REGULAR GRAPHS WITH REGULAR NEIGHBORHOODS

by ĽUBOMÍR ŠOLTÉS

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0. The existence of r-regular graphs such that each edge lies in exactly t triangles, for given integers $t < r$, is studied. If t is sufficiently close to r then each such connected graph has to be the complete multipartite graph. Relations to graphs with isomorphic neighborhoods are also considered.

1. In this paper we follow the notation of Behzad, Chartrand and Lesniak-Foster **[2]** with the exception of the following concepts. The *neighborhood* $N(v)$ of a nonisolated vertex v is the subgraph induced by vertices adjacent to v. By a graph with *regular* [*t-regular*, for an integer $t \geq 0$] *neighborhoods* we mean a graph such that the neighborhood of every vertex is a regular [t-regular, resp.] graph. Clearly G is a graph with t-regular neighborhoods if and only if each of its edges lies in exactly t triangles. Graphs with 1-regular neighborhoods were studied by Zelinka **[6]** and Fronček **[3]**. Regular graphs with regular neighborhoods were introduced by Antonucci **[1]** as quasi-strongly regular graphs.

If every neighborhood in G is isomorphic to a given graph H then G is a *locally-H graph*. These graphs have been often studied, after Zykov **[7]** asked for which H there is a locally-H graph. Regular graphs with regular neighborhoods play an important role in the study of line graphs with isomorphic neighborhoods.

Further we obtain the graph $K_r \overset{t}{\cup} K_r$ if we join two copies of K_r by t independent edges. Šoltés **[5]** has proved that a graph G is a locally-$(K_{r-1} \overset{t}{\cup} K_{r-1})$ graph $(r-1 \geq t > 0)$ if and only if it is the line graph of an r-regular graph with t-regular neighborhoods. Moreover, regular graphs with regular neighborhoods are the only graphs with a triangle such that their line graphs have isomorphic neighborhoods.

Here we study the question of the existence of r-regular graphs with t-regular neighborhoods for given integers r and t. The product rt being even and $0 \leq t < r$ are two trivial necessary conditions. In this paper we will prove that all regular graphs with sufficiently thick regular neighborhoods are complete multipartite graphs.

THEOREM 1. *Let t and r be integers such that*

$$t < r < t + \sqrt{\frac{8}{9}(t-1)} + \frac{4}{3} \tag{1}$$

holds. Then

(1): *every connected r-regular graph with t-regular neighborhoods is the complete $(1 + r/(r-t))$-partite graph in which every part has $r - t$ vertices;*

(2): *the line graph of the r-regular complete $(1 + r/(r-t))$-partite graph is the only connected locally-$\left(K_{r-1} \overset{t}{\cup} K_{r-1}\right)$ graph if $r \equiv 0 \bmod (r-t)$. Otherwise there is no locally-$\left(K_{r-1} \overset{t}{\cup} K_{r-1}\right)$ graph.*

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As Nedela [4] has pointed out, if $t \geq 1$ is a square then the conjunction $K_{2+\sqrt{t}} \wedge K_{2+\sqrt{t}}$ (vertices of the conjunction $K_{2+\sqrt{t}} \wedge K_{2+\sqrt{t}}$ are pairs (v_1, v_2) where both v_1 and v_2 are vertices in $K_{2+\sqrt{t}}$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent iff both $u_1 \neq v_1$ and $u_2 \neq v_2$ hold) is a $(t + 2\sqrt{t} + 1)$-regular graph with t-regular neighborhoods which is not a complete multipartite graph. Hence the coefficient of $\sqrt{t-1}$ in (1) cannot exceed two.

Before proving Theorem 1 we shall introduce the concept of an *admissible* graph. If G is a graph then $\overline{G^2}$ is the complement of the square of G. Two vertices are adjacent in $\overline{G^2}$ if and only if their distance in G is at least three. Let G be a regular graph and suppose the following two conditions hold.

(C1): Any two vertices with distance two in G lie in distinct components in $\overline{G^2}$.

(C2): For each pair of components α and β in $\overline{G^2}$ there is a number $c_{\alpha\beta}$ such that any two vertices a from α and b from β, not adjacent in G, have precisely $c_{\alpha\beta}$ common neighbors in G.

Then G is called an *admissible graph.*

By $|S|$ we mean the number of elements in a set S. The next lemma states that any neighborhood in a regular graph with regular neighborhoods has an admissible complement.

LEMMA 2. *Let G be an r-regular graph with t-regular neighborhoods. Then the neighborhood of any vertex is the complement of some $(r-t-1)$-regular admissible graph.*

Proof. Let G have n vertices. Then it is easy to see that its complement $\bar{G}$ is $(n-1-r)$-regular and every two of its non-adjacent vertices have precisely $\mu = n - 2r + t$ common neighbors.

If $\mu = 0$ then G is a complete multipartite graph and hence every neighborhood has admissible complement.

If $\mu \geq 1$ then $\operatorname{diam}(\bar{G}) = 2$. (If $\bar{G}$ is a complete graph then $r = 0$, which contradicts $r > t$). Let v be a fixed vertex. The complement of $N(v)$ is the subgraph H of $\bar{G}$ induced by all vertices not adjacent to v in $\bar{G}$. Clearly H is an $(r-t-1)$-regular graph. From now on we shall work only with the graph $\bar{G}$. Given a vertex h in H, let $f(h)$ be the μ-element set of vertices adjacent to both h and v.

First we note that if vertices h_1 and h_2 are adjacent in $\overline{H^2}$ then $f(h_1) = f(h_2)$ holds. In fact h_1 and h_2 have no common neighbors in H, hence all μ of their common neighbors are also neighbors of v and form sets $f(h_1)$ and $f(h_2)$. Hence whenever vertices h and g are connected in $\overline{H^2}$, the equality $f(h) = f(g)$ holds.

Observe that if vertices h and g have distance two in H then they have a common neighbor in H and hence $f(h) \cap f(g)$ has at most $\mu - 1$ elements which gives $f(h) \neq f(g)$ and so h and g lie in distinct components of $\overline{H^2}$. This proves (C1).

Further, two non-adjacent vertices g and h have in H precisely μ-$|f(h) \cap f(g)|$ common neighbors; this is a function depending only on components containing h, resp. g. Hence (C2) holds, which completes the proof. ∎

Clearly every graph with diameter at most two is admissible. But due to the following Lemma, every connected admissible graph has diameter at most three.

LEMMA 3. *Let G be a Δ-regular admissible graph on n vertices. Then either G is the union of several $K_{\Delta+1}$ or* $\operatorname{diam}(G) \leq 3$ *and we have*

$$n \leq \frac{9\Delta^2 + 2\Delta + 17}{8}. \tag{2}$$

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Proof. Let G be an admissible graph. If $\overline{G^2}$ is connected then, due to (C1), G has no two vertices with distance two; hence every component of G is a complete graph.

Otherwise, $\overline{G^2}$ is disconnected, and so G is connected. Note that for every shortest path P joining two vertices with distance four there are two of its vertices with distance two belonging to the same component of $\overline{G^2}$, which contradicts (C1). Hence we have proved that $\operatorname{diam}(G) \leq 3$. If moreover the radius of G is at most two then $n \leq 1 + \Delta^2$ and so (2) holds.

Now assume that G has radius three. Let v be a vertex in G such that its neighborhood has the maximal number of edges, say e, among all vertices. Let N_k be the number of vertices in G at distance k from v, for an integer k. Clearly

$$\begin{aligned} N_2 &\leq \Delta(\Delta - 1) - 2e, \\ 1 + N_1 + N_2 &\leq 1 + \Delta^2 - 2e. \end{aligned} \tag{3}$$

Let us say that a vertex is a *far-vertex* if its distance from v is three and let S be the subgraph of G induced by far-vertices. All far-vertices lie in the same component in $\overline{G^2}$ as v. Hence due to (C1) no two far-vertices have distance two and so every component of S is a complete graph. Later on, let a vertex x be adjacent to the maximal number of far-vertices, say $q \geq 1$ vertices, among all vertices which are not far.

Note that far-vertices adjacent to x lie in the same component of S, say F, having f vertices. Otherwise two non-adjacent far-vertices adjacent to x would have distance two, which contradicts (C1). Further

$$2e \geq q(q-1) \tag{4}$$

holds, because the graph F having $q(q-1)/2$ edges is a subgraph of $N(x)$. Now let A be the set of neighbors of x having distance two from v and let B be the set of far-vertices not lying in F. We shall compute the number of edges joining A and B. Note that $|A| = \Delta - q - a$, where a is the number of common neighbors of v and x. Further each vertex from A is adjacent to at most q vertices from B and the condition (C2) gives that every vertex from B is adjacent to exactly a vertices in A. Hence

$$|B| \leq \frac{q(\Delta - a - q)}{a} \tag{5}$$

holds, and for the number of far-vertices we have

$$N_3 = |B| + f. \tag{6}$$

Here we distinguish two cases.

Case 1. Let all vertices in F be adjacent to x. Then $f = q$ and, for the order of G, according to (3), (4), (5) and (6) we have

$$n \leq 1 + \Delta + \Delta(\Delta - 1) - q(q-1) + \frac{q(\Delta - a - q)}{a} + q.$$

On putting $a = 1$ we obtain

$$n \leq -2q^2 + (1 + \Delta)q + 1 + \Delta^2.$$

The last expression is maximal if $4q = 1 + \Delta$. Setting this for q we obtain (2).

Case 2. Let there be a vertex z in F not adjacent to x. Then $f \geq 2$ and all neighbors of x which lie in F are also adjacent to z and the condition (C2) gives $q \leq a$, which yields $|B| \leq \Delta - 2q$. Clearly $2e \geq (f-1)(f-2)$, as the neighborhood of z contains $F - z$. So for the order of G, according to (3), (4) and (6) we have

$$n \leq \Delta^2 + 1 - 2e + f + |B| \leq \Delta^2 + 1 - (f-1)(f-2) + f + \Delta - 2q$$

which is maximal if $f = 2$ and $q = 1$; so

$$n \leq \Delta^2 + \Delta + 1 \leq \frac{9\Delta^2 + 2\Delta + 17}{8}$$

which completes the proof. ■

Proof of Theorem 1. (1): Let G be an r-regular graph with t-regular neighborhoods which is not a complete multipartite graph. Then Lemmas 2 and 3 give that $8r \leq 9\Delta^2 + 2\Delta + 17$ where $\Delta := r - t - 1$. But setting $r := \Delta + t - 1$ we get $8(t-1) \leq 9(\Delta - \frac{1}{3})^2$ which gives $r \geq \frac{4}{3} + t + \sqrt{\frac{8}{9}(t-1)}$. This contradicts Lemma 2.

(2): It is proved in [5] that a graph G is locally-$\left(K_{r-1} \overset{t}{\cup} K_{r-1}\right)$ if and only if G is the line graph of an r-regular graph with t-regular neighborhoods. This together with part (1) completes the proof. ■

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KNOM MFF Comenius University
842 15 Bratislava
Czechoslovakia

Present address:
Katedra Matematiky CH7F STU
Radlinského 9
812 37 Bratislava
Czechoslovakia

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