EXTENSIONS OF ENDOMORPHISMS FROM THE HIGHER CENTRES

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Introduction. If $0 \to A \to C \to B \to 0$ is an exact sequence of abelian groups, if f is a 2-cocyle for this extension, if $\alpha \in \text{End}\, A$, and if $\beta \in \text{End}\, B$, then a necessary and sufficient condition that α extend to an endomorphism γ of C which induces β is that (M) αf and $f\beta$ be cohomologous; see Montgomery **(2)**. We shall extend this result to the case where $1 \to A \to G \to B \to 1$ is an exact sequence of groups and A is abelian. For α to extend to a γ which induces β and extends an endomorphism on the centralizer of A in G, it is (Theorem 2) necessary and sufficient (i) that, for each $b \in B$, b and βb be carried onto the same element of G modulo the centralizer of A in G; (ii) that α commute with the automorphisms of A induced by B via the extension; and (iii) that condition (M) hold. If α is an endomorphism on $Z_n G$, the nth member of the ascending central series of G, if each $Z_i G$, $0 \leqslant i < n$, is α-admissible, if β is an endomorphism on $G/Z_n G$, and if α can be extended to a $\gamma \in \text{End}\, G$ which induces β, then we shall show (Theorem 3) that the respective cohomology classes of the extensions

$$1 \to Z_1(G/Z_i G) \to G/Z_i G \to G/Z_{i+1} G \to 1 \qquad (0 \leqslant i < n)$$

lie in the kernels of suitable endomorphisms of the 2-cohomology groups $\mathfrak{H}^{(2)}(G/Z_{i+1}G, Z_{i+1}G/Z_i G)$, the endomorphisms generated in each case from α and β. Conversely, if essentially the cohomology classes lie in such kernels, then (Theorem 4) it is possible to modify α slightly so that the modification will extend to an endomorphism γ on G which induces β.

The principal device is that of an extended centrally compatible family of endomorphisms for $\alpha \in \text{End}\, Z_n G$, the family of endomorphisms induced on the various $Z_i(G/Z_j G)$, $0 \leqslant i + j \leqslant n$, by α. Theorem 1 provides us with such a family whenever all the $Z_i G$, $0 \leqslant i < n$, are α-admissible.

If A and B are groups or rings where B contains A, then $C(A, B)$ is to be the centralizer of A in B, a normal subgroup of B whenever A is a normal subgroup of B. If B is a group which operates on the abelian group A via some $\theta \in \text{Hom}(B, \text{Aut}\, A)$, we sometimes write $\mathfrak{H}^{(2)}(B, A; \theta)$ for $\mathfrak{H}^{(2)}(B, A)$ in order to emphasize the map θ. For a function ϕ from X to Y, $\phi | U$ means ϕ with its domain cut down to the subset U of X. The symbol δ denotes the usual coboundary operator. The symbol ι is reserved for the identity map of any set

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under consideration. If g is an element of the group G, then $\langle g\rangle$ is to be the inner automorphism on G given by $\langle g\rangle x = gxg^{-1}$ for each $x \in G$. If γ_1 and γ_2 are functions from a group G to a group H, $\gamma_1 + \gamma_2$ is to be that function γ from G to H which is given by $\gamma(g) = \gamma_1(g)\gamma_2(g)$ for each $g \in G$. Note that the order of summation is important. The function $-\gamma$ has its values given by $(-\gamma)(x) = (\gamma(x))^{-1}$. All maps are written to the left: if a product of maps $\prod_{j=1}^{n} \beta_j$ is given, it is to be understood that the first map to be applied is β_n and the last is β_1. Frequently, an inverse ϕ^{-1} will appear in a product of homomorphisms even though ϕ is not single valued. In each instance it will be found that $\ker \phi$ is such that the ambiguities disappear. In general, $\phi^{-1}(x)$ means the complete inverse image of x under ϕ.

1. Preliminaries. (a) Let A be an abelian group, let B be a group, and let θ be any member of $\mathrm{Hom}(B, \mathrm{Aut}\, A)$. Observe that $C(\mathrm{Im}\, \theta, \mathrm{End}\, A)$ is a subring of $\mathrm{End}\, A$. Let $\mathrm{End}_\theta\, B$ be the set of all $\beta \in \mathrm{End}\, B$ for which $\theta\beta = \theta$, a set which is closed under multiplication. If $\alpha \in C(\mathrm{Im}\, \theta, \mathrm{End}\, A)$ and if $\beta \in \mathrm{End}_\theta\, B$, then, each in a standard way **(1)**, α induces α^* and β induces $\beta^\#$, both in End $\mathfrak{H}^{(2)}(B, A; \theta)$. The map $(\ )^*$ is a ring homomorphism while $(\ )^\#$ preserves multiplication.

(b) Let $1 \to A \to G \xrightarrow{\Phi} B \to 1$ be any extension of A by B. Then there exist homomorphisms λ and μ such that the following diagram is commutative with exact rows and columns:

$$(C)\qquad \begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & A & \longrightarrow & A & \longrightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & C(A, G) & \longrightarrow & G & \xrightarrow{\lambda} & G/C(A, G) & \to & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \iota} & & \\
1 & \to & C(A, G)/A & \to & B & \xrightarrow{\mu} & G/C(A, G) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

Let $\mathrm{End}_\mu\, B$ be the set of all $\beta \in \mathrm{End}\, B$ for which $\mu\beta = \mu$, a set which is closed under multiplication.

(c) For a group G, let $Z_0\, G = 1$, and let $Z_i\, G$ be the ith member of the ascending central series of G, where $i = 1, 2, \ldots$. Let $J^0G = G$, let $J^1G = JG$, the inner automorphism group on G, and let $J^{i+1}G = J(J^jG)$ for $j = 1, 2, \ldots$. Let ϕ_j be the homomorphism which makes the sequence

$$(A_j)\qquad 1 \to Z_1\, J^{j-1}G \to J^{j-1}G \xrightarrow{\phi_j} J^jG \to 1$$

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exact, $j = 1, 2, \ldots$. If we let $\phi_{i,j} = \phi_j | Z_{i+1} J^{j-1} G$, then

$$(A_{i,j}) \qquad 1 \to Z_1 J^{j-1}G \to Z_{i+1} J^{j-1}G \xrightarrow{\phi_{i,j}} Z_i J^j G \to 1$$

is exact.

For a positive integer n and for $\alpha \in \operatorname{End}(Z_n G)$, we say that α is *centrally compatible on* $Z_n G$ if $\operatorname{Im}(\alpha | Z_m G) \leqslant Z_m G$ for each positive integer $m \leqslant n$. Given a centrally compatible α, write $\alpha_{n,0} = \alpha$, $\alpha_{m,0} = \alpha | Z_m G (1 \leqslant m \leqslant n)$ and $\alpha_{0,0} = 0$, the trivial (and sole) endomorphism on 1. The finite set $\{\alpha_{m,0}\}_{m=0}^{n}$ is called *the centrally compatible family for* (*centrally compatible*) α *on* $Z_n G$. Each $\beta = \alpha_{m,0}$ of such a family is centrally compatible on $Z_m G$, and if k is an integer for which $0 \leqslant k \leqslant m$, then $\beta_{k,0} = \alpha_{k,0}$. Later, we shall need the fact that the map

$$\sigma_i = \prod_{j=0}^{i-1} \phi_{i-j}$$

(multiplication proceeding from the left to right with increasing j) for each positive integer i makes

$$(A'_i) \qquad 1 \to Z_i G \to G \xrightarrow{\sigma_i} J^i G \to 1$$

exact.

THEOREM 1. *Let α be a centrally compatible endomorphism on $Z_n G$ where n is a positive integer. For each pair of non-negative integers i and j with $0 \leqslant i + j \leqslant n$ there exists an endomorphism $\alpha_{i,j}$ centrally compatible on $Z_i J^j G$ such that*

(i) $\alpha_{n,0} = \alpha$;

(ii) $\alpha_{i,j} | Z_m J^j G = \alpha_{m,j}$ *whenever* $0 \leqslant m \leqslant i$; *and*

(iii) *the following diagrams are commutative with exact rows:*

$$(B_{i,j}) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & Z_1 J^{j-1}G & \longrightarrow & Z_{i+1} J^{j-1}G & \xrightarrow{\phi_{i,j}} & Z_i J^j G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha_{1,j-1}} & & \downarrow{\scriptstyle \alpha_{i+1,j-1}} & & \downarrow{\scriptstyle \alpha_{i,j}} & & \\ 1 & \longrightarrow & Z_1 J^{j-1}G & \longrightarrow & Z_{i+1} J^{j-1}G & \xrightarrow{\phi_{i,j}} & Z_i J^j G & \longrightarrow & 1 \end{array}$$

for each pair of integers i and j satisfying $1 \leqslant i + j \leqslant n$, $0 \leqslant i$, *and* $1 \leqslant j$.

Proof. Let $\{\alpha_{m,0}\}_{m=0}^{n}$ be the centrally compatible family for α. Let $\alpha_{0,1}$ be the trivial (and only) endomorphism of $Z_0 JG = 1$. For each integer m such that $1 \leqslant m < n$, let $\alpha_{m,1} = \phi_{m,1} \alpha_{m+1,0} \phi_{m,1}^{-1}$. Even though $\phi_{m,1}^{-1}$ need not be single valued, the fact that $Z_1 G = \ker \phi_{m,1}$ makes the definition of $\alpha_{m,1} \in \operatorname{End} Z_m JG$ unambiguous. One readily checks that (ii) holds in that $\alpha_{m,1} | Z_k JG = \alpha_{k,1}$ for each integer k such that $0 \leqslant k \leqslant m < n$. Further, the diagram $(B_{m,1})$ is commutative with exact rows for each integer m with $1 \leqslant m < n$. Now suppose, inductively, that (1) there exist

$$\alpha_{n-l,l} \in \operatorname{End} Z_{n-l} J^l G$$

for each integer l such that $1 \leqslant l \leqslant j < n$; (2) there exist $\alpha_{m,l} \in \operatorname{End} Z_m J^l G$ for each integer m such that $0 \leqslant m \leqslant n-1$, where $\alpha_{m,l} = \alpha_{n-l,l}|Z_m J^l G$; and (3) $(B_{m,l})$ is commutative with exact rows for all such m and l; or (4) $j \geqslant n$. Then, if (4) holds for j, it holds for $j+1$. If $j < n$, let

$$\alpha_{n-j-1,j+1} = \phi_{n-j-1,j+1}\, \alpha_{n-j,j}\, \phi_{n-j-1,j+1}{}^{-1},$$

unambiguously defined as a member of $\operatorname{End} Z_{n-j-1} J^{j+1}G$, since $Z_1 J^j G = \ker \phi_{n-j-1,j+1}$, and since the induction assumption gives $\alpha_{n-j,j}|Z_1 J^j G = \alpha_{1,j}$. By construction and by the induction assumption, $(B_{n-j-1,j+1})$ is commutative with exact rows. If m is an integer for which $0 \leqslant m \leqslant n-j-1$, let

$$\alpha_{m,j+1} = \phi_{m,j+1}\, \alpha_{m+1,j}\, \phi_{m,j+1}{}^{-1},$$

unambiguously defined as a member of $\operatorname{End} Z_m J^{j+1}G$, since $Z_1 J^j G = \ker \phi_{m,j+1}$, and since $\alpha_{m+1,j}|Z_1 J^j G = \alpha_{1,j}$ by the induction assumption. By construction and by the induction assumption, $(B_{m,j+1})$ is commutative with exact rows. Finally, upon further appeal to the induction assumption, we have that, for each integer m such that $0 \leqslant m \leqslant n-j-1$,

$$\begin{aligned}\alpha_{n-j-1,j+1}|Z_m J^{j+1}G &= \phi_{n-j-1,j+1}\, \alpha_{n-j,j}\, \phi_{n-j-1,j+1}{}^{-1}|Z_m J^{j+1}G \\ &= \phi_{m,j+1}\, \alpha_{m+1,j}\, \phi_{m,j+1}{}^{-1} = \alpha_{m,j+1},\end{aligned}$$

and the proof is complete.

We call the set $\{\alpha_{i,j}\}$, $0 \leqslant i+j \leqslant n$, for non-negative integers i and j, *the extended centrally compatible family for* α *on* $Z_n G$.

2. The basic extension theorem.

THEOREM 2. *Let A be an abelian group, B be a group,* $\theta \in \operatorname{Hom}(B, \operatorname{Aut} A)$, *and let*

$$1 \to A \to G \xrightarrow{\Phi} B \to 1$$

be any extension of B by A which belongs to some $\mathrm{t} \in \mathfrak{H}^{(2)}(B, A; \theta)$. *Suppose that* $\alpha \in \operatorname{End} A$ *and that* $\beta \in \operatorname{End} B$. *Then there exists* $\gamma \in \operatorname{End} G$ *such that* $\operatorname{Im}(\gamma|C(A, G)) \leqslant C(A, G)$, *and that both*

$$(D) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{\Phi} & B & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\beta} & & \\ 1 & \longrightarrow & A & \longrightarrow & G & \xrightarrow{\Phi} & B & \longrightarrow & 1 \end{array}$$

and

$$(D') \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & C(A, G) & \longrightarrow & G & \xrightarrow{\lambda} & G/C(A, G) & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\gamma|C(A, G)} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\iota} & & \\ 1 & \longrightarrow & C(A, G) & \longrightarrow & G & \xrightarrow{\lambda} & G/C(A, G) & \longrightarrow & 1 \end{array}$$

are commutative diagrams with exact rows, if and only if

(i) $\beta \in \mathrm{End}_\mu B$,

(ii) $\alpha \in C(\mathrm{Im}\,\theta, \mathrm{End}\,A)$, *and*

(iii) $\mathrm{t} \in \ker(\alpha^* - \beta^\#)$.

Proof. Suppose that $C(A;G)$ is γ-admissible and that both (D) and (D') are commutative with exact rows. From the commutativity of (C), $\lambda = \mu\Phi$ whence $\mu\beta = (\mu\Phi)\Phi^{-1}\beta = \lambda\Phi^{-1}\beta$. But, from the commutativity of (D), $\beta = \Phi\gamma\Phi^{-1}$; and from the commutativity of (D'), $\lambda\gamma = \lambda$ so that

$$\lambda\Phi^{-1}\beta = (\lambda\gamma)\Phi^{-1} = \lambda\Phi^{-1} = (\mu\Phi)\Phi^{-1},$$

and $\mu\beta = \mu$, or, equivalently, $\beta \in \mathrm{End}_\mu B$, establishing (i). Since $\lambda = \mu\Phi$, and since $\mu = \mu\beta$,

$$\lambda\Phi^{-1} = \mu = \mu\beta = \mu\Phi\Phi^{-1}\beta = \lambda\Phi^{-1}\beta$$

so that $\theta\beta = \theta$, and $\beta \in \mathrm{End}_\theta B$.

Since $\Phi\gamma\Phi^{-1} = \beta$, since $\gamma|A = \alpha$, and since $\theta\beta = \theta$,

$$\alpha(\theta(b)) = \gamma(\theta(b)) = (\theta\beta(b))\alpha = \theta(b)\alpha$$

for each $b \in B$, and (ii) holds. Because of (i) and (ii), α^* and $\beta^\#$ are defined.

Choose $g_b \in \Phi^{-1}b$ for each $b \in B$; and, for all $x, y \in B$, let $t(x, y) = g_x g_y g_{xy}{}^{-1}$. Here, t is a 2-cocycle which represents t. Since $\beta = \Phi\gamma\Phi^{-1}$, $(\gamma g_b)(g_{\beta b})^{-1} = h_b \in A$ for every $b \in B$. Then,

$$\begin{aligned}\alpha t(x, y)h_{xy}\, g_{\beta(xy)} &= \gamma(t(x, y)g_{xy}) = \gamma(g_x\, g_y) = h_x\, g_{\beta x}\, h_y\, g_{\beta y}\\ &= h_x\, \theta(\beta x)(h_y)g_{\beta x}\, g_{\beta y} = h_x(\theta x)(h_y)t(\beta x, \beta y)g_{\beta(xy)}\end{aligned}$$

so that αt differs from $t\beta$ by δh, yielding (iii).

Conversely, suppose that $\mathrm{t} \in \ker(\alpha^* - \beta^\#)$ where (i) and (ii) hold for β and α. Choose a normalized transversal $\{g_b\}$ and form the representing, normalized, 2-cocycle t. Then $\alpha t - t\beta = \delta h$ where h is a normalized 1-cochain on B with values $h_b \in A$. Each member of G has a unique representation in the form ag_b where $a \in A$. Define $\gamma \in G^G$ via $\gamma(ag_b) = \alpha(a)h_b\, g_{\beta b}$. By its definition, γ makes (D) commutative. As in the proof above, $\beta \in \mathrm{End}_\theta B$. An easy consequence is that

$$g_b{}^{-1}g_{\beta b} \in \ker\lambda = C(A, G).$$

From this one shows that $\mathrm{Im}(\gamma|C(A, G)) \leqslant C(A, G)$, so that (D') is commutative. A routine check, employing (a) $\theta\beta = \theta$, (b) $\alpha t - t\beta = \delta h$, and (c) the values of $t\beta$ can be written in terms of the transversal elements, shows that $\gamma \in \mathrm{End}\,G$.

COROLLARY. (a) *If* $1 \to A \to G \xrightarrow{\Phi} B \to 1$ *is an extension of* A *by* B *where* $A \leqslant Z_1 G$, *if* $\alpha \in \mathrm{End}\,A$, *and if* $\beta \in \mathrm{End}\,B$, *then a necessary and sufficient condition that there exist* $\gamma \in \mathrm{End}\,G$ *making* (D) *commutative is that the coho-*

mology class $\mathfrak{t}$ *of the extension lie in* $\ker(\alpha^* - \beta^\#)$. (b) *Let* α *be an endomorphism of* $Z_1 G$. *A necessary and sufficient condition that* α *possess an extension* γ *which is a central endomorphism of* G ($\operatorname{Im}\gamma \leqslant Z_1 G$) *is that* $\mathfrak{t}_0$, *the cohomology class of the extension*

$$1 \to Z_1 G \to G \xrightarrow{\phi_1} JG \to 1,$$

be in $\ker \alpha^*$. *The set of all such* α *is a left ideal in* $\operatorname{End} Z_1 G$. (c) *Let* α *be an endomorphism of* $Z_1 G$. *A necessary and sufficient condition that* α *possess an extension which is a normal endomorphism of* G (*induces* ι *on* JG) *is that* $\alpha^*(\mathfrak{t}_0) = \mathfrak{t}_0$. (d) *If* $\beta \in \operatorname{End} JG$, *then there exists an extension of the identity automorphism on* $Z_1 G$ *to a* $\gamma \in \operatorname{End} G$ *which induces* $\beta \in \operatorname{End} JG$ *if and only if* $\beta^\#(\mathfrak{t}_0) = \mathfrak{t}_0$. (e) *If* $\beta \in \operatorname{End} JG$, *then there exists an extension of the trivial endomorphism of* $Z_1 G$ *to a* $\gamma \in \operatorname{End} G$ *which induces* $\beta \in \operatorname{End} JG$ *if and only if* $\mathfrak{t}_0 \in \ker \beta^\#$.

3. Endomorphisms on the higher centres.

THEOREM 3. *Let* n *be a positive integer; let* $\alpha \in \operatorname{End} Z_n G$ *be centrally compatible with extended family* $\{\alpha_{i,j}\}$. *For each integer* i *such that* $0 \leqslant i < n$, *let*

$$\mathfrak{t}_i \in \mathfrak{H}^{(2)}(J^{i+1}G, Z_1 J^i G)$$

be the cohomology class of the extension

$$(A_{i+1}) \quad 1 \to Z_1 J^i G \to J^i G \xrightarrow{\phi_{i+1}} J^{i+1} G \to 1;$$

and let β *be in* $\operatorname{End} J^n G$. *Suppose that there exist* $\gamma \in \operatorname{End} G$ *such that*

$$(E_n) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & Z_n G & \longrightarrow & G & \xrightarrow{\sigma_n} & J^n G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \\ 1 & \longrightarrow & Z_n G & \longrightarrow & G & \xrightarrow{\sigma_n} & J^n G & \longrightarrow & 1 \end{array}$$

is commutative. Let $\gamma^{(n)} = \beta$, *and let* $\gamma^{(i)} = \sigma_i \gamma \sigma_i^{-1} \in \operatorname{End} J^i G$ *be the map on* $J^i G$ *which is induced by* γ *via* (A'_i), $1 \leqslant i < n$. *Then* $\mathfrak{t}_i \in \ker(\alpha_{1,i}{}^* - \gamma^{(i+1)\#})$, $0 \leqslant i < n$.

Proof. Recall that $\sigma_1 = \phi_1$ and that $\sigma_{j+1} = \phi_{j+1}\sigma_j$, $j = 1, 2, \ldots$. By the definition of $\gamma^{(i)}$, $\gamma^{(i)}\sigma_i = \sigma_i \gamma$, $i = 1, 2, \ldots, n$. Hence, for $1 \leqslant i < n$,

$$\phi_{i+1}\gamma^{(i)}\sigma_i = \phi_{i+1}\sigma_i\gamma = \sigma_{i+1}\gamma = \gamma^{(i+1)}\sigma_{i+1} = \gamma^{(i+1)}\phi_{i+1}\sigma_i.$$

But σ_i is an epimorphism with image $J^i G$, so that $\phi_{i+1}\gamma^{(i)} = \gamma^{(i+1)}\phi_{i+1}$, $i = 1, 2, \ldots, n-1$. That is, the right rectangle of

$$(F_i) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & Z_1 J^i G & \longrightarrow & J^i G & \xrightarrow{\phi_{i+1}} & J^{i+1} G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha_{1,i}} & & \downarrow{\scriptstyle \gamma^{(i)}} & & \downarrow{\scriptstyle \gamma^{(i+1)}} & & \\ 1 & \longrightarrow & Z_1 J^i G & \longrightarrow & J^i G & \xrightarrow{\phi_{i+1}} & J^{i+1} G & \longrightarrow & 1 \end{array}$$

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is commutative for integers i such that $1 \leqslant i < n$. It is likewise commutative for $i = 0$ if we take $\gamma^{(0)} = \gamma$ and recall the definitions of $\gamma^{(1)}$, ϕ_1, and σ_1.

Since α is centrally compatible and since (E_n) is commutative, $\gamma|Z_1 G = \alpha|Z_1 G = \alpha_{1,0}$, making the left rectangle of (F_0) commutative. Suppose that $1 \leqslant t < n$. Then

$$\gamma^{(1)}\phi_{t,1} = \gamma^{(1)}\phi_1|Z_{t+1} G = \phi_1 \gamma|Z_{t+1} G = \phi_1 \alpha_{t+1,0} = \alpha_{t,1} \phi_{t,1},$$

since $(B_{t,1})$ is commutative. The fact that Im $\phi_{t,1} = Z_t JG$ leads to

$$\gamma^{(1)}|Z_t JG = \alpha_{t,1}, 1 \leqslant t < \text{n}.$$

Let (S_k) be the statement that if i is an integer such that $0 \leqslant i < k$, then (I) $i > n$, or (II) $\gamma^{(i)}|Z_t J^i G = \alpha_{t,i}$ whenever $1 \leqslant t \leqslant n - \text{i}$. From the commutativity of (E_n), (S_1) holds. If $k = 2$, then $\gamma^{(1)}|Z_t JG = \alpha_{t,1}$, $1 \leqslant t < n$, leads to (S_2). Now suppose, inductively, that $k \geqslant 2$ and that (S_j) holds for each integer j such that $1 \leqslant j \leqslant k$. Then, if $k \leqslant n$, we have for $0 < t \leqslant n - k$ that

$$\begin{aligned}\gamma^{(k)}\phi_{t,k}\, \phi_{t+1,k-1} &= \gamma^{(k)}\phi_k\, \phi_{t+1,k-1} = \phi_k\, \gamma^{(k-1)}\phi_{t+1,k-1}\\ &= \phi_k\, \gamma^{(k-1)}\phi_{k-1}|Z_{t+2}\, J^{k-2}G = \phi_k\, \phi_{k-1}\, \gamma^{(k-2)}|Z_{t+2}\, J^{k-2}G\\ &= \phi_k\, \phi_{k-1}\, \alpha_{t+2,k-2}\end{aligned}$$

(by the induction assumption since $0 \leqslant k - 2 < n$)

$$= \phi_{t,k}\, \phi_{t+1,k-1}\, \alpha_{t+2,k-2} = \alpha_{t,k}\, \phi_{t,k}\, \phi_{t+1,k-1}$$

(since $(B_{t+1,k-1})$ and $(B_{t,k})$ are commutative diagrams for the values of t and k allowed). Moreover, $\phi_{t,k}\, \phi_{t+1,k-1}$ is onto $Z_t J^k G$, and we now have $\gamma^{(k)}|Z_t J^k G = \alpha_{t,k}$ for $0 < t \leqslant n - k$ and $2 \leqslant k \leqslant n$. If $k > n$, then the induction assumption says that (S_n) holds, from which (S_{k+1}) holds. In either case, the induction is complete. In particular, $\gamma^{(i)}|Z_1 G = \alpha_{1,i}$ for each integer i such that $1 \leqslant i \leqslant n$, and the left rectangle of (F_i) is commutative, $0 \leqslant i < n$. By Theorem 2, Corollary (a), $\text{t}_i \in \ker(\alpha_{1,i}{}^* - \gamma^{(i+1)\#})$, $0 \leqslant i < n$, completing the proof.

Suppose that $\alpha^{(1)}$ and $\alpha^{(2)}$ are both centrally compatible endomorphisms on $Z_n G$. We say that $\alpha^{(1)}$ and $\alpha^{(2)}$ are *equivalent centrally compatible endomorphisms on* $Z_n G$ $(\alpha^{(1)} \sim \alpha^{(2)})$ if

$$\text{Im}\,(\alpha^{(1)}{}_{i,0} - \alpha^{(2)}{}_{i,0}) \leqslant Z_{i-1} G$$

for each integer i such that $1 \leqslant i \leqslant n$. That is,

$$\alpha^{(1)}{}_{i,0}(x)[\alpha^{(2)}{}_{i,0}(x)]^{-1} \in Z_{i-1} G$$

whenever $x \in Z_i G$, $1 \leqslant i \leqslant n$. If $\alpha^{(1)} \sim \alpha^{(2)}$, then $\alpha^{(1)}{}_{1,0} = \alpha^{(2)}{}_{1,0}$. The relation $\sim$ is an equivalence, and all the automorphisms of G are equivalent. For $g \in G$ and for α centrally compatible on $Z_n G$,

$$\alpha(\langle g\rangle|Z_n G) \sim \alpha \sim (\langle g\rangle|Z_n G)\alpha,$$

all centrally compatible endomorphisms on $Z_n G$.

Let ρ be any *decreasing crossed endomorphism on $Z_n G$ related to a centrally compatible* $\alpha \in \operatorname{End} Z_n G$. That is, if $x \in Z_i G$, $0 < i \leqslant n$, then $\rho(x) \in Z_{i-1} G$; $\rho(1_G) = 1_G$; and if $x_1, x_2 \in Z_n G$, then

$$\rho(x_1 x_2) = (\langle \alpha(x_2)^{-1} \rangle \rho(x_1)) \rho(x_2).$$

We have $\alpha + \rho \sim \alpha$ where, for every $x \in Z_n G$, $(\alpha + \rho)(x) = \alpha(x)\rho(x)$; and if $\alpha' \sim \alpha$, then there exists some decreasing crossed endomorphism ρ on $Z_n G$ related to α such that $\alpha' = \alpha + \rho$.

THEOREM 4. *Let n, α, β, and $\mathfrak{t}_0$ be as in the hypothesis of Theorem 3. Suppose that, for each integer i obeying $1 \leqslant i \leqslant n$, one can find $\Gamma^{(i)} \in \operatorname{End} J^i G$ with the properties*

(i) $\Gamma^{(n)} = \beta$,

(ii) $\mathfrak{t}_0 \in \ker(\alpha_{1,0}{}^* - \Gamma^{(1)\#})$, *and*

(iii) *the diagrams*

$$(K_i)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & Z_1 J^{i-1}G & \longrightarrow & J^{i-1}G & \xrightarrow{\phi_i} & J^i G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha_{1,i-1}} & & \downarrow{\scriptstyle \Gamma^{(i-1)}} & & \downarrow{\scriptstyle \Gamma^{(i)}} & & \\ 1 & \longrightarrow & Z_1 J^{i-1}G & \longrightarrow & J^{i-1}G & \xrightarrow{\phi_i} & J^i G & \longrightarrow & 1 \end{array}$$

$2 \leqslant i \leqslant n$, are commutative with exact rows. Then there exist $\gamma \in \operatorname{End} G$ and a decreasing crossed endomorphism ρ on $Z_n G$ related to α such that the diagram

$$(E'_n)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & Z_n G & \longrightarrow & G & \xrightarrow{\sigma_n} & J^n G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha + \rho} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} & & \\ 1 & \longrightarrow & Z_n G & \longrightarrow & G & \xrightarrow{\sigma_n} & J^n G & \longrightarrow & 1 \end{array}$$

is commutative.

Proof. From (ii) and from Theorem 2, Corollary (a), there exists $\Gamma^{(0)} = \gamma \in \operatorname{End} G$ such that (K_1) is commutative. From the commutativity of the left rectangle in (K_1),

$$\gamma_1 = \gamma | Z_1 G = \alpha_{1,0}.$$

Suppose, inductively, that, for a positive integer j, we have (I) $j > n$, or (II) $\gamma_j = \gamma | Z_j G$ is a centrally compatible endomorphism on $Z_j G$ for which $\gamma_j \sim \alpha_{j,0}$. If $j + 1 \leqslant n$, consider the diagram with exact rows

$$(E_j)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & Z_j G & \longrightarrow & G & \xrightarrow{\sigma_j} & J^j G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \gamma_j} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \Gamma^{(j)}} & & \\ 1 & \longrightarrow & Z_j G & \longrightarrow & G & \xrightarrow{\sigma_j} & J^j G & \longrightarrow & 1 \end{array}$$

the commutativity of the left rectangle of which is immediate from the induction hypothesis. Since the diagrams (K_i), $1 \leqslant i \leqslant n$, are commutative with exact rows, the right rectangle of (E_j) is commutative.

Consider the diagram with exact rows

$$(L_j) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & Z_j G & \longrightarrow & Z_{j+1} G & \xrightarrow{\phi^{(j)}} & Z_1 J^j G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \gamma_j} & & \downarrow{\scriptstyle \gamma_{j+1}} & & \downarrow{\scriptstyle \alpha_{1,j}} & & \\ 1 & \longrightarrow & Z_j G & \longrightarrow & Z_{j+1} G & \xrightarrow{\phi^{(j)}} & Z_1 J^j G & \longrightarrow & 1 \end{array}$$

where

$$\phi^{(j)} = \sigma_j | Z_{j+1} G = \prod_{k=1}^{j} \phi_{k,j-k+1},$$

multiplication proceeding with increasing k from left to right. The commutativity of (E_j) implies that $\Gamma^{(j)}\sigma_j = \sigma_j \gamma$. Now $\Gamma^{(j)}\sigma_j|Z_{j+1} G = \Gamma^{(j)}\phi^{(j)}$. From the commutativity of the left rectangle of (K_{j+1}), $\Gamma^{(j)}\phi^{(j)} = \alpha_{1,j}\,\phi^{(j)}$ so that $\mathrm{Im}(\sigma_j \gamma | Z_{j+1} G) \leqslant Z_1 J^j G$. From the fact that $\sigma_j^{-1}(Z_1 J^j G) = Z_{j+1} G$, one obtains $\mathrm{Im}(\gamma | Z_{j+1}) \leqslant Z_{j+1} G$, so that $\gamma_{j+1} \in \mathrm{End}\, Z_{j+1}$, and (L_j) is commutative.

From the commutativity of the diagrams $(B_{k,j-k+1})$, $1 \leqslant k \leqslant j$, the commutativity of the diagram

$$(L'_j) \qquad \begin{array}{ccccccccc} 1 & \longrightarrow & Z_i G & \longrightarrow & Z_{j+1} G & \xrightarrow{\phi^{(j)}} & Z_1 J^j G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \alpha_{j,0}} & & \downarrow{\scriptstyle \alpha_{j+1,0}} & & \downarrow{\scriptstyle \alpha_{1,j}} & & \\ 1 & \longrightarrow & Z_i G & \longrightarrow & Z_{j+1} G & \xrightarrow{\phi^{(j)}} & Z_1 J^j G & \longrightarrow & 1 \end{array}$$

follows. From the commutativity of the right rectangles of both (L_j) and (L'_j),

$$\phi^{(j)}\alpha_{j+1,0} = \alpha_{1,j}\,\phi^{(j)} = \phi^{(j)}\gamma_{j+1},$$

so that $\mathrm{Im}(\alpha_{j+1,0} - \gamma_{j+1}) \leqslant \ker \phi^{(j)} = Z_j G$, completing the proof.

References

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