ORDER AND SCHWARTZ DISTRIBUTIONS†

by W. A. FELDMAN and J. F. PORTER

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Introduction. The space $\mathscr{D}'$ of Schwartz distributions on the unit circle Γ in the plane is topologically a considerable generalization of the space $\mathscr{D}'_0$ of regular, finite Borel measures on Γ. However, the order structure of $\mathscr{D}'$ is usually taken to be the same as that of $\mathscr{D}'_0$: there are no "positive" distributions which are not measures. This perhaps warrants consideration, since the order structure of $\mathscr{D}'_0$ generates its topology. In this paper we construct a system of order structures for $\mathscr{D}'$ which is a more natural complement in the intermediate stages to the topology of $\mathscr{D}'$ and which provides an interpretation of $\mathscr{D}'$ with its Schwartz topology as a quotient of a generalized base norm space V'. Where $\mathscr{D}_0$ denotes the space of continuous functions on Γ with its supremum norm topology, V' is the dual of $\prod_{n=0}^{\infty} \mathscr{D}_0$. The space $\Pi\,\mathscr{D}_0$ contains the infinitely differentiable functions on Γ with their usual topology, and (*via* the pointwise ordering on $\mathscr{D}_0$) $\Pi\,\mathscr{D}_0$ in its product ordering is realized as a generalized order unit space. Some consequences for harmonic functions are discussed.

We cite **[5]** and **[9]** for general information on partial orderings for topological vector spaces and **[10]** for general information on Schwartz distributions.

1. Partial Order Structures on $\mathscr{D}_n$ and $\mathscr{D}'_n$. We recall that the Banach space $\mathscr{D}_0$ of real continuous functions on Γ with the norm

$$\|f\|_0 = \sup\{|f(x)| : x \in \Gamma\}$$

for f in $\mathscr{D}_0$ and positive cone $\mathscr{D}_{0+}$ of pointwise non-negative members of $\mathscr{D}_0$ is an order unit space (and an M-space). Where $\mathscr{D}'_{0+}$ denotes the set of linear functionals ϕ on $\mathscr{D}_0$ satisfying $\phi(f) \geqq 0$ for all f in $\mathscr{D}_{0+}$ and $(\mathscr{D}'_0, \|\cdot\|'_0)$ is the Banach dual of $(\mathscr{D}_0, \|\cdot\|_0)$, the system $(\mathscr{D}'_0, \mathscr{D}'_{0+}, \|\cdot\|'_0)$ is a base norm space (and an L-space). The order unit 1 in $\mathscr{D}_{0+}$ can be taken as the generator of $\|\cdot\|_0$ (i.e., $\|\cdot\|_0$ is the Minkowski functional on the order interval

$$[-1, 1]_0 = \{f \in \mathscr{D}_0 : -1 \leqq f \leqq 1\}).$$

The Riesz Representation Theorem states that $(\mathscr{D}'_0, \|\cdot\|'_0)$ can be identified with the space of regular, finite Borel measures on Γ with total variation norm and that $\mathscr{D}'_{0+}$ can be identified with the non-negative measures in $\mathscr{D}'_0$. The base of probability measures on Γ for $\mathscr{D}'_{0+}$ corresponds to the order unit 1.

Let $\mathscr{D}_n$ $(n = 1, 2, \ldots)$ be the space of n-times continuously differentiable functions on Γ with the norm

$$|||f|||_n = ||f||_0 + ||f^{(1)}||_0 + \ldots + ||f^{(n)}||_0,$$

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where $f^{(j)}$ is the jth derivative of the function f in $\mathscr{D}_n$. Then $\{\mathscr{D}_n\}_{n=0}^{\infty}$ is a decreasing sequence of Banach spaces with successively increasing norms. The corresponding dual spaces form an increasing sequence $\{\mathscr{D}_n'\}_{n=0}^{\infty}$ with successively decreasing norms, which we will denote by $|||\cdot|||_n'$ for $n = 1, 2, \ldots$. Let $\mathscr{D}$ denote $\bigcap_{n=0}^{\infty} \mathscr{D}_n$ with the projective limit (Schwartz) topology. The space $\mathscr{D}'$ of Schwartz distributions on Γ is the topological dual of $\mathscr{D}$. Moreover, $\mathscr{D}' = \bigcup_{n=0}^{\infty} \mathscr{D}_n'$ and the Schwartz topology on $\mathscr{D}'$ is the inductive limit topology. One defines " derivatives " of members ϕ of $\mathscr{D}'$ by stipulating that

$$\phi^{(n)}(f) = (-1)^n \phi(f^{(n)})$$

for all f in $\mathscr{D}$, this being essentially an extension of the formula for integration by parts.

We induce the order structures of $\mathscr{D}_0$ and $\mathscr{D}_0'$ onto $\mathscr{D}_n$ and $\mathscr{D}_n'$, respectively, in a way which is compatible both to their usual topologies and with the natural embeddings $\mathscr{D}_n \subseteq \mathscr{D}_{n-1}$ and $\mathscr{D}_n' \supseteq \mathscr{D}_{n-1}'$. All order structures to be considered evolve from the map M of the following proposition. Here, $\bar{f}$ denotes the mean value

$$(1/2\pi)\int_0^{2\pi} f(t)\,dt$$

and $f^{(n)}$ denotes the nth derivative of f. The choice of M is motivated by the desire to convert differentiation in $\mathscr{D}$ into an isomorphism whose dual produces the derivatives in $\mathscr{D}'$ and whose inverse preserves the order structure of $\mathscr{D}_0$.

PROPOSITION 1. *Let $M : \mathscr{D}_1 \to \mathscr{D}_0$ be defined by $M(f) = \bar{f} - \pi f^{(1)}$. Then $M(1) = 1$ and for each $n = 1, 2, \ldots$*

(1) *$M \mid \mathscr{D}_n$ is an isomorphism of $\mathscr{D}_0$ onto $\mathscr{D}_{n-1}$;*
(2) *M establishes an isomorphism M^n of $\mathscr{D}_n$ onto $\mathscr{D}_0$, with $M^n(f) = \bar{f} + (-\pi)^n f^{(n)}$.*

Proof. Trivially, $M(1) = 1$. For (1), M and hence $M \mid \mathscr{D}_n$ for each n is clearly linear. If $\bar{f} - \pi f^{(1)} = 0$, then $f^{(1)}$ is the constant function $\bar{f}/\pi$. Since, by Rolle's Theorem, $f^{(1)}(x_1) = 0$ for some x_1 in Γ, then $f^{(1)}(x) = \bar{f} = 0$ for all x in Γ and f is a constant. This constant, being its own mean value, must be zero. Then M, and hence $M \mid \mathscr{D}_n$ for each n, is one-to-one. To see that M is onto, let g be in $\mathscr{D}_0$ and let

$$k_g = \left(\int_0^{2\pi} dx \int_0^x g(t)\,dt\right) \Big/ 2\pi^2.$$

Define

$$f(x) = \left[\left(x\bar{g} - \int_0^x g(t)\,dt\right) \Big/ \pi\right] + k_g.$$

Then f is in $\mathscr{D}_1$ and $\bar{f} = \bar{g}$, so that $M(f) = \bar{f} - \pi f^{(1)} = g$. For g in $\mathscr{D}_{n-1}$ this construction produces a function f in $\mathscr{D}_n$; hence, $M \mid \mathscr{D}_n$ is onto $\mathscr{D}_{n-1}$ for each n. For (2), define

$M^n : \mathscr{D}_n \to \mathscr{D}_0$ to be the composite

$$(M \mid \mathscr{D}_1) \circ (M \mid \mathscr{D}_2) \circ \ldots \circ (M \mid \mathscr{D}_n).$$

Since each factor is an isomorphism, so is M^n. The formula for M^n follows by induction.

It will be convenient also to denote the restriction $M^n \mid \mathscr{D}_{n+j}$ for all integers $j \geqq 0$ by M^n; with the conventions that M^0 is the identity operator on $\mathscr{D}_0$ and that M^{-n} is the inverse of M^n, we will use the laws of integral exponents freely on M. Of course, $M^n(1) = 1$ for all n. Since M^{-n} is an isomorphism ($n = 1, 2, \ldots$), the set $M^{-n}\mathscr{D}_{0+}$ is a positive cone for $\mathscr{D}_n$. We will denote this positive cone by $\mathscr{D}_{n+}$; thus f is in $\mathscr{D}_{n+}$ if and only if

$$\bar{f} + (-\pi)^n f^{(n)} = M^n f \geqq 0.$$

We also define a norm for $\mathscr{D}_n$ by

$$\|f\|_n = \|M^n f\|_0 = \sup\{|\bar{f} + (-\pi)^n f^{(n)}(x)| : x \in \Gamma\}.$$

Thus $(\mathscr{D}_n, \mathscr{D}_{n+}, \|\cdot\|_n)$ is an M-space with order unit 1 isometric and order-isomorphic to $(\mathscr{D}_0, \mathscr{D}_{0+}, \|\cdot\|_0)$.

PROPOSITION 2. *The positive cones* $\{\mathscr{D}_{n+}\}_{n=0}^{\infty}$ *form a decreasing sequence. The norms* $\{\|\cdot\|_n\}_{n=0}^{\infty}$ *successively increase.*

Proof. Let f be in $\mathscr{D}_{1+}$ and suppose $f(x_0) \leqq 0$ for some x_0 in Γ. By translating f we can assume $x_0 = 0$. Since $Mf \geqq 0$, the function

$$h(x) = \int_0^x (Mf)(t)\,dt = x\bar{f} - \pi[f(x) - f(0)]$$

is non-negative and non-decreasing, with $\bar{h} = \pi f(0)$. Since $f(0) \leqq 0$ then h must vanish—i.e., $x\bar{f} - \pi[f(x) - f(0)] = 0$; since f is periodic it must also vanish. Thus $\mathscr{D}_{1+} \subseteq \mathscr{D}_{0+}$. If f is in $\mathscr{D}_{n+}$ for $n > 1$ then $M^{n-1}f$ is in $\mathscr{D}_{1+}$ and hence in $\mathscr{D}_{0+}$ so that f is in $\mathscr{D}_{(n-1)+}$. Thus $\{\mathscr{D}_{n+}\}_{n=0}^{\infty}$ is decreasing. It now follows that the order intervals

$$[-1, 1]_n = \{f \in \mathscr{D}_n : -1 \leqq M^n f \leqq 1\}$$

form a decreasing sequence

$$[-1, 1]_0 \supseteq [-1, 1]_1 \supseteq \ldots \supseteq [-1, 1]_n \supseteq \ldots.$$

Since $[-1, 1]_n$ is also the unit ball in $(\mathscr{D}_n, \|\cdot\|_n)$ ($n = 0, 1, 2, \ldots$) the norms increase with n.

PROPOSITION 3. *The norms* $\|\cdot\|_n$ *and* $|||\cdot|||_n$ *are equivalent.*

Proof. (Suprema will be taken over all x in Γ.) Clearly

$$\|f\|_0 \leqq |||f|||_n \quad \text{and} \quad \|f^{(n)}\|_0 \leqq |||f|||_n.$$

Since $\bar{f} = f(x_0)$ for some x_0 in Γ,

$$|\bar{f}| \leqq \|f\|_0 \leqq |||f|||_n.$$

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Thus

$$\begin{aligned}\|f\|_n &= \sup|\bar{f}+(-\pi)^n f^{(n)}(x)| \\ &\leqq |\bar{f}|+\pi^n \sup|f^{(n)}(x)| \\ &\leqq (1+\pi^n)|||f|||_n.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|f\|_n &= \sup|\bar{f}+(-\pi)^n f^{(n)}(x)| \\ &\geqq \sup(\pi^n|f^{(n)}(x)|-|\bar{f}|) \\ &= \pi^n\|f^{(n)}\|_0-|\bar{f}|.\end{aligned}$$

Thus

$$\pi^n\|f^{(n)}\|_0 \leqq \|f\|_n+|\bar{f}| \leqq 2\|f\|_n,$$

so that $\|f^{(n)}\|_0 \leqq 2\pi^{-n}\|f\|_n$. Then

$$\begin{aligned}|||f|||_n &= \|f\|_0+\|f^{(1)}\|_0+\ldots+\|f^{(n)}\|_0 \\ &\leqq \|f\|_0+2\pi^{-1}\|f\|_1+\ldots+2\pi^{-n}\|f\|_n \\ &\leqq (1+2\pi^{-1}+\ldots+2\pi^{-n})\|f\|_n,\end{aligned}$$

this last step by Proposition 2.

We now wish to dualize and summarize the results of this section. For a linear functional ϕ on $\mathscr{D}_n$, let

$$\|\phi\|_n' = \sup\{|\phi(f)|: f \in \mathscr{D}_n, \|f\| \leqq 1\}.$$

Let us presume the Schwartz notation $\mathscr{D}_n'$ for the set of those ϕ for which $\|\phi\|_n' < +\infty$ and denote by $\mathscr{D}_{n+}'$ the set of those ϕ for which $\phi(f) \geqq 0$ whenever f is in $\mathscr{D}_{n+}$. Let $L^n: \mathscr{D}_0' \to \mathscr{D}_n'$ denote the dual of $M^n: \mathscr{D}_n \to \mathscr{D}_0$, i.e. $L^n(\sigma) = \sigma \circ M^n$ for all σ in $\mathscr{D}_0'$. For the sake of emphasis, Proposition 3 is incorporated into the next theorem.

THEOREM 1. *Let* $n = 1, 2, \ldots$.

(1) $(\mathscr{D}_n, \mathscr{D}_{n+}, \|\cdot\|_n)$ *is an M-space with order unit* 1, *isometric and order-isomorphic to* $(\mathscr{D}_0, \mathscr{D}_{0+}, \|\cdot\|_0)$. *The norm* $\|\cdot\|_n$ *is equivalent to the frequently employed norm* $|||\cdot|||_n$ *for* $\mathscr{D}_n$ *and increases with n.*

(2) $(\mathscr{D}_n', \mathscr{D}_{n+}', \|\cdot\|_n')$ *is an L-space isometric and order-isomorphic to* $(\mathscr{D}_0', \mathscr{D}_{0+}', \|\cdot\|_0'$. $\mathscr{D}_n'$ *is the space of* nth *order Schwartz distributions on* Γ. *The norm* $\|\cdot\|_n'$ *is equivalent to* $|||\cdot|||_n'$ *and decreases as n increases.*

Proof. Part (1) contains nothing new. For part (2) we remark that it follows from part (1) that the order-theoretic and Banach duals of $(\mathscr{D}_n, \mathscr{D}_{n+}, \|\cdot\|_n)$ coincide and that $(\mathscr{D}_n', \mathscr{D}_{n+}', \|\cdot\|_n')$ is an L-space. That $\mathscr{D}_n'$ is indeed the space of nth order distributions on Γ follows from Proposition 3, which also implies that $\|\cdot\|_n'$ and $|||\cdot|||_n'$ are equivalent. That $\{\mathscr{D}_{n+}'\}_{n=0}^{\infty}$ increases is dual to Proposition 2, as is the fact that the norms $\{\|\cdot\|_n'\}_{n=0}^{\infty}$ decrease.

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(Here we use the convention that if ϕ is not in $\mathscr{D}'_n$, then $\|\phi\|_n = +\infty$.) The mapping L^n, being the dual of an isometry and order-isomorphism, is an isometry and order-isomorphism, with $L^n\mathscr{D}'_{0+} = \mathscr{D}'_{n+}$ and $\|L^n\sigma\|'_n = \|\sigma\|'_0$.

Utilizing the formula $\phi^{(n)}(f) = (-1)^n\phi(f^{(n)})$ mentioned at the beginning of the paper, we can obtain an explicit description for L^n. Here, and throughout the paper, m will be the Lebesgue measure normalized to have $m\Gamma = 1$. For f in $\mathscr{D}_n$ and σ in $\mathscr{D}'_0$,

$$\begin{aligned}(L^n\sigma)(f) &= \sigma(M^nf) = \sigma(\bar{f}+(-\pi)^nf^{(n)})\\ &= \int_\Gamma \bar{f}\,d\sigma+\pi^n\sigma^{(n)}(f) = \sigma\Gamma m(f)+\pi^n\sigma^{(n)}(f)\\ &= (\pi^n\sigma^{(n)}+\sigma\Gamma m)(f),\end{aligned}$$

by Proposition 1; thus

$$L^n(\sigma) = \pi^n\sigma^{(n)}+\sigma\Gamma m.$$

We also note that $L^nm = m$ and if $L^n\sigma = \sigma$ for σ in $\mathscr{D}'_0$, then σ is a multiple of m.

One can modify the proof of Proposition 1 to show that $f\mapsto\bar{f}-a\pi f^{(1)}$ is an isomorphism of $\mathscr{D}_1$ onto $\mathscr{D}_0$ for all real $a \neq 0$ and leads to structures similar to ours. However,

$$\{f\in\mathscr{D}_1 : \bar{f}-a\pi f^{(1)} \geqq 0\}$$

is contained in $\mathscr{D}_{0+}$ if and only if $|a| \geqq 1$. In this sense $\mathscr{D}_{1+}$ is maximal in $\mathscr{D}_{0+}$.

2. Partially Ordered Structures on $\mathscr{D}$ and $\mathscr{D}'$. In order to apply the results of §1 to the locally convex spaces $\mathscr{D}$ and $\mathscr{D}'$, we utilize generalizations of order unit and base norm spaces (see [2], [3] and [4]). A positive element u in a real vector lattice V is called a *semiorder-unit* (sou) if for each v in V there is a $\lambda > 0$ such that $v \wedge nu \leqq \lambda u$ for all positive integers n. An Archimedean vector lattice V is called a *semiorder-unit space* (sou space) if it is endowed with the topology generated by all seminorms

$$p_u(x) = \inf\{\lambda > 0 : |x| \wedge nu \leqq \lambda u\,(n = 1, 2, \ldots)\}$$

for u a sou in V, called the *sou topology* for V.

Let V be a real Archimedean vector lattice and V^0 its order dual. A convex set S of positive elements in V is called a *semibase* if it is $\sigma(V, V^0)$-bounded (weakly), the ideal $I(S)$ generated by S is a projective band (see [8]) and S is a base for the positive cone of $I(S)$. The space V is called a *semibase space* if it is the union of the ideals generated by its semibases and if its semibases are directed in the following sense: For each pair S' and S'' of semibases there is a semibase S such that $I(S)$ contains $I(S')$ and $I(S'')$. The topology generated by all seminorms

$$p_S(x) = \inf\{\lambda > 0 : \rho_S(x) \in \lambda I(S)\},$$

where S is a semibase and ρ_S is the projection mapping from V into $I(S)$, is called the *semibase topology* for V.

PROPOSITION 4. *The countable product* $\prod_{n=0}^{\infty} \mathcal{D}_0 (= \mathcal{D}_0^N)$ *in its product ordering is a sou space whose topology is the product topology. Dually, the countable direct sum* $\bigoplus_{n=0}^{\infty} \mathcal{D}_0'$ *in its direct sum ordering is a semibase space whose topology is the direct sum topology.*

Proof. Clearly $\Pi \mathcal{D}_0$ is an Archimedean vector lattice. Let u be a sou in $\Pi \mathcal{D}_0$ and let ρ_n denote the projection map from $\Pi \mathcal{D}_0$ into the nth factor $\mathcal{D}_0$. The fact that u is a sou implies that for some integer $M > 0$,

$$[\{n\rho_n(u)\}_{n=1}^{\infty}] \wedge mu \leqq Mu \quad (m = 1, 2, \ldots).$$

Since ρ_n is a lattice homomorphism, we obtain

$$[n\rho_n(u)] \wedge mu \leqq M\rho_n(u),$$

which for large m implies $n\rho_n(u) \leqq M\rho_n(u)$. Thus for $n > M$, $\rho_n(u) = 0$. We note that, if $\rho_n(u) \neq 0$, it is a sou in $\mathcal{D}_0$. In fact, if $\rho_n(u) \neq 0$, it is an order unit. (Since $\rho_n(u)$ is a sou in $\mathcal{D}_0$ there is a $\lambda > 0$ such that

$$1 \wedge m\rho_n(u) \leqq \lambda\rho_n(u) \quad (n = 1, 2, \ldots),$$

so that $[\rho_n(u)](x) \geqq 1/\lambda$ whenever $[\rho_n(u)](x) \neq 0$.) To verify that the sou topology is the product topology we note that if the sou u has $\rho_n(u) = 0$ for $n > M$, it is dominated by a multiple of that sou v_M which has 1 for its first M entries and 0 elsewhere. The seminorms p_{v_M} obviously generate the product topology.

The dual result is a consequence of Theorem 1 of [4], or can be proved directly as follows. The set B_0 of probability measures on Γ is a $\sigma(\mathcal{D}_0', \mathcal{D}_0'^0)$-bounded base for $\mathcal{D}_{0+}'$ whose base norm is $\| \cdot \|_0'$. Let B be any semibase in $\oplus \mathcal{D}_0'$. If $\rho_n(B) = 0$, define $f_n = 0$; if $\rho_n(B)$ contains a positive element ϕ_n, define f_n to be some element in $\mathcal{D}_{0+}$ for which $\phi_n(f_n) > n$. For the element $\{f_n\}$ of $\Pi \mathcal{D}_0$, as a positive linear functional on $\oplus \mathcal{D}_0'$, to be bounded on B, $\rho_n(B)$ must be zero for all but finitely many n. Thus from the $\sigma(\oplus \mathcal{D}_0', [\oplus \mathcal{D}_0']^0)$-boundedness of B we conclude that B is contained in a multiple of a sum of finitely many copies of B_0; i.e.,

$$B \subseteq \lambda \left[\bigoplus_{n=0}^{M} B_0 \right]$$

for some $\lambda > 0$ and integer M. Each such $\lambda \left[\bigoplus_{n=0}^{M} B_0 \right]$ is clearly a semibase. The fact that $\oplus \mathcal{D}_0'$ is a semibase space and the equivalence of the topologies can now be easily verified.

One could readily prove the following generalization of Proposition 4: A countable product of order unit spaces is a sou space whose topology agrees with the product topology, and a countable direct sum of base norm spaces is a semibase space whose topology agrees with the direct sum topology.

We recall that B_0, *the set of probability measures on* Γ, is a weak* compact base for $\mathcal{D}_{0+}'$ and the closed convex hull of the closed set E_0 of extreme points consisting of the unit point-measures on Γ. The set $\bigoplus_{n=0}^{\infty} B_0$ is a base for the positive cone of $\oplus \mathcal{D}_0'$. This base, though not weak* compact, is the closed convex hull of the set of extreme points $\bigoplus_{n=0}^{\infty} E_0$.

We can now apply the order structures of §1 to $\mathscr{D}$ and $\mathscr{D}'$. We let $[\Pi\mathscr{D}_0]_+$ and $[\oplus\mathscr{D}_0']_+$ denote the positive cones of Proposition 4.

THEOREM 2. *There is a topological isomorphism i from the space $\mathscr{D}$ of infinitely differentiable functions on Γ with its Schwartz topology into the sou space $(\Pi\mathscr{D}_0, [\Pi\mathscr{D}_0]_+)$. Dually, the space $\mathscr{D}'$ of distributions on Γ with its Schwartz topology is topologically isomorphic to the quotient of the semibase space $(\oplus\mathscr{D}_0', [\oplus\mathscr{D}_0']_+)$ by the kernel of the adjoint of i.*

Proof. Let $i:\mathscr{D}_0\to\Pi\mathscr{D}_0$ be defined by setting $i(f)=\{M^nf\}_{n=0}^{\infty}$. The theorem is a consequence of Proposition 4 and the results of §1. (Recall that $\mathscr{D}_n$ and $\mathscr{D}_n'$ have been identified with $\mathscr{D}_0$ and $\mathscr{D}_0'$ by the maps M^n and L^n, and $\mathscr{D}$ and $\mathscr{D}'$ are the appropriate projective limit topologies.)

The maps i and i^* naturally induce positive cones K and P on $\mathscr{D}$ and $\mathscr{D}'$ respectively, the usual subspace and quotient cones. It is easy to see that $K=\bigcap_{n=0}^{\infty}\mathscr{D}_{n+}$ and $P=\bigcup_{n=0}^{\infty}\mathscr{D}_{n+}'$. We will give explicit characterizations for these cones.

PROPOSITION 5. *K consists of the nonnegative constant functions on Γ and P is $\{\phi\in\mathscr{D}':\phi(1)>0\}\cup\{0\}$.*

Proof. That $P\subseteq\{\phi\in\mathscr{D}':\phi(1)>0\}\cup\{0\}$ can be argued as follows. If θ is in $P=\bigcup_{n=0}^{\infty}\mathscr{D}_{n+}'$ then θ is in $\mathscr{D}_{m+}'$ for some integer $m\geqq 0$, so that $\theta=L^m\sigma$ for some (unique) measure σ in $\mathscr{D}_{0+}'$. If $\sigma\Gamma=0$ then $\sigma\equiv 0$, so that $\theta\equiv 0$. Otherwise, $\theta(1)=\sigma\Gamma>0$. For the converse, let ϕ be in $\mathscr{D}'$ with $\phi(1)>0$. Then ϕ is in $\mathscr{D}_M'$ for some integer $M\geqq 0$, and so $\phi=L^M\sigma$ for some σ in $\mathscr{D}_0'$. Let

$$\sum_{k=-\infty}^{+\infty} a_k e^{ikx}$$

be the Fourier series for σ. Then $a_0=\sigma\Gamma>0$. For each $n=1,2,\ldots$, the Fourier series

$$f_n(x)=a_0+\sum_{k\neq 0}(\pi ik)^{-n}a_k e^{ikx}$$

defines an absolutely continuous measure μ_n with nth distributional derivative $(\sigma-\sigma\Gamma m)/\pi^n$. For $n\geqq 2$,

$$|f_n(x)-a_0|\leqq\sum_{k\neq 0}|a_k|/(\pi^n|k|^n).$$

Since $|a_k|\leqq\|\sigma\|_0$ we can write $|f_n(x)-a_0|\leqq\|\sigma\|_0 d_n$, where

$$d_n=(2/\pi^n)\sum_{k=1}^{\infty}k^{-n}\leqq 1/(3\pi^{n-2})\quad(\text{for } n\geqq 2).$$

Because $d_n\to 0$ as $n\to\infty$ and $a_0>0$, we can choose N large enough so that $0<d_N\leqq a_0/\|\sigma\|_0$. But then $|f_N(x)-a_0|\leqq a_0$. In particular $f_N(x)\geqq 0$, implying that μ_N is in $\mathscr{D}_{0+}'$. Since

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$\mu_N^{(N)} = (\sigma - \sigma\Gamma m)/\pi^N$ and $\mu_N\Gamma = a_0 = \sigma\Gamma$, it follows that

$$\sigma = \pi^N \mu_N^{(N)} + \mu_N \Gamma m = L^N \mu_N.$$

Thus $\phi = L^M\sigma = L^{M+N}\mu_N$ so that ϕ is in $\mathscr{D}'_{(N+M)+}$ and consequently in P.

To characterize K, we note that the set of nonnegative constant functions on Γ is contained in K, and K is contained in

$$\{f \in \mathscr{D} : f(P) \geqq 0\}.$$

Since for each $x \in \Gamma$ and $\lambda > 0$, the measures m, $\lambda\delta_x + (1-\lambda)m$ and $(1+\lambda)m - \lambda\delta_x$ are in P, we obtain that each member of $\{f \in \mathscr{D} : f(P) \geqq 0\}$ satisfies

$$\left(1 - \frac{1}{\lambda}\right)\bar{f} \leqq f(x) \leqq \left(1 + \frac{1}{\lambda}\right)\bar{f} \quad \text{and} \quad \bar{f} \geqq 0,$$

and is thus a nonnegative constant function.

The cone K is the dual in $\mathscr{D}$ of P but P is not the dual of K. The dual $\mathscr{K}$ of K is just the set of distributions ϕ in $\mathscr{D}'$ having $\phi(1) \geqq 0$. It is not difficult to show that $\mathscr{K}$ is the closure of P in the Schwartz topology. $\mathscr{K}$ fails to be a positive cone for $\mathscr{D}'$ since $\mathscr{K} \cap (-\mathscr{K})$ contains all ϕ in $\mathscr{D}'$ having $\phi(1) = 0$.

Let $B = \bigcup_{n=0}^{\infty} L^n B_0$ in $\mathscr{D}'$. Then B is a base for P and it follows from Proposition 5 that

$$B = \{\phi \in \mathscr{D}' : \phi(1) = 1\}.$$

B is not linearly compact. Thus (see **[9]**) $\mathscr{D}'$ is not lattice ordered by P. Moreover, B has no extreme points and is, of course, not compact in any locally convex topology for $\mathscr{D}'$ (in contrast to the base $\oplus B_0$ discussed before Theorem 2).

We consider an interpretation of the above structures for the space H of real-valued harmonic functions on the unit disc Δ in the plane. Let H_+ denote the cone of pointwise nonnegative members of H and let H^0 be $H_+ - H_+$. We denote by H^n the linear span of derivatives

$$u^{(j)} = \frac{\partial^j u}{\partial \theta^j} \quad (j = 0, 1, \ldots, n)$$

of functions in H^0 (written in polar coordinates) and let $\mathscr{H} = \bigcup_{n=0}^{\infty} H^n$. Each H^n is isomorphic to $\mathscr{D}'_n$ (see **[6]** and **[7]**). The correspondence is obtained as follows. For $0 \leqq r < 1$ the Poisson function

$$\mathbf{P}_r(\theta) = \frac{1-r}{1 - 2r\cos\theta + r^2}$$

is in $\mathscr{D}$. For each ϕ in $\mathscr{D}'$ one obtains by convolutions $h_r = \mathbf{P}_r * \phi$ a harmonic function $h(r, \theta) = h_r(\theta)$. Thus $\mathscr{H}$ is isomorphic to $\mathscr{D}'$. The map L defined from H into H by

$$L(u) = \pi u^{(1)} + u(0),$$

when restricted to $\mathscr{H}$, corresponds to the map L discussed previously. Theorem 2 adapted to this context says that $\mathscr{H}$ is isomorphic to a quotient of $\bigoplus_{n=0}^{\infty} H^0$ in its direct sum ordering. The quotient cone P induced on $\mathscr{H}$ is $\bigcup_{n=0}^{\infty} L^n H_+$, and, as in Proposition 5 (since

$$h(0) = \mathbf{P}_0 * \phi = \phi(1)),$$
$$P = \{h \in \mathscr{H} : h(0) > 0\} \cup \{0\}.$$

We now have the following extension of the classical Herglotz theorem for H_+ (see [**1**]).

PROPOSITION 6. *For each* $h \in \mathscr{H}$ *having* $h(0) > 0$ *there is a least integer* n *and a unique positive measure* μ *in* $\mathscr{D}'_{0+}$ *such that*

$$h(r, \theta) = \int_\Gamma (L^n\mathbf{P})(r, \theta - t)\, d\mu(t).$$

Proof. Let h be in $\mathscr{H}$ with $h(0) > 0$; i.e., let h be in P. Since $P = \bigcup L^n H_+$ there exists a least integer n and a unique element $u \in H_+$ such that $h = L^n u$. The classical Herglotz theorem implies that

$$u(r, \theta) = \int_\Gamma \mathbf{P}(r, \theta - t)\, d\mu(t)$$

for a unique μ in $\mathscr{D}'_{0+}$, and thus

$$h(r, \theta) = (L^n u)(r, \theta) = \int_\Gamma (L^n\mathbf{P})(r, \theta - t)\, d\mu(t).$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ARKANSAS
FAYETTEVILLE, ARKANSAS 72701, U.S.A.

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