HOW COMPLETE ARE CATEGORIES OF ALGEBRAS?

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Completeness properties of (i) the category $Alg(T)$ of T-algebras
over a functor $T : X \to X$ and (ii) the subcategory X^T in the case
where $T = (T, \mu, \eta)$ is a monad, are investigated. It is known that
if X is compact, then each X^T is compact; we present a functor
$T : Set \to Set$ such that $Alg(T)$ is non-compact, although it is
hypercomplete. If T either preserves epis or has a rank, we
prove that $Alg(T)$ and X^T are topologically algebraic over X
provided X satisfies mild additional hypotheses. Nevertheless, a
natural monad over the category of Δ-complete posets is exhibited
such that its category of algebras is solid, but not topologically
algebraic, over Set.

Introduction

Two examples of categories of algebras form the background of the
present paper. One is a hypercomplete category which is not compact, thus
answering the problem as to whether these two properties are equivalent,
put by Börger, Tholen, Wischnewsky and Wolff [6]. The other is a solid

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category of unary ordered algebras which is not topologically-algebraic
over Set , the first "natural" example of this sort.

By categories of algebras are meant the categories $Alg(T)$ given
by a functor $T : X \to X$: objects are T-algebras, that is maps
$TX \xrightarrow{x} X$ (no axions), morphisms from $TX \xrightarrow{x} X$ to $TY \xrightarrow{y} Y$ are maps
$f : X \to Y$ in X with $f.x = y.Tf$. The four properties mentioned above
are studied for categories $Alg(T)$ and, in the case where T is the
functor-part of a monad T , for their full subcategories X^T . For the
latter, compactness as well as hypercompleteness are lifted from X to
X^T , see [15]. The same is, consequently, true for $Alg(T)$ if T is a
varietor, that is, if each X-object generates a free T-algebra : then
$Alg(T) = X^{T^*}$ for the free monad T^* generated by T , see [4]. The
quotient P^* of the covariant power-set functor obtained by merging
finite sets has the property that $Alg(P^*)$ is hypercomplete but non-
compact. It is an open problem whether compactness of $Alg(T)$ implies
that T is a varietor.

The concepts of solid (= semi-topological) and topologically-algebraic
categories have both been created in order to express the common features
of topological and algebraic concrete categories. But whereas each
topological category is clearly topologically-algebraic (which is slightly
stronger than solid, see [5] and [7]), the situation with algebraic
categories is not that clear. For "reasonable" base categories we prove
that $Alg(T)$ and X^T are topologically-algebraic whenever T either
preserves epis, or has a rank. However, the non-cocomplete monadic
category over graphs, presented in [1], is not solid (since solid
functors lift colimits). We present a monad T over the category X of
Δ-complete posets such that X^T , as a concrete category over Set , is
solid, but not topologically-algebraic. However, it is not known whether
a monadic category X^T is topologically-algebraic (over X) whenever it
is solid.

1. Compactness and Hypercompleteness

1.1 Recall from [6] that a category K is said to be <u>hypercomplete</u>
if it has limits of all (possibly large) diagrams D with the property
that for each $K \in K$ the collection of all compatible sources of D with
the domain K is small. (Since these sources are in a bijective corres-
pondence with hom$(K, \lim D)$, this is the strongest completeness possible
in a locally small category.)

1.2 A class M of monos in a cocomplete category X is said to be
<u>chain-cocomplete</u> if for each M-chain the colimit maps are in M , and the
factorization map of any compatible source of M-monos is also a M-mono.

1.3 PROPOSITION. *Let X be a cocomplete, hypercomplete, wellpowered*
category, and let M be a chain-cocomplete class of monos containing
all coproduct injections. Then Alg(T) is hypercomplete for each functor
$T : X \to X$ preserving M-monos ($T M \subseteq M$) and such that there is an object
X isomorphic to $TX + Y$ for some separator Y .

Proof. It is proved in [18] that the separator Y generates a free
T-algebra, that is, there is a T-algebra K and a universal arrow $Y \to UK$
(where $U : $ Alg$(T) \to X$ denotes the forgetful functor). Let D be a
diagram of T-algebras satisfying the condition of 1.1 in Alg(T) . Then
the underlying diagram $U.D$ satisfies the corresponding condition in Set:

(a)   the compatible sources of $U.D$ with the domain Y form a small
collection because they are in a bijective correspondence with the
compatible sources of D with domain K , and

(b)   the compatible sources of $U.D$ with any other fixed domain Y'
form a small collection because Y' is a quotient of a co-power of Y .
This proves that $\lim U.D$ exists in X , and since U creates limits
[14], $\lim D$ exists also.

COROLLARY. *Alg(T) for a set functor T is hypercomplete whenever*
there is a set $X \neq \emptyset$ with card $TX \leq$ card $X - 1$.

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In fact, the hypotheses of the proposition are satisfied for M = monos. Each set functor T preserves non-empty monos, and we can redefine T on the empty set, putting $T'X = TX$ for all $X \neq \emptyset$ and $T'\emptyset = \emptyset$ (and analogously on maps) to obtain a set functor T' preserving all monos. The hypercompleteness of $Alg(T)$ is clearly equivalent to that of $Alg(T')$.

Open problem. If $Alg(T)$ is hypercomplete for a set functor T , does it follow that there is a set $X \neq \emptyset$ with card $TX \leq$ card $X - 1$?

1.4 EXAMPLE. For the power-set functor $P[PX = \exp X$ and $Pf : M \rightarrow f(M)]$, the category $Alg(P)$ is not hypercomplete.

In fact, for each cardinal $n > 0$ we can choose a map $x_n : Pn \rightarrow n$ satisfying

$$x_n(M) \notin M \quad \text{for any set } M \subsetneq n .$$

Then the large discrete diagram of all of these algebras (n, x_n) does not have a compatible source in $Alg(P)$. In fact, consider an arbitrary P-algebra (Y, y) . Given a homomorphism $f : (Y, y) \rightarrow (n, x_n)$ for some cardinal n , then

$$x_n(PfP(Y)) = f.y(Y) \in Pf(Y)$$

which implies $Pf(Y) = n$, that is, f is onto. Thus, card $Y \leq n$.

On the other hand, in a hypercomplete category each diagram obviously has a compatible source.

1.5 Recall from [11] that a category K is said to be <u>compact</u> if each functor with domain K preserving all (possibly large) colimits existing in K is a left adjoint.

1.6 PROPOSITION. *$Alg(P^*)$ is hypercomplete but non-compact for the quotient P^* of the power-set functor modulo finite sets, that is, the following functor*

$$P^*X = \{M | M \subseteq X \text{ and } M \text{ is infinite, or } M = \emptyset\}$$

and

$$P^*f(M) = Pf(M) \quad \text{if} \quad Pf(M) \text{ is infinite, else,} \quad P^*f(M) = \emptyset .$$

Proof. $\text{Alg}(P^*)$ is hypercomplete by 1.3, since for a two-point set X we have card $P^*X = 1 = \text{card } X - 1$.

Let $\tilde{P}$ denote the quotient of the power-set functor modulo singletons, that is, $\tilde{P}X = \{M \mid M \subseteq X \text{ and card } M \neq 1\}$ and $\tilde{P}f(M) = Pf(M)$ if card $Pf(M) \neq 1$, else $\tilde{P}f(M) = \emptyset$. We shall prove that the natural full embedding

$$E : \text{Alg}(P^*) \to \text{Alg}(\tilde{P})$$

assigning to each P^*-algebra $P^*X \xrightarrow{x} X$ the corresponding $\tilde{P}$-algebra

$\tilde{P}X \xrightarrow{\varepsilon X} P^*X \xrightarrow{x} X$ (for the obvious epitransformation $\varepsilon : \tilde{P} \to P^*$) preserves colimits, but is not a left adjoint. The latter is obvious : for $X = \{0, 1, 2\}$ define $x : \tilde{P}X \to X$ by $x(M) = 0$ for all $M \neq X$ and $x(X) = 1$. Then the $\tilde{P}$-algebra (X, x) does not have a co-universal arrow from E because both $\{0, 1\}$ and $\{0, 2\}$ are subalgebras lying in $E(\text{Alg}(P^*))$ but the algebra (X, x) itself does not lie there.

To prove that E preserves colimits, we can restrict ourselves to non-empty diagrams $D : \mathcal{D} \to \text{Alg}(P^*)$, since E clearly preserves initial (= singleton) algebras. We prove the following statement :

(*) If D has a colimit $e_d : Dd = (X_d, x_d) \to (Y, y)$, $d \in \mathcal{D}^{obj}$

then there exists d_o such that e_{d_o} is onto.

Once (*) is established, we shall see that the diagram $E.D$ has the colimit $e_d : (X_d, x_d.\varepsilon_{X_d}) \to (Y, y.\varepsilon_Y)$ as follows. Let $f_d : (X, x_d.\varepsilon_{X_d}) \to (Z, z)$ be a compatible family in $\text{Alg}(\tilde{P})$, then defining $\overline{z} : P^*Z \to Z$ by $\overline{z}(M) = z(M)$ for all $M \in P^*Z$, we have clearly a compatible family $f_d : (X_d, x_d) \to (Z, \overline{z})$ in $\text{Alg}(P^*)$. There is a unique P^*-homomorphism $f : (Y, y) \to (Z, \overline{z})$ with $f_d = f.e_d$, and it is sufficient to show that $f : (Y, y.\varepsilon_Y) \to (Z, z)$ is a $\tilde{P}$-homomorphism. For

each $M \in \tilde{P}Y$ we are to show that $z.\tilde{P}f(M) = f.y.\varepsilon_Y(M)$. This is clear if $f(M)$ is infinite, since then $\tilde{P}f(M) = P^*f(M)$. For $f(M)$ finite there exists $M' \subseteq X_{d_o}$ finite with $f(M) = f_{d_o}(M')$, see $(*)$, and hence,

$$z.\tilde{P}f(M) = z.\tilde{P}f_{d_o}(M') = f_{d_o}.x_{d_o}.\varepsilon_{X_{d_o}}(M') = f_{d_o}.x_{d_o}(\emptyset)$$

because f_{d_o} is a $\tilde{P}$-homomorphism. Since e_{d_o} is a $\tilde{P}$-homomorphism, we have $e_{d_o}.x_{d_o}(\emptyset) = y(\emptyset)$ and thus,

$$z.\tilde{P}f(M) = f.e_{d_o}.x_{d_o}(\emptyset) = f.y(\emptyset) = f.y.\varepsilon_Y(M) .$$

In order to prove $(*)$, we assume the contrary, and derive a contradiction. Let n be a cardinal larger than $\operatorname{card} Y$ and assume, for simplicity, that $Y \cap n = \emptyset$, that is, Y does not contain ordinals $i < n$. Define a P^*-algebra $(Y \cup n, \overline{y})$ by

$$\overline{y}(M) = \begin{cases} y(M) & \text{for all } M \in P^*X, M \neq Y \\ i & \text{for } M = Y \cup i , i < \eta \\ 0 & \text{otherwise} . \end{cases}$$

Denote by $e_d' : X_d \to Y \cup n$ the range-extension of e_d , then $e_d' : (X_d, x_d) \to (Y \cup n, \overline{y})$ is a compatible family of the given diagram : for each $M \in P^*X_d$ we have $P^*e_d'(M) \neq Y$ (else, e_d would be onto, contrary to our assumption) and hence, $\overline{y}.P^*e_d'(M) = y.Pe_d(M)$, which proves that e_d' is a P^*-homomorphism. Let $h : (Y, y) \to (Y \cup n, \overline{y})$ be the unique P^*-homomorphism with $h.e_d = e_d'$, that is, with

$$h(x) = x \quad \text{for all} \quad x \in \cup e_d(X_d) .$$

We shall prove by induction on $i < n$ that $Y \cup i \subseteq h(Y)$; this clearly contradicts $\operatorname{card} h(Y) \leq \operatorname{card} Y < n$.

First step: $Y \subseteq h(Y)$. The set $Y_o = Y \cap h(Y)$ is a subalgebra of (Y, y) . In fact, for each $M \in P^*Y_o$ there is $N \in P^*Y$ with

$M = P^*f(N)$, and we can suppose that $\bar{y}(M) = y(M)$ (for otherwise $M = Y$ which implies $Y \subseteq h(Y)$ anyhow). Thus,

$$y(M) = \bar{y}.P^*f(N) = h.y(N) \in h(Y) ,$$

and this implies $y(M) \in Y_O$. The subalgebra Y_O contains $\cup\ e_d(X_d)$ and hence, we can form range-restrictions of the P^*-homomorphisms e_d to Y_O . This clearly implies that $Y_O = Y$, that is, that $Y \subseteq h(Y)$.

At a successor ordinal we show that $Y \cup i \subseteq h(Y)$ implies $i \in h(Y)$ [and thus, $Y \cup (i + 1) \subseteq h(Y)$] . Let $M \in P^*Y$ be a set with $Ph(M) = Y \cup i$ then

$$i = \bar{y}.P^*h(M) = h.y(M) \in h(Y) .$$

The limit steps are clear.

2. Topological-algebraicity and solidness

2.1 A concrete category K over a base-category X is said to be topologically-algebraic if it has free objects and (epi, initial)-factorization of sources. This concept, introduced by Y.H. Hong [10] and S.S. Hong [9] , aims to express the common features of topological and algebraic categories. The same is the aim of the following, somewhat weaker, property : K is said to be solid if each structured sink $|K_i| \xrightarrow{f_i} X(i \in I)$, $K_i \in K$, has a semifinal lift, that is an object $K \in K$ and a morphism $\eta : X \to |K|$ such that

(a)   all $\eta.f_i : K_i \to K$ are K-morphisms and

(b)   if $h : X \to |L|$ is a map such that all $h.f_i : K_i \to L$ are K-morphisms, then there is unique $h^{\#} : K \to L$ in K with $h = h^{\#}.\eta$. The concept of a solid category was introduced by [8], [16] and [17], and has often been called a semi-topological category. The following has been proved in [5] and [7] for each cocomplete, cowellpowered base-category X :

(i)   Topologically-algebraic $\Rightarrow$ solid $\Rightarrow$ cocomplete and has free objects; and

(ii) The reverse implications hold whenever the concrete category K is cowellpowered.

Moreover, each reflective full subcategory of a solid category is solid, and solid forgetful functors are closed under composition (and they form the composition hull of topologically-algebraic forgetful functors).

2.2 LEMMA. *In each monadic concrete category, monosources are initial.*

Proof. Let $m_i : (X, x) \to (Y_i, y_i)$, $i \in I$, be a monosource in X^T .
Then $m_i : X \to Y_i$ $(i \to I)$ is a monosource in X . If $p, q : Z \to X$ fulfil $m_i \cdot p = m_i \cdot q$, then for the homomorphisms $x.Tp, x.Tq : (TZ, \mu_Z) \to (X, x)$ we have

$$m_i.x.Tp = y_i.T(m_i.p) = y_i.T(m_i.q) = m_i.x.Tq ,$$

and hence $x.Tp = x.Tq$. Consequently

$$p = x.\eta_X.p = x.Tp.\eta_Z = x.Tq.\eta_Z = q .$$

Let (Z, z) be a T -algebra, and let $h : Z \to X$ be a map such that each $m_i.h$ is a homomorphism, that is,

$$m_i.h.z = y_i.Tm_i.Th = m_i.x.Th \quad \text{(for all } i \text{)} .$$

Then $h.z = x.Th$, that is, h is a homomorphism.

2.3 PROPOSITION. *Let X be a complete category with intersections. Then each monadic category over X which is cowellpowered with respect to extremal epis is topologically-algebraic.*

Proof. By the preceding Lemma, it is sufficient to show that each category X^T , cowellpowered with respect to extremal epis, has (epi, monosource)-factorizations of sources.

Since monadic functors create (large) limits, see [14], the category X^T is complete and has intersections. Consequently, it has (extremal epi,

mono)-factorizations of morphisms; it follows immediately that each small source $f_i : A \to A_i$ $(i \in I)$ has an (extremal epi, monosource)-factoriz-ation: factor the corresponding morphism $A \to \prod\limits_{i \in I} A_i$. Finally, for large sources, we factor a small subsource representing all of the extremal co-images of the members of the source.

EXAMPLE. There is a monad T over Pos, the category of posets, such that Pos^T is not topologically-algebraic. In fact, a monad T such that Pos^T is not cocomplete is exhibited in [3], and topologically-algebraic functors lift colimits.
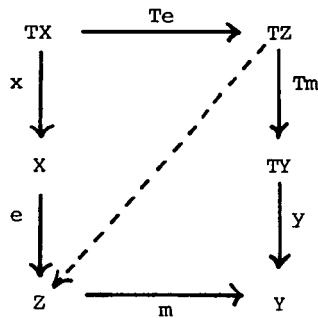
2.4 PROPOSITION. *Let X be a cocomplete, E-cowellpowered category with a morphism factorization system (E, M) . For each functor $T : X \to X$ preserving E-epis (that is, $TE \subseteq E$), equivalent are:*

(i) T is a varietor;

(ii) $Alg(T)$ is topologically-algebraic.

Remark. Furthermore, for each monad T preserving E-epis, X^T is topologically-algebraic - the proof is analogous.

Proof. ii $\to$ i is clear since a topologically-algebraic category always has free objects. To prove i $\to$ ii, it is sufficient to show that if T is a varietor then $Alg(T)$ has $(E$, monosource)-factorization of source - since $E \subseteq Epi$ and each monosource is initial, this will prove that $Alg(T)$ is topologically-algebraic.

It is proved in [3] that $Alg(T)$ is cocomplete. It has (E, M)-factorizations of morphisms because for each homomorphism $f : (X, x) \to (Y, y)$ with (E, M)-factorization $f = m.e$ in X , the diagonal fill-in yields an operation map such that m and e become homomorphisms:

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$$
\begin{array}{ccc}
TX & \xrightarrow{\ \ Te\ \ } & TZ \\
\downarrow{\scriptstyle x} & & \downarrow{\scriptstyle Tm} \\
X & & TY \\
\downarrow{\scriptstyle e} & & \downarrow{\scriptstyle y} \\
Z & \xrightarrow{\ \ m\ \ } & Y
\end{array}
$$

For each source $f_i : A \to B_i (i \in I)$ let $e : A \to A'$ be the cointersection of all E-quotients of A in $\mathrm{Alg}(T)$ through which each f_i factors, then $e \in E$ and we have factorizations $f_i = f_i'.e\ (i \in I)$. The source $f_i' : A' \to B_i (i \in I)$ is a monosource : if $p, q \ \vdots\ C \to A'$ are merged by each f_i' , then each f_i' factors through the coequalizer c of p and in $\mathrm{Alg}(T)$, and hence, each f_i factors through $c.e \in E$ - consequently, c is an isomorphism.

2.5 DEFINITION. Let X be a category with a morphism factorization system (E, M) and with colimits of M-chains.

A functor $T : X \to X$ is said to <u>have rank</u> k , where k is a regular cardinal, if for each chain $V : k \to X$ of M-monos the canonical morphism colim $(T.V) \to T$ (colim V) is an E-epi.

2.6 Remark. It is often the case that the class M of all monos is chain-cocomplete (see 1.2), and that rank k implies that colimits of k-chains of M-monos are actually preserved (for example, this holds for mono-preserving functors on $X = \mathrm{Set}, \mathrm{Top}, \mathrm{Pos},$ varieties of finitary algebras.)

For functors T with rank, the category $\mathrm{Alg}(T)$ was proved to be solid by Koubek and Reiterman [13] (assuming X cocomplete and E-cowellpowered). Using the technique developed by Barr [4], we are able to prove more:

2.7 THEOREM. *Let* X *be a cocomplete, E-cowellpowered category with a factorization system* (E,M) *such that* M *is chain-cocomplete. Then* $Alg(T)$ *is topologically-algebraic for each functor* $T : X \to X$ *with rank.*

Remark. Furthermore, for each monad T with rank, X^T is topologically-algebraic. The proof is analogous.

Proof. It has been proved by Koubek and Reiterman in [13] that T is a varietor and $Alg(T)$ is cocomplete.

Thus, it is sufficient to prove that $Alg(T)$ is cowellpowered with respect to extremal epis. Then $Alg(T)$ has (extremal epi, monosource)-factorization of sources : for each source $f_i : A \to B_i$ $(i \in I)$ let $e : A \to A'$ denote the cointersection of all extremal quotients of A in $Alg(T)$ through which each f_i factors $(i \in I)$. Then each f_i factors as $f_i = f'_i.e$, and the source $f'_i : A' \to B_i$ $(i \in I)$ is clearly a monosource. Since T is a varietor, monosources are initial in $Alg(T)$ by 2.2, and this will establish $Alg(T)$ as a topologically-algebraic category.

Let k denote the rank of T . For each object X of X define a small set $X^* = \underset{i<k}{\cup} M_i$ of objects inductively as follows:

$$M_0 \quad = \{X\}$$

$$M_{i+1} = \begin{cases} \text{closure of } M_i \text{ under } T\text{-images and} \\ \text{colimits of diagrams with less than} \\ k \text{ objects } (i \text{ even});\\ \text{a choice set of all } E\text{-quotients of} \\ \text{all } M_i\text{-objects } (i \text{ odd}), \end{cases}$$

$$M_i \quad = \underset{j<i}{\cup} M_j \quad \text{for limit ordinals } i .$$

The proof will be concluded if we verify that for each extremal epi $f : (X, x) \to (Y, y)$ in $Alg(T)$, Y is isomorphic to an object in X^* : since X^* is a small set, it follows that $Alg(T)$ is cowellpowered with respect to extremal epis.

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Define a chain $V : k + 1 \to X$ and a natural M-transformation $m_i : V_i \to Y$ $(i \leq k)$ by the following induction:

(i)   factor f as $e_0 : X \to V_0$ in E followed by $m_0 : V_0 \to Y$ in M ;

(ii)   given $m_i : V_i \to Y$, factor the morphism $(m_i, y.Tm_i) : V_i + TV_i \to Y$ as an E-epi (having compoents $V_{i,i+1} : V_i \to V_{i+1}$ and $d_i : TV_i \to V_{i+1}$)
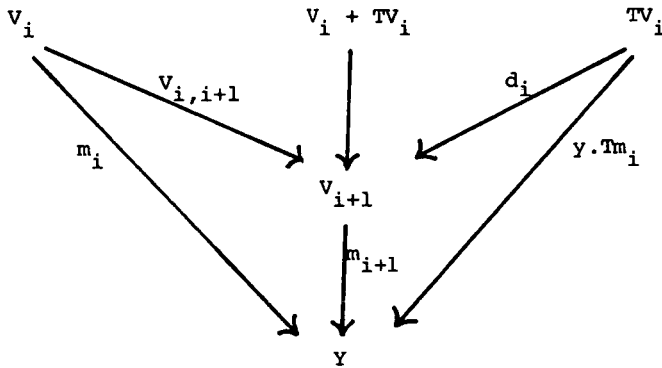
followed by $m_{i+1} : V_{i+1} \to Y$ in M ;

(iii)   for each limit ordinal i , $V_i = \operatorname*{colim}_{j < i} V_j$ and $m_i = \operatorname*{colim}_{j < i} m_j$ (which is in M since M is chain-cocomplete).

Observe that $V_i \in X^*$ for each $i \leq k$. Since T has rank k , the canonical morphism

$$c : \operatorname*{colim}_{i < k} TV_i \to TV_k$$

is an E-epi. Let $d : \operatorname{colim} TV_i \to V_k$ be the colimit map of the morphisms $d_i : TV_i \to V_{i+1} (i < k)$, then the square below commutes. In fact, the components of $y.Tm_k.c$ are $y.Tm_k.TV_{ik} = y.Tm_i = m_{i+1}.d_i$, and $m_{i+1}.d_i$ are the components of $m_k.d$. By the diagonal fill-in, there

$$\begin{array}{ccc}
\text{colim } TV_i & \xrightarrow{\quad c \quad} & TV_k \\
\downarrow d & & \downarrow y.Tm_k \\
V_k & \xrightarrow{\quad m_k \quad} & Y
\end{array}$$

is $b_k : TV_k \to V_k$ such that

$$m_k : (V_k, v_k) \to (Y, y)$$

becomes a homomorphism. Since f factors through this monomorphism in X, it factors also in Alg(T) (because monos are initial) and hence, m_k is an isomorphism. Thus, Y is isomorphic to an object of X^*.

In order to study the solidness of the categories of T-algebras, we shall work with the concept of a <u>partial T-algebra</u> as a partial morphism from TX to X. Thus, a partial T-algebra is represented by a span $TX \xleftarrow{m} X_0 \xrightarrow{x} X$ with m a mono. The category P Alg(T) of partial T-algebras has as morphisms from $TX \xleftarrow{m} X_0 \xrightarrow{x} X$ into $TX' \xleftarrow{m'} X_0' \xrightarrow{x'} X'$ those X-maps $f : X \to X'$ for which there exists a (unique) $f_0 : X_0 \to X_0'$ such that the following diagram commutes. This category contains Alg(T)

$$\begin{array}{ccccc}
TX & \xleftarrow{\ m\ } & X_0 & \xrightarrow{\ x\ } & X \\
\downarrow Tf & & \downarrow f_0 & & \downarrow f \\
TX' & \xleftarrow{\ m'\ } & X_0' & \xrightarrow{\ x'\ } & X'
\end{array}$$

as a full subcategory in the obvious sense.

2.9 PROPOSITION. *Let X be a finitely complete category with intersections. Then for each $T : X \to X$ the category P Alg(T) is monotopological, that is, has initial lifts of structured monosources.*

Proof. Let us work with a larger category $R \, \text{Alg}(T)$ of relational algebras, that is, relations from TX to X (represented by a span $TX \xleftarrow{m} X_0 \xrightarrow{x} X$ such that m and x are collectively mono) with morphisms defined as above. We prove that $R \, \text{Alg}(T)$ is topological and then we conclude that $P \, \text{Alg}(T)$ is monotopological as follows : given a structured monosource in $P \, \text{Alg}(T)$:

$$X \xrightarrow{f_i} (TY_i \xleftarrow{m_i} X_i' \xrightarrow{y_i} Y_i) \quad (i \in I)$$

we can find its initial lift $TX \xleftarrow{m} X' \xrightarrow{x} X$ in $R \, \text{Alg}(T)$ and then m is a mono, hence, this is an initial lift in $P \, \text{Alg}(T)$. In fact, let $m.a_1 = m.a_2$:
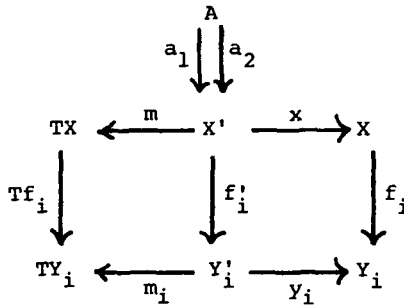
For each $i \in I$ we have $f_i'.a_1 = f_i'.a_2$ (since m_i is a mono, and $m_i.f_i'.a_1 = m_i.f_i'.a_2$) and hence, $f_i.(x.a_1) = f_i.(x.a_2)$, and since (f_i) is a monosource, we get $x.a_1 = x.a_2$. Since x and m are collectively mono, this proves $a_1 = a_2$.

To prove that $R \, \text{Alg}(T)$ is topological, we use that fact that X has (extremal epi, mono)-factorizations of morphisms. Let a structured sink be given in $R \, \text{Alg}(T)$:

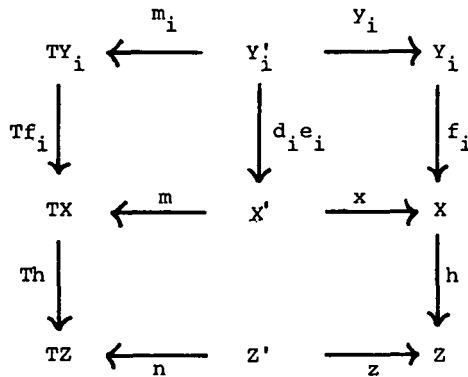$$(TY_i \xleftarrow{m_i} Y_i' \xrightarrow{y_i} Y_i) \xrightarrow{f_i} X \quad (i \in I) .$$

Denote by $\bar{m}_i : Y_i' \to Y_i \times TY_i$ the mono with components y_i and m_i , and

let us factor $(f_i \times Tf_i).\bar{m}_i$ as an extremal epi $e_i : Y'_i \to Y_i{}^*$ followed by a mono $m_i{}^* : Y_i{}^* \to X \times TX$. Since X has intersections, the lattice of subobjects of $X \times TX$ is large-complete, and hence, there is a union

$$m^* = \bigcup_{i \in I} m_i{}^*, \quad m_i{}^* \doteq m^*.d_i \quad \text{for} \quad i \in I .$$

The mono m^* represents a relation $TX \xleftarrow{\ m\ } X' \xrightarrow{\ x\ } X$, and we verify that this is the final lift of the given sink in $R \text{ Alg}(T)$:
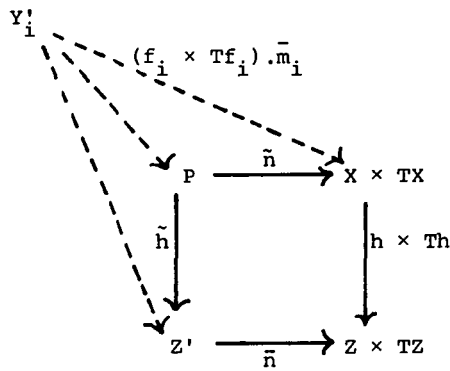
$$
\begin{array}{ccccc}
TY_i & \xleftarrow{\ m_i\ } & Y'_i & \xrightarrow{\ y_i\ } & Y_i \\
{\scriptstyle Tf_i}\downarrow & & \downarrow{\scriptstyle d_i e_i} & & \downarrow{\scriptstyle f_i} \\
TX & \xleftarrow{\ m\ } & X' & \xrightarrow{\ x\ } & X \\
{\scriptstyle Th}\downarrow & & & & \downarrow{\scriptstyle h} \\
TZ & \xleftarrow{\ n\ } & Z' & \xrightarrow{\ z\ } & Z
\end{array}
$$

First, $f_i : (TY \xleftarrow{\ m_i\ } Y'_i \xrightarrow{\ y_i\ } Y) \to (TX \xleftarrow{\ m\ } X' \xrightarrow{\ x\ } X)$ is a homomorphism, that is $(f_i \times Tf_i).\bar{m}_i$ factors through m^* : in fact, $(f_i \times Tf_i).\bar{m}_i = m_i{}^*.e_i = m^*.d_i.e_i$. Next, let $TZ \xleftarrow{\ n\ } Z' \xrightarrow{\ z\ } Z$ be an object and $h : X \to Z$ a morphism such that each $h.f_i (i \in I)$ is a homomorphism. This means that for each $i \in I$,

$$(h \times Th).(f_i \times Tf_i).\bar{m}_i \quad \text{factors through} \quad \bar{n} ,$$

where $\bar{n} : Z' \to Z \times TZ$ is the mono with the components z and n. Let us form the pullback $(\tilde{n}, \tilde{h})$ of $\bar{n}$ and $h \times Th$, see below.

For each i , we see that $(f_i \times Tf_i).\bar{m}_i = m_i{}^*.e_i$ factors through the mono $\tilde{n}$ and hence, $m_i{}^*$ factors through $\tilde{n}$. Therefore, m^* factors through $\tilde{n}$, and this proves that $(h \times Th).m^*$ factors through $\bar{n}$. In

$$Y_i' \dashrightarrow P \xrightarrow{\tilde{n}} X \times TX$$

with $(f_i \times Tf_i)\cdot\bar{m}_i$, $\tilde{h}$, $h \times Th$, $Z' \xrightarrow{\bar{n}} Z \times TZ$

other words, $h : (TX \xleftarrow{m} X' \xrightarrow{x} X) \to (TZ \xleftarrow{n} Z' \xrightarrow{z} Z)$ is a homomorphism.

COROLLARY. *Let* X *be complete and wellpowered. Then each* X^T *and* $Alg(T)$ *is a strongly fibre-small category, that is, every* $X \in X$ *has a small set of representative structured maps from* X *to algebras.*

In fact, since X is wellpowered, P $Alg(T)$ is fibre-small and, being monotopological, it is thus strongly fibresmall. Consequently, each full subcategory is strongly fibresmall.

The equivalence of (ii) and (iii) in the following proposition has been established, under more restrictive hypotheses, in [13].

2.10 PROPOSITION. *Let* $T = (T, \eta, \mu)$ *be a monad on a finitely complete category* X *with intersections and coequalizers. Equivalent are:*
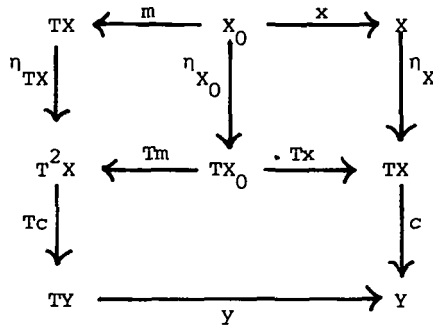
(i) X^T *is solid;*

(ii) X^T *has coequalizers;*

(iii) partial T*-algebras have free* T*-completions, that is,* X^T *is reflective in* P $Alg(T)$.

Proof. $i \to ii$ is clear since solidness implies lifting of colimits.

$ii \to iii$ For each partial T-algebra $TX \xleftarrow{m} X_0 \xrightarrow{x} X$ we form the coequalizer in X^T :

$$(TX_0, \mu) \; \underset{\mu.Tm}{\overset{Tx}{\rightrightarrows}} \; (TX, \mu) \; \overset{c}{\longrightarrow} \; (Y, \; y) \; .$$

Then the following diagram

commutes, and since $y.Tc = c.\mu_X$ implies that $y.TC.\eta_{TX} = y.Tc \; . \; T\eta_X$, we conclude that

$$c.\eta_X \; : \; (TX \xleftarrow{\; m \;} X_0 \xrightarrow{\; x \;} X) \; \to \; (Y, \; y)$$

is a homomorphism in P Alg(T) . To verify that this is the reflection of the given partial T-algebra, let

$$h \; : \; (TX \xleftarrow{\; m \;} X_0 \xrightarrow{\; x \;} X) \; \to \; (Y', \; y')$$

be a homomemorphism in P Alg(T) . Then the T-homomorphism $y'.Th \; : \; (TX, \mu) \to (Y', \; y')$ merges Tx and $\mu.Tm$: since h is a homomorphism, that is $h.x = y'.Th.m$, we have

$$(y'.Th).Tx = y'.T(h.x)$$
$$= y'.Ty'.T^2h.Tm$$
$$= y'.\mu_{Y'}.T^2h.Tm$$
$$= (y'.Th).\mu_X.Tm \; .$$

Thus, $y'.Th$ factors through c , and hence, h factors through $c.\eta_X$.

iii $\to$ i This follows from the preceding proposition.

COROLLARY. *For X as above, Alg(T) is solid if and only if*
T : X → X is a varietor such that Alg(T) has coequalizers.

2.11 EXAMPLE. A concrete category over Set which is solid, but is
not topologically-algebraic.

Let $X = \Delta$Pos denote the category of Δ-complete posets, (that is,
such that each non-empty up-directed set has a join) and Δ-continuous
maps (that is, those preserving directed joins). Let $T : X \to X$ be the
"discretization" functor, assigning to each poset X the discretely
ordered poset TX with the same underlying set. Then Alg(T) is the
category of ordered unary algebras where the order is Δ-complete (and
unrelated to the operation). We shall prove that Alg(T) , as a concrete
category over Set , is solid but not topologically-algebraic.

(a)   Alg(T) is solid over ΔPos .

In fact, Alg(T) is reflexive in P Alg(T) : for each partial
algebra (X, α) (where X is a Δ-complete poset, and $\alpha : D \to X$ is a
map, $D \subseteq X$) the free completion is the following algebra

$$(X + (X - D) \times \omega, \bar{\alpha})$$

where $(X - D) \times \omega$ is discretely ordered, $X + (X - D) \times \omega$ is the
(cardinal) sum, and

$$\bar{\alpha}(x) = \alpha(x) \quad \text{if} \quad x \in D$$
$$\bar{\alpha}(x) = (x, 0) \quad \text{if} \quad x \in X - D$$
$$\bar{\alpha}(x, i) = (x, i + 1) \quad \text{if} \quad (x, i) \in (X - D) \times \omega .$$

(b) ΔPos is solid over Set .

This follows from [2, Theorem 2E3]: for solidness over Set it is
sufficient to have concrete products, small fibres, and bounded generation.
The first two properties are obvious, for the last one we observe that
given a subset M of a Δ-complete poset X , then the closure $\bar{M}$ of X
under all existing joins in X is a subobject of X with
card $\bar{M} \leq 2^{\text{card } M}$.

(c)   Alg(T) is not topologically-algebraic over Set .

For each infinite ordinal i we denote by $(i + 1, \gamma_i)$ the T-algebra with the usual ordering on $i + 1 = \{0, 1, ..., i\}$ and with $\gamma_i(j) = j + 1$ for $j < i$, $\gamma_i(i) = i$. We have a source of inclusion maps.

$$f_i : (X, \alpha) \to (i + 1, \gamma_i) \quad (i \in \text{Ord}, i \geq \omega)$$

where X is the discretely ordered set of all natural numbers, and α is the successor function. Let $g_i : (Y, \beta) \to (i + 1, \gamma_i)$ be an initial source through which our source factors $(f_i = g_i.e)$ – we shall derive a contradiction by exhibiting, for each ordinal i , a strictly increasing chain $y_0 < y_1 < ... < y_j < ...$ $(j < i)$ in Y with $g_k(y_j) = j$ for all $k \geq j$.

First, put $y_0 = e(0)$. We have $g_k(y_0) = f_k(0) = 0$.

Given such an i-chain $y_0 < y_1 < ... < y_j < ...,$ put

$$y_i = \bigvee_{j<i} y_j \quad \text{if } i \text{ is a limit ordinal}$$

$$y_i = \beta(y_{i-1}) \quad \text{if } i \text{ is isolated .}$$

In the first case, we obtain a strictly increasing $(i + 1)$-chain, and by the Δ-continuity of g_k we have $g_k(y_i) = \bigvee_{j<i} g_k(y_j) = \bigvee_{j<i} j = i$. In the latter one, we have $g_k(y_i) = g_k.\beta(y_{i-1}) = \gamma_k.g_k(y_{i-1}) = \gamma_k(i-1) = i$. It remains to prove that $y_{i-1} < y_i$. First observe that $y_{i-1} \neq y_i$ (because if $y_{i-1} = y_i$, then y_i is a fixpoint of β and hence, $g_k(y_i)$ is a fixpoint of γ_k , but this is not true for $k > i$). Define a T-algebra $A = (X + \{0, 1, ..., i\}, \delta)$ on the coproduct of X and the chain $i + 1$ as follows : δ agrees with α on X , and δ is the successor map on the second summand except that $\delta(i) = 0 \in X$. Define $h : X + \{0, 1, ..., i\} \to Y$ by $h(n) = \beta^n(y_i)$ for $n \in X$ and $h_j = y_j$ for $j = 0, 1, ..., i$. This map clearly has the property that $g_k.h : A \to (k + 1, \gamma_k)$ are homomorphisms for all $k \geq \omega$. Consequently,

$h : A \rightarrow (Y, \beta)$ is a homomorphism, and thus, $y_{i-1} = h(i - 1) \leq h(i) = y_i$.

Remark. A solid category over Set which is not topologically algebraic was first exhibited by Börger and Tholen [5].

Open problem. Does there exist a monad on a "reasonable" category X such that the Eilenberg-Moore category is solid, but not topologically-algebraic over X ?

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