified since uniqueness is immediate. □

Theorem 2.3. *Suppose $a \neq -b$ and let φ be a homomorphism from $N_{a,b}(X)$ into $N_{a,b}(Y)$ with the property that $\varphi\langle 1\rangle = \langle 1\rangle$. Then there exists a unique continuous function h from Y into X such that $\varphi(f) = f \circ h$ for all $f \in N_{a,b}(X)$.*

Proof: For each $y \in Y$, define a homomorphism ψ_y from $N_{a,b}(X)$ into $R_{a,b}$ by $\psi_y(f) = \varphi(f)(y)$. The homomorphism ψ_y is nonzero since $\varphi\langle 1\rangle = \langle 1\rangle$ and according to Lemma 2.2, there exists a unique $x \in X$ so that $\psi_y(f) = f(x)$. We define $h(y) = x$ and we note that $\varphi(f)(y) = \psi_y(f) = f(h(y))$ for all $y \in Y$. In other words, we have $\varphi(f) = f \circ h$ and it follows from [**2**, Theorem 3.8, p.40] that h is continuous. □

Theorem 2.4. *Suppose $a \neq -b$ and let φ be an isomorphism from $N_{a,b}(X)$ onto $N_{a,b}(Y)$. Then there exists a unique homeomorphism h from Y onto X such that $\varphi(f) = f \circ h$ for each $f \in N_{a,b}(X)$.*

Proof: There exists a $g \in N_{a,b}(X)$ such that $\varphi(g) = \langle 1\rangle$. Again define, for each $y \in Y$, $\psi_y(f) = \varphi(f)(y)$. Since $\psi_y(g) = \varphi(g)(y) = \langle 1\rangle(y) = 1$, we see that ψ_y is a nonzero homomorphism from $N_{a,b}(X)$ to $R_{a,b}$ and it follows from (2.2.1) that $\varphi\langle 1\rangle(y) = \psi_y\langle 1\rangle = 1$ for all $y \in Y$. In other words, $\varphi\langle 1\rangle = \langle 1\rangle$. We now conclude from Theorem 2.3 that there exists a unique continuous map h from Y into X such that $\varphi(f) = f \circ h$ for all $f \in N_{a,b}(X)$. Similarly, there exists a unique continuous map k from X into Y such that $\varphi^{-1}(g) = g \circ k$ for all $g \in N_{a,b}(Y)$. It readily follows that both $h \circ k$ and $k \circ h$ are identity maps which means $h = k^{-1}$ and h is a homeomorphism. □

In the next result, Aut $N_{a,b}(X)$ will denote the automorphism group of $N_{a,b}(X)$ and $H(X)$ will denote the homeomorphism group of X.

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THEOREM 2.5. *Suppose $a \neq -b$. Then* Aut $N_{a,b}(X)$ *is isomorphic to* $H(X)$.

PROOF: Let $\varphi \in$ Aut $N_{a,b}(X)$. Then according to Theorem 2.4, there exists a unique homeomorphism h from X onto X so that $\varphi(f) = f \circ h$ for each $f \in N_{a,b}(X)$. One easily verifies that the mapping Φ which is defined by $\Phi(\varphi) = h^{-1}$, is an isomorphism from Aut $N_{a,b}(X)$ onto $H(X)$. ∎

The proof of the next lemma is routine and, for that reason, is omitted.

LEMMA 2.6. *Let $f \in N_{a,b}(X)$ and suppose $a \neq 0 \neq b$. Then $f^2 = f$ if and only if for each $x \in X$, we have either $f(x) = 1/a$, $f(x) = 1/b$, or $f(x) = 0$. If $a = 0$, then $f^2 = f$ if and only if for each $x \in X$, we have either $f(x) = 1/b$ or $f(x) = 0$. Similarly, if $b = 0$, then $f^2 = f$ if and only if for each $x \in X$, we have either $f(x) = 1/a$ or $f(x) = 0$.*

LEMMA 2.7. *Let φ be a homomorphism from $N_{a,b}(X)$ into $N_{a,b}(Y)$. Suppose $a \neq 0$ and $\varphi\langle 1/a\rangle(y) = 0$. Then $\varphi(f)(y) = 0$ for all $f \in N_{a,b}(X)$. Similarly, if $b \neq 0$ and $\varphi\langle 1/b\rangle(y) = 0$, then $\varphi(f)(y) = 0$ for all $f \in N_{a,b}(X)$.*

PROOF: Suppose first that $a \neq 0$ and $\varphi\langle 1/a\rangle(y) = 0$. One readily checks that if $g(x) \leqslant 0$ for all $x \in X$, then $\langle 1/a\rangle g = g$. Now, for any $f \in N_{a,b}(X)$, define

$$f_1(x) = \begin{cases} f(x) & \text{if } f(x) \leqslant 0 \\ 0 & \text{if } f(x) > 0 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 0 & \text{if } f(x) \leqslant 0 \\ -f(x) & \text{if } f(x) > 0. \end{cases}$$

Then $f_1(x) \leqslant 0$ and $f_2(x) \leqslant 0$ for all $x \in X$ and $f = f_1 - f_2$. For any $f \in N_{a,b}(X)$ and any $y \in Y$, we get

$$\begin{aligned} \varphi(f)(y) &= \varphi(f_1 - f_2)(y) = \big(\varphi(f_1) - \varphi(f_2)\big)(y) = \varphi(f_1)(y) - \varphi(f_2)(y) \\ &= \varphi(\langle 1/a\rangle f_1)(y) - \varphi(\langle 1/a\rangle f_2)(y) = \big(\varphi\langle 1/a\rangle\varphi(f_1)\big)(y) - \big(\varphi\langle 1/a\rangle\varphi(f_2)\big)(y) \\ &= \big(\varphi\langle 1/a\rangle(y) * \varphi(f_1)(y)\big) - \big(\varphi\langle 1/a\rangle(y) * \varphi(f_2)(y)\big) = 0. \end{aligned}$$

Now suppose $b \neq 0$ and $\varphi\langle 1/b\rangle(y) = 0$. It is immediate that if $g(x) \geqslant 0$ for all $x \in X$, then $\langle 1/b\rangle g = g$. This time define

$$f_1(x) = \begin{cases} f(x) & \text{if } f(x) \geqslant 0 \\ 0 & \text{if } f(x) < 0 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 0 & \text{if } f(x) \geqslant 0 \\ -f(x) & \text{if } f(x) < 0. \end{cases}$$

Then $f_1(x) \geqslant 0$ and $f_2(x) \geqslant 0$ for all $x \in X$ and again we have $f = f_1 - f_2$. For any $f \in N_{a,b}(X)$ and any $y \in Y$, we get

$$\begin{aligned} \varphi(f)(y) &= \varphi(f_1 - f_2)(y) = \big(\varphi(f_1) - \varphi(f_2)\big)(y) = \varphi(f_1)(y) - \varphi(f_2)(y) \\ &= \varphi(\langle 1/b\rangle f_1)(y) - \varphi(\langle 1/b\rangle f_2)(y) = \big(\varphi\langle 1/b\rangle\varphi(f_1)\big)(y) - \big(\varphi\langle 1/b\rangle\varphi(f_2)\big)(y) \\ &= \big(\varphi\langle 1/b\rangle(y) * \varphi(f_1)(y)\big) - \big(\varphi\langle 1/b\rangle(y) * \varphi(f_2)(y)\big) = 0. \end{aligned}$$ ∎

THEOREM 2.8. *Suppose $a \neq -b$ and let φ be a nonzero homomorphism from $N_{a,b}(X)$ into $N_{a,b}(Y)$. Then there exists a continuous function h from a nonempty clopen (simultaneously closed and open) subspace Z of Y into X such that*

$$\varphi(f)(y) = \begin{cases} f(h(y)) & \text{for } y \in Z \\ 0 & \text{for } y \in Y \backslash Z. \end{cases}$$

PROOF: We have either $a \neq 0$ or $b \neq 0$. Suppose first that $a \neq 0$. We want to show that even if $b \neq 0$, we cannot have $\varphi\langle 1/a\rangle(y) = 1/b$ for any $y \in Y$. Suppose, to the contrary, that there is a $y \in Y$ for which $\varphi\langle 1/a\rangle(y) = 1/b$ and define a homomorphism ψ_y from $N_{a,b}(X)$ into $R_{a,b}$ by $\psi_y(f) = \varphi(f)(y)$. Then ψ_y is nonzero since $\psi_y(\langle 1/a\rangle) = 1/b$ and according to Lemma 2.2, there exists an $x \in X$ such that $\psi_y(f) = f(x)$ for all $f \in N_{a,b}(X)$. From this, we get

$$1/a = \langle 1/a\rangle(y) = \psi_y(\langle 1/a\rangle) = \varphi\langle 1/a\rangle(y) = 1/b$$

which results in the contradiction that $a = b$. Consequently, $\varphi\langle 1/a\rangle(y) \neq 1/b$ for all $y \in Y$ and it follows from Lemma 2.6 that either $\varphi\langle 1/a\rangle(y) = 1/a$ or $\varphi\langle 1/a\rangle(y) = 0$ for all $y \in Y$. Next, define

$$\begin{aligned} Z &= \{y \in Y : \varphi\langle 1/a\rangle(y) = 1/a\} \\ A &= \{y \in Y : \varphi\langle 1/a\rangle(y) = 0\}. \end{aligned}$$

It follows from what we have just observed and Lemma 2.6 that Z and A are disjoint clopen sets whose union is X. Moreover, since φ is nonzero, it follows from Lemma 2.7 that $Z \neq \emptyset$. For each $y \in Z$ define a homomorphism ψ_y from $N_{a,b}(X)$ into $R_{a,b}$ by $\psi_y(f) = \varphi(f)(y)$. The homomorphism ψ_y is nonzero since $\psi_y\langle 1/a\rangle = \varphi\langle 1/a\rangle(y) = 1/a$ and it follows from Lemma 2.2 that there exists a unique point $x \in X$ such that $\psi_y(f) = f(x)$ for all $f \in N_{a,b}(X)$. Define $h(y) = x$ and note that $\varphi(f)(y) = \psi_y(f) = f(h(y))$ for all $f \in N_{a,b}(X)$ and $y \in Z$. Moreover, it follows from [**2**, Theorem 3.8, p.40], that h is continuous. Finally, it follows from Lemma 2.7 that $\varphi(f)(y) = 0$ for $y \in Y \backslash Z$. The case where $b \neq 0$ (and perhaps $a = 0$) is handled in much the same manner as the case we have just completed so we omit the details. ∎

THEOREM 2.9. *Suppose $a \neq -b$ and let Y be any topological space. Then for each space X and each nonzero homomorphism φ from $N_{a,b}(X)$ into $N_{a,b}(Y)$, there exists a continuous function h from Y to X such that $\varphi(f) = f \circ h$ for all $f \in N_{a,b}(X)$ if and only if Y is connected.*

PROOF: Suppose first that Y is connected and that φ is a nonzero homomorphism from $N_{a,b}(X)$ into $N_{a,b}(Y)$ for some space X. According to Lemma 2.8, there exists a nonempty clopen subspace Z of Y such that $\varphi(f)(y) = f \circ h(y)$ for all $f \in N_{a,b}(X)$ and all $y \in Z$. But we must have $Z = Y$ since Y is connected.

Now suppose that Y is not connected. We must exhibit a space X and a nonzero homomorphism from $N_{a,b}(X)$ into $N_{a,b}(Y)$ which is not of the form described in the theorem. Take $X = \{p, q\}$ to be the two point discrete space. Since Y is not connected, we have $Y = A \cup B$ for two disjoint nonempty clopen subsets A and B. Define a map φ from $N_{a,b}(X)$ into $N_{a,b}(Y)$ by

$$\varphi(f)(y) = \begin{cases} f(p) & \text{for } y \in A \\ 0 & \text{for } y \in B. \end{cases} \tag{2.9.1}$$

One verifies in a routine manner that φ is, indeed, a nonzero homomorphism. However, there exists no continuous function h from Y to X such that $\varphi(f) = f \circ h$ for in such a situation, we would have $\varphi\langle 1\rangle = \langle 1\rangle$ and this is certainly not the case. ∎

In the next result End $N_{a,b}(X)$ will denote the semigroup of all endomorphisms of the near-ring $N_{a,b}(X)$. The symbol $S(X)$ denotes the semigroup, under composition, of all continuous selfmaps of X. The dual of $S(X)$ will be denoted by $S_D(X)$. The product $f \bullet g$ of $f, g \in S_D(X)$ is therefore defined by $f \bullet g = g \circ f$. Finally, the symbol $S_D(X)^0$ will denote the semigroup $S_D(X)$ with a zero element adjoined and the zero endomorphism of a near-ring will be denoted by φ_0.

THEOREM 2.10. *Suppose X is a continuum and $a \neq -b$. Then* End $N_{a,b}(X)$ *is isomorphic to $S_D(X)^0$.*

PROOF: Suppose $\varphi \in$ End $N_{a,b}(X)$ and $\varphi \neq \varphi_0$. According to Theorem 2.9, there exists a necessarily unique continuous selfmap h of X such that $\varphi(f) = f \circ h$ for all $f \in N_{a,b}(X)$. Define a map Φ from End $N_{a,b}(X)$ into $S_D(X)^0$ by $\Phi(\varphi_0) = 0$ and $\Phi(\varphi) = h$. Suppose $\Phi(\varphi_1) = h_1$ and $\Phi(\varphi_2) = h_2$. Then $(\varphi_1 \circ \varphi_2)(f) = \varphi_1(\varphi_2(f)) = \varphi_1(f \circ h_2) = f \circ (h_2 \circ h_1) = f \circ (h_1 \bullet h_2)$. But this means $\Phi(\varphi_1 \circ \varphi_2) = h_1 \bullet h_2 = \Phi(\varphi_1) \bullet \Phi(\varphi_2)$. It follows easily that Φ is bijective and the proof is complete. ∎

THEOREM 2.11. *Suppose X and Y are continua which contain arcs and suppose also that $a \neq -b$. Then the following statements are equivalent:*

(2.11.1) End $N_{a,b}(X)$ *and* End $N_{a,b}(Y)$ *are isomorphic,*
(2.11.2) *The near-rings $N_{a,b}(X)$ and $N_{a,b}(Y)$ are isomorphic,*
(2.11.3) *The semigroups $S(X)$ and $S(Y)$ are isomorphic,*
(2.11.4) *The spaces X and Y are homeomorphic.*

PROOF: Theorem 2.10 assures us that (2.11.1) implies (2.11.3). Because X and Y are both continua which contain arcs, it follows that both X and Y are generated spaces as defined in [**3**, Definition 2.2, p.198]. It then follows from [**3**, Theorem 2.3, p.198], that the spaces X and Y are homeomorphic. That is, (2.11.3) implies (2.11.4). Since it is immediate that (2.11.4) implies (2.11.2) and (2.11.2) implies (2.11.1), the proof is complete. $\square$

Now we turn our attention to the case where $a = -b \neq 0$.

LEMMA 2.12. *Suppose $b > 0$ and φ is a nonzero homomorphism from $N_{-b,b}(X)$ into $R_{-b,b}$. Then there exists a unique $x \in X$ such that either $\varphi(f) = f(x)$ for all $f \in N_{-b,b}(X)$ or $\varphi(f) = -f(x)$ for all $f \in N_{-b,b}(X)$.*

PROOF: We can take Y in Lemma 2.7 to be the one point space and we can then identify $N_{-b,b}(Y)$ with $R_{-b,b}$ and it then follows from Lemma 2.7 that $\varphi\langle 1/b\rangle \neq 0$. Then define $\psi(x) = \langle x\rangle$. Then $\varphi\circ\psi$ is a nonzero endomorphism of $R_{-b,b}$ and according to Theorem 2.1, either $\varphi\circ\psi(r) = r$ for all $r \in R_{-b,b}$ or $\varphi\circ\psi(r) = -r$ for all $r \in R_{-b,b}$. This means that either

$$\varphi\langle r\rangle = r \text{ for all } r \in R_{-b,b} \tag{2.12.1}$$

or

$$\varphi\langle r\rangle = -r \text{ for all } r \in R_{-b,b}. \tag{2.12.2}$$

In either event φ is a homomorphism from $N_{-b,b}(X)$ onto $R_{-b,b}$ and since $R_{-b,b}$ is simple, Ker φ is a maximal ideal of $N_{-b,b}(X)$ and according to [5, Theorem 2.3], there exists a unique $x \in X$ such that Ker $\varphi = M_x = \{f \in N_{-b,b}(X) : f(x) = 0\}$. Again, we have $f - \langle f(x)\rangle \in M_x$ and $0 = \varphi(f - \langle f(x)\rangle) = \varphi(f) - f(x)$ in case (2.12.1) holds, whereas $0 = \varphi(f - \langle f(x)\rangle) = \varphi(f) + f(x)$ in case (2.12.2) holds. This completes the proof. $\square$

THEOREM 2.13. *Let $b > 0$ and let φ be a nonzero homomorphism from $N_{-b,b}(X)$ to $N_{-b,b}(Y)$. Then there exist two clopen subsets A and B of Y such that $Z = A \cup B \neq \emptyset$ and a continuous function h from Z into X such that*

$$\varphi(f)(y) = \begin{cases} f(h(y)) & \text{for } y \in A \\ -f(h(y)) & \text{for } y \in B \\ 0 & \text{for } y \in Y \backslash Z. \end{cases}$$

PROOF: Evidently, we have $\langle 1/b\rangle\langle 1/b\rangle = \langle 1/b\rangle$ so that $\varphi\langle 1/b\rangle\varphi\langle 1/b\rangle = \varphi\langle 1/b\rangle$ and it follows from Lemma 2.6 that for each $y \in Y$, we have either $\varphi\langle 1/b\rangle(y) = 1/b$ or $\varphi\langle 1/b\rangle(y) = -1/b$ or $\varphi\langle 1/b\rangle(y) = 0$. Let

$$A = \{y \in Y : \varphi\langle 1/b\rangle(y) = 1/b\}$$
$$B = \{y \in Y : \varphi\langle 1/b\rangle(y) = -(1/b)\}$$
$$C = \{y \in Y : \varphi\langle 1/b\rangle(y) = 0\}.$$

Now A, B and C are mutually disjoint clopen subsets whose union is Y. Since φ is nonzero, it follows from Lemma 2.7 that $Z \neq \emptyset$. Now suppose $y \in A$. Define a homomorphism ψ_y from $N_{-b,b}(X)$ to $R_{-b,b}$ by $\psi_y(f) = \varphi(f)(y)$. Since $y \in A$, we have $\psi_y\langle 1/b\rangle = 1/b$. By Lemma 2.12 there exists a point $x \in X$ such that either $\psi_y(f) = f(x)$ for all $f \in N_{-b,b}(X)$ or $\psi_y(f) = -f(x)$ for all $f \in N_{-b,b}(X)$. In view of the fact that $\psi_y\langle 1/b\rangle = 1/b$, we see that $\psi_y(f) = f(x)$ for all $f \in N_{-b,b}(X)$. We define $h(y) = x$ and we note that $\varphi(f)(y) = \psi_y(f) = f(h(y))$ for all $y \in A$ and $f \in N_{-b,b}(X)$. In a similar manner, one extends the function h over B also but with the difference that $\varphi(f)(y) = -f(h(y))$ for all $y \in B$ and $f \in N_{-b,b}(X)$. Of course, Lemma 2.7 tells us that $\varphi(f)(y) = 0$ for all $y \in C = Y \backslash Z$. Finally, It follows from [**2**, Theorem 3.8, p.40], that the restrictions of h to both A and B are continuous and since both those sets are clopen, it follows that h is a continuous function from Z to X. ∎

Theorem 2.14. *Suppose $b > 0$ and Y is any topological space. Then for each space X and each nonzero homomorphism from $N_{-b,b}(X)$ into $N_{-b,b}(Y)$, there exists a continuous function h from Y into X such that either $\varphi(f) = f \circ h$ for all $f \in N_{-b,b}(X)$ or $\varphi(f) = -f \circ h$ for all $f \in N_{-b,b}(X)$ if and only if Y is connected.*

Proof: Suppose first, that Y is connected and let the sets A, B and C be defined as in the previous proof. As we noted there, these sets are all clopen and since Y is connected, this means that either $Y = A$ or $Y = B$ or $Y = C$. But $Y \neq C$ since φ is a nonzero homomorphism. If $Y = A$, it is immediate that $\varphi(f) = f \circ h$ for all $f \in N_{-b,b}(X)$ and if $Y = B$, it is immediate that $\varphi(f) = -f \circ h$ for all $f \in N_{-b,b}(X)$. If Y is not connected, one defines a homomorphism from $N_{-b,b}(X)$ to $N_{-b,b}(Y)$ where X is the two point discrete space, just as in (2.9.1). In this case also, there exists no continuous function h such that either $\varphi(f) = f \circ h$ for all $f \in N_{-b,b}(X)$ or $\varphi(f) = -f \circ h$ for all $f \in N_{-b,b}(X)$ since this would imply that either $\varphi\langle 1\rangle = \langle 1\rangle$ or $\varphi\langle 1\rangle = \langle -1\rangle$ and neither of these is the case. ∎

Theorem 2.15. *Suppose $b > 0$, Y is a continuum and φ is an isomorphism from $N_{-b,b}(X)$ onto $N_{-b,b}(Y)$. Then there exists a homeomorphism h from Y onto X such that either $\varphi(f) = f \circ h$ for all $f \in N_{-b,b}(X)$ or $\varphi(f) = -f \circ h$ for all $f \in N_{-b,b}(X)$.*

Proof: According to Theorem 2.14, there exists a continuous function h from Y into X such that either $\varphi(f) = f \circ h$ for all $f \in N_{-b,b}(X)$ or $\varphi(f) = -f \circ h$ for all $f \in N_{-b,b}(X)$. Similarly, there exists a continuous function k from X into Y such that either $\varphi^{-1}(g) = g \circ k$ for all $g \in N_{-b,b}(Y)$ or $\varphi^{-1}(g) = -g \circ k$ for all $g \in N_{-b,b}(Y)$. Suppose $\varphi(f) = -f \circ h$ for all $f \in N_{-b,b}(X)$ while $\varphi^{-1}(g) = g \circ k$ for all $g \in N_{-b,b}(Y)$. Then $f = (\varphi^{-1} \circ \varphi)(f) = -f \circ (h \circ k)$ for all $f \in N_{-b,b}(X)$. But this would imply, for example, that $\langle 1\rangle = -\langle 1\rangle \circ (h \circ k) = -\langle 1\rangle$ which is a contradiction. It readily follows that if $\varphi(f) = f \circ h$ for all $f \in N_{-b,b}(X)$, then $\varphi^{-1}(g) = g \circ k$ for all $g \in N_{-b,b}(Y)$ and similarly if $\varphi(f) = -f \circ h$, then $\varphi^{-1}(g) = -g \circ k$. In either event, we have $f = f \circ (h \circ k)$ for all $f \in N_{-b,b}(X)$ and $g = g \circ (k \circ h)$ for all $g \in N_{-b,b}(Y)$. This implies that $h \circ k$ is the identity map on Y and $k \circ h$ is the identity map on X. Consequently, h is a homeomorphism from Y onto X. ∎

In the next result, Z_2 will denote the cyclic group of order two.

THEOREM 2.16. *Suppose $b > 0$ and X is a continuum. Then* End $N_{-b,b}(X)$ *is isomorphic to $(Z_2 \times S_D(X))^0$ and* Aut $N_{-b,b}(X)$ *is isomorphic to $Z_2 \times H(X)$.*

PROOF: Let $Z_2 = \{-1, 1\}$ where the operation is ordinary multiplication. Let $\varphi \in$ End $N_{-b,b}(X)$ and suppose $\varphi \neq \varphi_0$. According to Theorem 2.9, there exists a continuous selfmap h of X such that either $\varphi(f) = f \circ h$ for all $f \in N_{-b,b}(X)$ or $\varphi(f) = -f \circ h$ for all $f \in N_{-b,b}(X)$. Define a map Φ from End $N_{-b,b}(X)$ into $Z_2 \times S_D(X)$ by $\Phi(\varphi) = (1, h)$ in the former case and $\Phi(\varphi) = (-1, h)$ in the latter. As for φ_0, we define $\Phi(\varphi_0) = 0$. One verifies in a routine manner that Φ is in fact an isomorphism. For example, suppose $\Phi(\varphi_1) = (-1, h_1)$ and $\Phi(\varphi_2) = (-1, h_2)$. Then

$$\Phi(\varphi_1)\Phi(\varphi_2) = (-1, h_1)(-1, h_2) = (1, h_1 \bullet h_2) = (1, h_2 \circ h_1).$$

But

$$\varphi_1 \circ \varphi_2(f) = \varphi_1(\varphi_2(f)) = \varphi_1(-f \circ h_2) = f \circ (h_2 \circ h_1)$$

which means that

$$\Phi(\varphi_1 \circ \varphi_2) = (1, h_2 \circ h_1) = \Phi(\varphi_1)\Phi(\varphi_2).$$

One verifies the remaining cases in a similar manner and it is also easily verified that Φ is a bijection. As for Aut $N_{-b,b}(X)$, let φ be any automorphism of $N_{-b,b}(X)$. According to Theorem 2.11, there exists a homeomorphism from X onto X such that either $\varphi(f) = f \circ h$ for all $f \in N_{-b,b}(X)$ or $\varphi(f) = -f \circ h$ for all $f \in N_{-b,b}(X)$. In the former case, define $\Phi(\varphi) = (1, h^{-1})$ and in the latter case, define $\Phi(\varphi) = (-1, h^{-1})$. One readily verifies that Φ is an isomorphism from Aut $N_{-b,b}(X)$ onto $Z_2 \times H(X)$. ∎

Before we present and verify our concluding result, it will be convenient to have at our disposal the following

LEMMA 2.17. *An element $(n, f) \in Z_2 \times S_D(X)$ is of the form $(1, \langle x \rangle)$ for some $x \in X$ if and only if it is idempotent and it is a right zero for all squares of elements in $Z_2 \times S_D(X)$.*

PROOF: $(1, \langle x \rangle)$ is certainly idempotent and for every $(m, g) \in Z_2 \times S_D(X)$, we have

$$(m, g)^2 (1, \langle x \rangle) = (m^2, g \circ g)(1, \langle x \rangle) = (1, (g \circ g) \bullet \langle x \rangle) = (1, \langle x \rangle \circ g \circ g) = (1, \langle x \rangle)$$

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and we see that $(1, \langle x \rangle)$ is a right zero for every square of $Z_2 \times S_D(X)$. Suppose, conversely, that (n, f) is idempotent and is a right zero for all squares of $Z_2 \times S_D(X)$. Then $(n, f) = (n, f)^2 = (n^2, f \circ f)$. Thus $n = n^2$ which means $n = 1$ and we have $(n, f) = (1, f)$. We must yet show that f is a constant function. To this end, choose any $x \in X$. Since $(1, \langle x \rangle)^2 = (1, \langle x \rangle)$, we have

$$(1, f) = (1, \langle x \rangle)(1, f) = (1, \langle x \rangle \bullet f) = (1, f \circ \langle x \rangle) = (1, \langle f(x) \rangle).$$

This means that $f = \langle f(x) \rangle$ and the proof is complete. ∎

THEOREM 2.18. *Suppose $b > 0$ and let X and Y be continua which contain arcs. Then the following statements are equivalent:*

(2.18.1) *End $N_{-b,b}(X)$ and End $N_{-b,b}(Y)$ are isomorphic,*
(2.18.2) *The near-rings $N_{-b,b}(X)$ and $N_{-b,b}(Y)$ are isomorphic,*
(2.18.3) *The semigroups $S(X)$ and $S(Y)$ are isomorphic,*
(2.18.4) *The spaces X and Y are homeomorphic.*

PROOF: If (2.18.1) holds, Theorem 2.16 tells us that $(Z_2 \times S_D(X))^0$ is isomorphic to $(Z_2 \times S_D(Y))^0$ which implies that $Z_2 \times S_D(X)$ is isomorphic to $Z_2 \times S_D(Y)$. Let ψ be any isomorphism from $Z_2 \times S_D(X)$ onto $Z_2 \times S_D(Y)$. Suppose $\psi(1, f) = (-1, g)$ for some $f \in S_D(X)$ and $g \in S_D(Y)$. Choose any $x \in X$. The elements of the form $(1, \langle x \rangle)$ have been characterised algebraically in Lemma 2.17 so it follows that $\psi(1, \langle x \rangle) = (1, \langle y \rangle)$ for some $y \in Y$. From this we get

$$\begin{aligned}(1, \langle y \rangle) &= \psi(1, \langle x \rangle) = \psi(1, \langle x \rangle \circ f) = \psi(1, f \bullet \langle x \rangle) = \psi\big((1, f)(1, \langle x \rangle)\big) = \psi(1, f)\psi(1, \langle x \rangle) \\ &= (-1, g)(1, \langle y \rangle) = (-1, g \bullet \langle y \rangle) = (-1, \langle y \rangle \circ g) = (-1, \langle y \rangle).\end{aligned}$$

This, of course, is a contradiction and we conclude that $\psi(1, f) = (1, g)$ for each $f \in S_D(X)$ and an appropriate $g \in S_D(Y)$. Let $P(X) = \{(1, f) : f \in S_D(X)\}$ and $P(Y) = \{(1, g) : g \in S_D(Y)\}$. Evidently, $P(X)$ and $P(Y)$ are subsemigroups of $Z_2 \times S_D(X)$ and $Z_2 \times S_D(Y)$ respectively and it follows from what we have just verified that ψ maps $P(X)$ isomorphically onto $P(Y)$. Now define a monomorphism α from $S_D(X)$ into $Z_2 \times S_D(X)$ by $\alpha(f) = (1, f)$ and an epimorphism β from $Z_2 \times S_D(Y)$ onto $S_D(Y)$ by $\beta(m, g) = g$. It readily follows that $\beta \circ \psi \circ \alpha$ is an isomorphism from $S_D(X)$ onto $S_D(Y)$ and since the latter two semigroups are isomorphic, it is immediate that $S(X)$ and $S(Y)$ are isomorphic. The remainder of the proof is just as in the case of Theorem 2.11. Recall that X and Y are generated spaces and are therefore homeomorphic by [**3**, Theorem 2.3, p.198]. Thus, (2.18.3) implies (2.18.4). As in the case for Theorem 2.11, it is immediate that (2.18.4) implies (2.18.2) which, in turn, implies (2.18.1) and this completes the verification. ∎

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