ENDOMORPHISM REGULAR OCKHAM ALGEBRAS OF FINITE BOOLEAN TYPE

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Abstract. If $(L;f)$ is an Ockham algebra with dual space $(X;g)$, then it is known that the semigroup of Ockham endomorphisms on L is (anti-)isomorphic to the semigroup $\Lambda(X;g)$ of continuous order-preserving mappings on X that commute with g. Here we consider the case where L is a finite boolean lattice and f is a bijection. We begin by determining the size of $\Lambda(X;g)$, and obtain necessary and sufficient conditions for this semigroup to be regular or orthodox. We also describe its structure when it is a group, or an inverse semigroup that is not a group. In the former case it is a cartesian product of cyclic groups and in the latter a cartesian product of cyclic groups each with a zero adjoined.

An *Ockham algebra* $(L;f)$ is a bounded distributive lattice L on which there is defined a dual endomorphism f. For the basic properties of Ockham algebras we refer the reader to [**1**]. The most obvious example of an Ockham algebra is, of course, a boolean algebra $(B;')$. In general, a boolean lattice can be made into an Ockham algebra in many different ways. Throughout what follows we shall assume that the Ockham algebra $(L;f)$ is of *finite boolean type*, in the sense that L is a finite boolean lattice and f is a dual automorphism. Then $(L;f)$ necessarily belongs to the Berman class $\mathbf{K}_{p,0}$ for some p. If $L \simeq \mathbf{2}^k$ then, by [**1**, Chapter 4], the dual space $(X;g)$ is such that X is discretely ordered with $|X| = k$, and g is a permutation on X such that $g^{2p} = \mathrm{id}_X$. We shall denote by $X_1, \dots, X_m$ the orbits of g. For each i we choose and fix a representative $x_i \in X_i$. Defining $c_i = |X_i|$, for each i, we therefore have

$$X_i = \{x_i, g(x_i), g^2(x_i), \dots, g^{c_i-1}(x_i)\}.$$

Consider the set $\Lambda(X;g)$ consisting of those mappings $\vartheta: X \to X$ that commute with g. By duality [**2**] we know that $(\Lambda(X;g), \circ)$ is a semigroup that is (anti-)isomorphic to the semigroup $\mathrm{End}((L;f), \circ)$ of Ockham endomorphisms on L. By considering $\Lambda(X;g)$ we can therefore obtain properties of $\mathrm{End}(L;f)$. Our principal objective is to determine precisely when this semigroup is regular. For this purpose, we begin by observing the following results.

THEOREM 1. *If $\vartheta \in \Lambda(X;g)$, then for every X_i there exists an X_j such that $\vartheta(X_i) = X_j$.*

Proof. Given x_i, let j be such that $\vartheta(x_i) \in X_j$. Observe first that every $y \in \vartheta(X_i)$ is of the form $y = \vartheta(g^{r_i}(x_i))$, where $0 \leqslant r_i < c_i$. Since ϑ, g commute, we have $y = g^{r_i}(\vartheta(x_i)) \in X_j$. It follows that $\vartheta(X_i) \subseteq X_j$. On the other hand, since $\vartheta(x_i) \in X_j$ we have $\vartheta(x_i) = g^{s_i}(x_j)$, where $0 \leqslant s_i < c_j$. It follows that, since $g^{2p} = \mathrm{id}_X$,

$$x_j = g^{2p-s_i}(\vartheta(x_i)) = \vartheta(g^{2p-s_i}(x_i)) \in \vartheta(X_i),$$

whence $X_j \subseteq \vartheta(X_i)$. Combining these observations, we obtain $\vartheta(X_i) = X_j$. $\diamond$

THEOREM 2. *If $b_1, \dots, b_m \in X$, with $b_i \in X_{t_i}$ for each i, then the following statements are equivalent*:

(1) *there exists a (necessarily unique)* $\vartheta \in \Lambda(X; g)$ *such that* $\vartheta(x_i) = b_i$ *for each* i;

(2) $c_{t_i} \mid c_i$ *for each* i.

Proof. (1)$\Rightarrow$(2). From $g^{c_i}(x_i) = x_i$ we obtain $g^{c_i}(\vartheta(x_i)) = \vartheta(x_i)$. If (1) holds, then $g^{c_i}(b_i) = b_i$. Since $b_j \in X_{t_i}$ we deduce that $c_{t_i} \mid c_i$.

(2)$\Rightarrow$(1). Every $y \in X$ belongs to a unique X_i and so is of the form $y = g^r(x_i)$ with $0 \leqslant r < c_i$. We can therefore define mapping $\vartheta : X \to X$ by

$$\vartheta(y) = \vartheta(g^r(x_i)) = g^r(b_i).$$

Clearly, $\vartheta(x_i) = \vartheta(g^0(x_i)) = g^0(b_i) = b_i$.

To see that $\vartheta \in \Lambda(X; g)$, we shall make use of the observation that

$$g^{c_i}(b_i) = b_i. \tag{A}$$

In fact, by (2) we can write $c_i = d_i c_{t_i}$ whence, using the fact that $g^{c_{t_i}}(b_i) = b_i$, we have $g^{c_i}(b_i) = g^{d_i c_{t_i}}(b_i) = b_i$.

There are two cases to consider. Suppose that $y = g^r(x_i)$ is an arbitrary element of X.

(a) If $r < c_i - 1$, then

$$g\vartheta(y) = g(g^r(b_i)) = g^{r+1}(b_i);$$
$$\vartheta g(y) = \vartheta(g^{r+1}(x_i)) = g^{r+1}(b_i).$$

(b) If $r = c_i - 1$, then

$$g\vartheta(y) = g(g^{c_i - 1}(b_i)) = g^{c_i}(b_i) = b_i \text{ [by (A)]};$$
$$\vartheta g(y) = \vartheta g(g^{c_i - 1}(x_i)) = \vartheta(x_i) = b_i.$$

In each case we have $g\vartheta(y) = \vartheta g(y)$ and so $\vartheta \in \Lambda(X; g)$.

That ϑ is unique follows from the fact that if $\varphi \in \Lambda(X; g)$ is also such that $\varphi(x_i) = b_i$ then, since the x_i form a set of representatives of the orbits of g, we have $\varphi = \vartheta$. ◇

Corollary. *For each* i, *define* $J_i = \{j; c_j \mid c_i\}$. *Then*

$$|\Lambda(X; g)| = \prod_{i=1}^{m} \left(\sum_{j \in J_i} c_j \right).$$

Proof. Let $B_i = \bigcup_{j \in J_i} X_j$. Then we can define a mapping $\psi : \Lambda(X; g) \to \mathop{\times}_{i=1}^{m} B_i$ by the prescription $\psi(\vartheta) = (\vartheta(x_1), \ldots, \vartheta(x_m))$. It follows immediately from Theorem 2 that ψ is a bijection. Consequently,

$$|\Lambda(X; g)| = \prod_{i=1}^{m} |B_i| = \prod_{i=1}^{m} \left(\sum_{j \in J_i} c_j \right). \quad \diamond$$

In order to investigate the regularity of $\Lambda(X; g)$ we require the following concepts.

Definition. With g, m, c_i as above, consider the set $P_g = \{c_i; i = 1, \ldots, m\}$ ordered by divisibility. The length of $(P_g; |)$ will be called the *dimension* of g and denoted by $\dim g$.

Definition. For $i, j, k \in \{1, \ldots, m\}$ we shall say that (i, j, k) is a *pathological triple* associated with g if i, j, k are distinct with $c_k = c_j \mid c_i$ and $c_j \neq c_i$.

THEOREM 3. *The semigroup* $\Lambda(X;g)$ *is regular if and only if* $\dim g < 2$ *and there is no pathological triple associated with g.*

Proof. $\Rightarrow$. Suppose, by way of obtaining a contradiction, that $\dim g \geq 2$ or that there is a pathological triple associated with g. Then there exist distinct i, j, k such that $c_k \mid c_j \mid c_i$ and $c_j < c_i$. By Theorem 2, we can define $\vartheta \in \Lambda(X;g)$ by setting

$$\vartheta(x_k) = \vartheta(x_j) = x_k, \qquad \vartheta(x_i) = x_j, \qquad \vartheta(x_t) = x_t (t \notin \{i,j,k\}).$$

Then, by Theorem 1, we have

$$\vartheta(X_k) = \vartheta(X_j) = X_k, \qquad \vartheta(X_i) = X_j, \qquad \vartheta(X_t) = X_t (t \notin \{i,j,k\}).$$

Since, by hypothesis, $\Lambda(X;g)$ is regular, there exists $\varphi \in \Lambda(X;g)$ such that $\vartheta\varphi\vartheta = \vartheta$. Applying each side to x_i we obtain $\vartheta\varphi(x_j) = x_j$. It follows from this that $\varphi(x_j) \in X_i$. Theorem 2 now gives the contradiction $c_i \mid c_j$.

$\Leftarrow$. Conversely, suppose that $\dim g < 2$ and there is no pathological triple associated with g. Observe first that if $\vartheta \in \Lambda(X;g)$ then, for every i,

$$|\vartheta(X_i)| \neq |X_i| \Rightarrow \vartheta^2(X_i) = \vartheta(X_i). \tag{1}$$

In fact, given $\vartheta \in \Lambda(X;g)$ let i be such that $|\vartheta(X_i)| \neq |X_i|$. By Theorem 1 there exist j, k such that $\vartheta(X_i) = X_j$ and $\vartheta^2(X_i) = \vartheta(X_j) = X_k$; also by Theorem 2, we have $c_k \mid c_j \mid c_i$. Note that $j \neq i$ since, by hypothesis, $c_j \neq c_i$ and consequently, since $\dim g < 2$, we have $c_k = c_j < c_i$. It follows that $k \neq i$. In fact, we have $k = j$, for otherwise (i, j, k) would be a pathological triple, contradicting the hypothesis. Consequently, $\vartheta^2(X_i) = \vartheta(X_j) = X_k = X_j$ and therefore $\vartheta^2(X_i) = \vartheta(X_i)$.

Now let $\vartheta \in \Lambda(X;g)$ and consider the set

$$A = \{j; x_j \in \vartheta(X)\}.$$

Let $\pi : A \to \{1, \ldots, m\}$ be given by

$$\pi(j) = \min\{i; x_j \in \vartheta(X_i)\}.$$

Since $x_j \in \vartheta(X_{\pi(j)})$ it follows from Theorem 1 that $\vartheta(X_{\pi(j)}) = X_j$. Consequently, for every $j \in A$, there is a unique $s_j \in \{0, \ldots, c_j - 1\}$ such that

$$\vartheta(x_{\pi(j)}) = g^{s_j}(x_j).$$

Define

$$l_j = \begin{cases} 0, & \text{if } s_j = 0; \\ c_j - s_j, & \text{otherwise.} \end{cases}$$

Then from the above we have

$$x_j = \vartheta(g^{l_j}(x_{\pi(j)})). \tag{2}$$

Consider next the sets

$$A_1 = \{j \in A; c_j = c_{\pi(j)}\}, \qquad A_2 = A \setminus A_1.$$

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Note that for every $j \in A_2$ there is a unique $v_j \in \{0, \ldots, c_j - 1\}$ such that

$$\vartheta(x_j) = g^{v_j}(x_j). \tag{3}$$

In fact, we have $|\vartheta(X_{\pi(j)})| = |X_j| = c_j \neq c_{\pi(j)} = |X_{\pi(j)}|$ and so, by (1), we deduce that $\vartheta^2(X_{\pi(j)}) = \vartheta(X_{\pi(j)})$; that is $\vartheta(X_j) = X_j$, whence (3) follows.

Using Theorem 2 we can now define $\varphi \in \Lambda(X; g)$ by setting

$$\varphi(x_j) = \begin{cases} x_j, & \text{if } j \notin A_i, \\ g^{l_j}(x_{\pi(j)}), & \text{if } j \in A_1; \\ g^{v_j(c_j-1)}(x_j), & \text{if } j \in A_2. \end{cases}$$

We show as follows that $\vartheta\varphi\vartheta = \vartheta$, whence $\Lambda(X; g)$ is regular. For this purpose, observe that φ satisfies the property

$$(\forall j \in A) \qquad \vartheta\varphi(x_j) = x_j. \tag{4}$$

In fact, if $j \in A_1$ then, by (2), we have

$$\vartheta\varphi(x_j) = \vartheta(g^{l_j}(x_{\pi(j)})) = x_j;$$

also if $j \in A_2$ then, by (3),

$$\begin{aligned} \vartheta\varphi(x_j) = \vartheta(g^{v_j(c_j-1)}(x_j)) &= g^{v_j(c_j-1)}(\vartheta(x_j)) \\ &= g^{v_j(c_j-1)}(g^{v_j}(x_j)) \\ &= g^{v_jc_j}(x_j) \\ &= \underbrace{g^{c_j} \ldots g^{c_j}}_{v_j}(x_j) = x_j. \end{aligned}$$

Now let $x \in X$ with $\vartheta(x) \in X_j$. Then $\vartheta(x) = g^r(x_j)$ for some r with $0 \leqslant r \leqslant c_j - 1$, whence $g^{2p-r}(\vartheta(x)) = \vartheta(g^{2p-r}(x)) = x_j$. It follows that $j \in A$. Applying $\vartheta\varphi$, we deduce by (4) that

$$g^{2p-r}\vartheta\varphi\vartheta(x) = \vartheta\varphi(x_j) = x_j = g^{2p-r}\vartheta(x),$$

whence, g being a bijection, $\vartheta\varphi\vartheta(x) = \vartheta(x)$ and so $\vartheta\varphi\vartheta = \vartheta$, as required. ◇

Example 1. By Theorem 3, the smallest X such that $\Lambda(X; g)$ is not regular arises when $m = 3$ and

$$X_1 = \{x_1\}, \qquad X_2 = \{x_2\}, \qquad X_3 = \{x_3, g(x_3)\}.$$

Hence $|X| = 4$ with $c_1 = 1$, $c_2 = 1$, and $c_3 = 2$. By the Corollary of Theorem 2, $|\Lambda(X; g)| = 2 \,.\, 2 \,.\, 4 = 16$.

We now proceed to consider the question of when, as a regular semigroup, $\Lambda(X; g)$

is orthodox, inverse, or a group. We begin with the orthodox case, and for this purpose we shall use the following results.

THEOREM 4. *If* $c_i = c_j > 1$ *for some* i, j *with* $i \neq j$, *then* $\Lambda(X; g)$ *is not orthodox.*

Proof. Without loss of generality, we may suppose that $c_1 = c_2 > 1$. Using Theorem 2, we can define $\alpha, \beta \in \Lambda(X; g)$ as follows:

$$\alpha(x_1) = x_2, \qquad \alpha(x_i) = x_i (i \geq 2);$$
$$\beta(x_1) = x_1, \qquad \beta(x_2) = g(x_1), \qquad \beta(x_i) = x_i (i \geq 3).$$

Clearly, α and β are idempotent, but $\alpha\beta$ is not. To see this, observe that

$$\alpha\beta(x_2) = \alpha g(x_1) = g\alpha(x_1) = g(x_2);$$
$$\alpha\beta\alpha\beta(x_2) = \alpha\beta g(x_2) = g\alpha\beta(x_2) = g^2(x_2).$$

Since $c_2 > 1$ we have $g^2(x_2) \neq g(x_2)$ and therefore $\alpha\beta$ is not idempotent. ◇

THEOREM 5. *If* $|X| > 2$ *and* $c_i = c_j = 1$ *for some* i, j *with* $i \neq j$, *then* $\Lambda(X; g)$ *is not orthodox.*

Proof. Since $|X| > 2$ there are at least three distinct orbits. Without loss of generality we may assume that $c_1 = c_2 = 1$. Using Theorem 2, we can define $\alpha, \beta \in \Lambda(X; g)$ as follows:

$$\alpha(x_1) = x_1, \qquad \alpha(x_2) = x_2, \qquad \alpha(x_3) = x_1, \qquad \alpha(x_i) = x_i, \ (i \geq 4);$$
$$\beta(x_1) = x_2, \qquad \beta(x_i) = x_i, \ (i \geq 2).$$

Clearly, α and β are idempotent. However

$$\alpha\beta(x_3) = \alpha(x_3) = x_1,$$
$$\alpha\beta\alpha\beta(x_3) = \alpha\beta(x_1) = \alpha(x_2) = x_2,$$

and so $\alpha\beta$ is not idempotent. ◇

If M_g denotes the set of minimal elements of the ordered set $(P_g, |)$, let

$$\Delta_g = \{i; c_i \in M_g\}.$$

For each $\vartheta \in \Lambda(X; g)$ define also

$$I_\vartheta = \{i; \vartheta(X_i) = X_i\}.$$

THEOREM 6. *If the* c_i *are distinct, then*

$$\Delta_g = \bigcap_\vartheta I_\vartheta.$$

Proof. Let $i \in \Delta_g$ and $\vartheta \in \Lambda(X; g)$. By Theorem 1, we have $\vartheta(X_i) = X_j$ for some j; also, by Theorem 2, we have $c_j \mid c_i$. Since c_i is minimal in P_g, it follows that $c_j = c_i$, whence $j = i$. Thus $\vartheta(X_i) = X_i$ and consequently $i \in I_\vartheta$. Hence we see that $\Delta_g \subseteq \bigcap_\vartheta I_\vartheta$.

To obtain the reverse inclusion, let Δ'_g denote the complement of Δ_g in $\{1, \ldots, m\}$ and observe from the definitions that

$$(\forall i \in \Delta'_g)(\exists j \in \Delta_g)\, c_j \mid c_i.$$

There therefore exists a mapping $\pi:\Delta'_g \to \Delta_g$ with the property that $c_{\pi(i)} \mid c_i$. Consequently, by Theorem 2, we can define $\varphi \in \Lambda(X;g)$ by setting

$$\varphi(x_i) = \begin{cases} x_i, & \text{if } i \in \Delta_g; \\ x_{\pi(i)}, & \text{if } i \notin \Delta_g. \end{cases}$$

It suffices to prove that $I_\varphi \subseteq \Delta_g$. Suppose, by way of obtaining a contradiction, that $i \in I_\varphi$ and $i \notin \Delta_g$. Then since $i \in I_\varphi$ we have $\varphi(X_i) = X_i$; also, since $i \notin \Delta_g$, we have $\varphi(x_i) = x_{\pi(i)}$ whence, by Theorem 1, $\varphi(X_i) = X_{\pi(i)}$. We conclude that $X_i = X_{\pi(i)}$, whence $i = \pi(i)$, which is absurd since by hypothesis $i \notin \Delta_g$ and $\pi(i) \in \Delta_g$. ◇

THEOREM 7. *Suppose that* $\dim g < 2$ *and the* c_i *are distinct. If* $\vartheta(X_i) \neq X_i$, *then* $\vartheta(X_i) = X_j$ *for some* $j \in \Delta_g$.

Proof. By Theorem 1, there exists j such that $\vartheta(X_i) = X_j$ with clearly, $j \neq i$. Also, by Theorem 2, we have $c_j \mid c_i$ with $c_j < c_i$. The hypothesis that $\dim g < 2$ now gives $c_j \in M_g$ and therefore $j \in \Delta_g$. ◇

THEOREM 8. *If* $\dim g < 2$ *and the* c_i *are distinct, then*

$$I_\vartheta \cap I_\varphi = I_{\vartheta\varphi} \qquad (\forall \vartheta, \varphi \in \Lambda(X;g))$$

Proof. If $i \in I_\vartheta \cap I_\varphi$, then $\vartheta\varphi(X_i) = \vartheta(X_i) = X_i$ gives $i \in I_{\vartheta\varphi}$. Conversely, if $i \in I_{\vartheta\varphi}$, then we have

$$\vartheta\varphi(X_i) = X_i. \tag{B}$$

Suppose, by way of obtaining a contradiction that $i \notin I_\varphi$. Then $\varphi(X_i) \neq X_i$ and so, by Theorem 7, there exists $j \in \Delta_g$ such that $\varphi(X_i) = X_j$. By Theorem 6, we have $j \in I_\vartheta$ and so $\vartheta(X_j) = X_j$. Consequently, $\vartheta\varphi(X_i) = \vartheta(X_j) = X_j$, which contradicts (B) since $i \neq j$. Hence $i \in I_\varphi$ and so $\varphi(X_i) = X_i$. It now follows from (B) that $\vartheta(X_i) = X_i$ and so $i \in I_\vartheta$. Hence $i \in I_\vartheta \cap I_\varphi$. ◇

In order to determine when $\Lambda(X;g)$ is orthodox we must consider separately the cases $|X| \leq 2$ and $|X| > 2$. In the former case, $\Lambda(X;g)$ is always orthodox. In fact, $\Lambda(X;g)$ reduces to the trivial group when $|X| = 1$, and when $|X| = 2$ there are two possibilities.

(1) *g has a single orbit* (*that is*, $m = 1$).

In this case $\Lambda(X;g)$ is isomorphic to the group $\mathbb{Z}_2$.

(2) *g has two orbits* (*that is*, $m = 2$).

In this case, g is the identity map on X and $\Lambda(X;g)$ is the full transformation semigroup on X, and for $|X| = 2$ this is orthodox.

The situation when $|X| > 2$ is as follows.

THEOREM 9. *If* $|X| > 2$, *then the semigroup* $\Lambda(X;g)$ *is orthodox if and only if* $\dim g < 2$ *and the* c_i *are distinct.*

Proof. $\Rightarrow$. Suppose that $\Lambda(X;g)$ is orthodox. Then, by Theorem 3, $\dim g < 2$ and, by Theorems 4 and 5, the c_i are distinct.

$\Leftarrow$. Conversely, suppose that the conditions hold. Since the c_i are distinct, there can be no pathological triple associated with g and therefore, by Theorem 3, $\Lambda(X;g)$ is regular. To see that it is orthodox, let ϑ, φ be idempotents of $\Lambda(X;g)$. Then we have

$$\vartheta\varphi(X) \subseteq \varphi(X). \tag{C}$$

To see this, observe that

(1) if $i \in I_{\vartheta\varphi}$, then $\vartheta\varphi(X_i) = X_i$ and so, by Theorem 8, $\vartheta(X_i) = X_i$ and $\varphi(X_i) = X_i$, whence $\vartheta\varphi(X_i) = \varphi(X_i) \subseteq \varphi(X)$;

(2) If $i \notin I_{\vartheta\varphi}$, then by Theorem 7 there exists $j \in \Delta_g$ such that $\vartheta\varphi(X_i) = X_j$. But, by Theorem 6, $\varphi(X_j) = X_j$. Hence we have $\vartheta\varphi(X_i) = \varphi(X_j) \subseteq \varphi(X)$.

Thus, in all cases, $\vartheta\varphi(X_i) \subseteq \varphi(X)$ and therefore $\vartheta\varphi(X) \subseteq \varphi(X)$.

Using (C) we now have, for every $x \in X$, $\vartheta\varphi(x) \in \varphi(X)$ and so there exists $t \in X$ such that $\vartheta\varphi(x) = \varphi(t)$. Consequently,

$$\varphi\vartheta\varphi(x) = \varphi^2(t) = \varphi(t) = \vartheta\varphi(x)$$

and so $\varphi\vartheta\varphi = \vartheta\varphi$. It follows from this that $\vartheta\varphi$ is idempotent, whence $\Lambda(X;g)$ is orthodox. ◇

Example 2. The smallest X such that $\Lambda(X;g)$ is regular but not orthodox arises when $m = 3$ and $c_1 = c_2 = c_3 = 1$. In this case we have $|X| = 3$ and $|\Lambda(X;g)| = 3 . 3 . 3 = 27$.

We now proceed to consider the question of when $\Lambda(X;g)$ is a group and, more generally, an inverse semigroup. As we shall see, in each of these cases it is commutative. For this purpose we shall use the following results.

Theorem 10. *If $c_i = c_j$, for some i, j with $i \neq j$, then $\Lambda(X;g)$ contains a pair of idempotents that do not commute.*

Proof. Suppose, without loss of generality, that $c_1 = c_2$. Using Theorem 2 we can define $\alpha, \beta \in \Lambda(X;g)$ as follows:

$$\alpha(x_1) = \alpha(x_2) = x_1, \qquad \alpha(x_i) = x_i \ (i \geq 3);$$
$$\beta(x_1) = \beta(x_2) = x_2, \qquad \beta(x_i) = x_i \ (i \geq 3).$$

Clearly, α and β are idempotent. Moreover, $\alpha\beta(x_1) = x_1$ and $\beta\alpha(x_1) = x_2$, so that $\alpha\beta \neq \beta\alpha$. ◇

Theorem 11. *If $1 \neq c_i \mid c_j$, for some i, j with $i \neq j$, then $\Lambda(X;g)$ contains a pair of idempotents that do not commute.*

Proof. Suppose, without loss of generality, that $1 \neq c_1 \mid c_2$. Using Theorem 2 we can define $\alpha, \beta \in \Lambda(X;g)$ as follows;

$$\alpha(x_1) = \alpha(x_2) = x_1, \qquad \alpha(x_i) = x_i \ (i \geq 3);$$
$$\beta(x_1) = x_1, \qquad \beta(x_2) = g(x_1), \qquad \beta(x_i) = x_i \ (i \geq 3).$$

Clearly, α and β are idempotent. Moreover, $\alpha\beta(x_2) = g(x_1)$ and $\beta\alpha(x_2) = x_1$. Since $c_1 \neq 1$ it follows that $\alpha\beta \neq \beta\alpha$. ◇

Theorem 12. *The following statements are equivalent*:

(1) *$\Lambda(X;g)$ is a group*;

(2) $\dim g = 0$ *and the* c_i *are distinct.*

Moreover, as a group, $\Lambda(X;g) \simeq \bigtimes_{i=1}^{m} \mathbb{Z}_{c_i}$.

Proof. (1)$\Rightarrow$(2). Suppose that $\Lambda(X;g)$ is a group. If $m=1$, then $X = X_1$ and so $\dim g = 0$. Also, by Theorem 2, every $\vartheta \in (X;g)$ is given by $\vartheta(x_1) = g^r(x_1)$, where $0 \leqslant r < |X|$. Since g has a single orbit it follows that $\vartheta = g^r$ and therefore $\Lambda(X;g) \simeq \mathbb{Z}_{|X|}$.

If now $m > 1$ then, by Theorem 10, the c_i are distinct. Moreover, $c_i \neq 1$ for every i. To see this, suppose that (say) $c_1 = 1$. Then, applying Theorem 2 with $b_1 = b_2 = x_1$ and $b_i = x_i$ for $i \geqslant 3$, we obtain $\vartheta \in \Lambda(X;g)$ such that

$$\vartheta(x_1) = \vartheta(x_2) = x_1, \qquad \vartheta(x_i) = x_i \ (i \geqslant 3).$$

This is not possible since by hypothesis $\Lambda(X;g)$ is a group and ϑ is not a bijection.

Suppose now, by way of obtaining a contradiction, that $\dim g \geqslant 1$. Then there exist c_i, c_j with $c_i \mid c_j$ and $c_i < c_j$. Since $c_i \neq 1$ we can apply Theorem 11 to produce a pair of non-commuting idempotents in $\Lambda(X;g)$. This contradicts the hypothesis that $\Lambda(X;g)$ is a group, and so we conclude that $\dim g = 0$.

(2)$\Rightarrow$(1). Conversely, suppose that $\dim g = 0$ and that the c_i are distinct. Then for every $\vartheta \in \Lambda(X;g)$ we have, for all i,

$$\vartheta(X_i) = X_i. \tag{D}$$

In fact, by Theorem 1, for each i there exists j such that $\vartheta(X_i) = X_j$. If $j \neq i$, then by Theorem 2 we have $c_j \mid c_i$, whence the contradiction $\dim g \geqslant 1$. Thus we must have $j = i$ and so $\vartheta(X_i) = X_i$.

It follows from (D) that ϑ is surjective, and therefore bijective, since X is finite. Hence $\Lambda(X;g)$ is a group.

As for the final statement, observe by (D) that every $\vartheta \in \Lambda(X;g)$ induces a mapping $\vartheta_i : X_i \to X_i$ given by $\vartheta_i(x) = \vartheta(x)$. Let g_i be the bijection induced on X_i by g and consider the mapping

$$\psi : \Lambda(X;g) \to \bigtimes_{i=1}^{m} \Lambda(X_i;g_i)$$

given by the prescription $\psi(\vartheta) = (\vartheta_1, \ldots, \vartheta_m)$.

To see that ψ is surjective observe that if we choose $\varphi_1, \ldots, \varphi_m$ with $\varphi_i \in \Lambda(X_i;g_i)$ for each i, then we can define $\varphi \in \Lambda(X;g)$ by setting $\varphi(x) = \varphi_i(x)$ when $x \in X_i$, thereby obtaining $\psi(\varphi) = (\varphi_1, \ldots, \varphi_m)$. That ψ is injective is immediate from the fact that if ϑ, φ induce the same mapping on the orbits then they are equal. Thus ψ is a bijection. Since clearly $(\vartheta\varphi)_i = \vartheta_i\varphi_i$, it follows that ψ is an isomorphism.

Finally, since in the Ockham space $(X_i;g_i)$ the bijection g_i has only one orbit we have, from the above, $\Lambda(X_i;g_i) \simeq \mathbb{Z}_{|X_i|} = \mathbb{Z}_{c_i}$. We conclude that $\Lambda(X;g) \simeq \bigtimes_{i=1}^{m} \mathbb{Z}_{c_i}$. ◇

COROLLARY 1. *If* $\Lambda(X;g)$ *is a group, then*

$$|\Lambda(X;g)| = \prod_{i=1}^{m} c_i. \quad \diamond$$

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COROLLARY 2. *For every integer $k \geqslant 2$ there exists an Ockham algebra $(L;f)$ of finite boolean type such that $(L;f) \in \mathbf{P}_{k,0}$ and* End$(L;f)$ *is a group isomorphic to* $\mathbb{Z}_k$.

Proof. Consider the prime factorisation $k = \prod_{i=1}^{m} p_i^{\alpha_i}$. For $i = 1, \ldots, m$ let $X_i = \{x_{i,1}, \ldots, x_{i,c_i}\}$ be a set of cardinality $c_i = p_i^{\alpha_i}$ and choose the X_i to be pairwise disjoint. Let $X = \bigcup_{i=1}^{m} X_i$ and define $g: X \to X$ by

$$g(x_{i,j}) = x_{i,j+1} \qquad (\text{mod } c_i).$$

Then clearly g is a permutation on X with orbits $X_1, \ldots, X_m$. Now since $k = \text{lcm}\,\{c_i; i = 1, \ldots, m\}$ we have $g^k = \text{id}_X$ and therefore the Ockham space $(X;g)$ is such that its dual algebra $(L;f)$ is of finite boolean type and belongs to the class $\mathbf{P}_{k,0}$. Since clearly $\dim g = 0$ and the c_i are distinct, it follows from Theorem 12 that $\Lambda(X;g)$ is a group, of the same cardinality as the group of Ockham automorphisms on L. For each i we have

$$J_i = \{j; c_j \mid c_i\} = \{j; p_j^{\alpha_j} \mid p_i^{\alpha_i}\} = \{i\}.$$

Consequently, by the Corollary of Theorem 2,

$$|\Lambda(X;g)| = \prod_{i=1}^{m} \sum_{j \in J_i} c_j = \prod_{i=1}^{m} c_i = \prod_{i=1}^{m} p_i^{\alpha_i} = k,$$

and, by Theorem 12, $\Lambda(X;g) \simeq \mathbb{Z}_k$. $\diamond$

THEOREM 13. *The following statements are equivalent*:
(1) $\Lambda(X;g)$ *is an inverse semigroup that is not a group*;
(2) $\dim g = 1$, *the c_i are distinct, and g has a* (*necessarily unique*) *fixed point.*

Proof. (1)$\Rightarrow$(2). If (1) holds then, by Theorem 3, $\dim g < 2$ and, by Theorem 10, the c_i are distinct. Since $\Lambda(X;g)$ is not a group, it follows by Theorem 12 that $\dim g = 1$.

To prove that g has a fixed point, we show that at least one $c_i = 1$ (whence only one $c_i = 1$). Suppose, to the contrary, that every $c_i \neq 1$. Since $\dim g = 1$ there exist c_i, c_j with $c_i \mid c_j$. It follows from Theorem 11 that $\Lambda(X;g)$ is not an inverse semigroup, a contradiction. Hence g has a fixed point.

(2)$\Rightarrow$(1). If (2) holds then, by Theorem 3, $\Lambda(X;g)$ is regular and, by Theorem 12, it is not a group. Let x_1 be the fixed point of g. Then for every $\vartheta \in \Lambda(X;g)$ we have

$$\vartheta(X_i) = X_i \quad \text{or} \quad \vartheta(X_i) = X_1 = \{x_1\}. \tag{E}$$

In fact, by Theorem 1 we have $\vartheta(X_i) = X_j$ for some j. Suppose that $j \neq i$. Then, by Theorem 2, $c_j \mid c_i$. We must then have $c_j = 1$ since otherwise $\dim g \geqslant 2$, a contradiction. Hence $X_j = X_1$ and consequently we have that $\vartheta(X_i) = X_1$.

We now use (E) to prove that $\Lambda(X;g)$ is inverse. For this purpose, let $\alpha, \beta \in \Lambda(X;g)$. We shall show that α, β commute, and for this it suffices to prove that $\alpha\beta$ and $\beta\alpha$ coincide on every X_i.

Since x_1 is the only fixed point of g, it is clear that $\alpha\beta$ and $\beta\alpha$ agree on $X_1 = \{x_1\}$. To show that they agree on X_i with $i \neq 1$, we must consider several cases.

(a) $\beta(X_i) = X_i$.

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In this case $\beta(x_i)=g^r(x_i)$, where $0\leq r<c_i$ and $\alpha\beta(x_i)=g^r(\alpha(x_i))$. Two sub-cases, according to (E), arise:

(a_1) $\alpha(X_i)=X_i$.

Here $\alpha(x_i)=g^s(x_i)$ where $0\leq s<c_i$, and we have

$$\alpha\beta(x_i)=g^{r+s}(x_i);$$
$$\beta\alpha(x_i)=\beta(g^s(x_i))=g^s(\beta(x_i))=g^{r+s}(x_i).$$

(a_2) $\alpha(X_i)=X_1$.

Here we have

$$\alpha\beta(x_i)=g^r(\alpha(x_i))=g^r(x_1)=x_1;$$
$$\beta\alpha(x_i)=\beta(x_1)=x_1.$$

(b) $\beta(X_i)=X_1$.

In this case similar calculations to the above reveal that $\alpha\beta(x_i)=\beta\alpha(x_i)$.

It follows from the above that $\alpha\beta=\beta\alpha$. Hence $\Lambda(X;g)$ is a commutative regular semigroup and therefore is inverse. ◇

COROLLARY. *If $\Lambda(X;g)$ is inverse, then it is commutative.*

Proof. This follows from the above and Theorem 12. ◇

EXAMPLE 3. By Theorem 13, the smallest X such that $\Lambda(X;g)$ is inverse but not a group arises when $m=2$ and $c_1=1$, $c_2=2$. Here $|X|=3$ and $|\Lambda(X;g)|=3$. The smallest corresponding algebra is such that $L\simeq\mathbf{2}^3$.

EXAMPLE 4. The smallest X such that $\Lambda(X;g)$ is orthodox but not inverse arises when $m=2$ and $c_1=2$, $c_2=4$. Here $|X|=6$ and $|\Lambda(X;g)|=2(2+4)=12$.

When $\Lambda(X;g)$ is inverse but not a group, we can describe its structure in terms of a cartesian product of cyclic groups each with a zero adjoined.

THEOREM 14. *If $\Lambda(X;g)$ is inverse but not a group, then*

$$\Lambda(X;g)\simeq\mathop{\times}_{i=2}^{m}\mathbb{Z}^0_{c_i}.$$

Proof. As before, let x_1 be the fixed point of g. For each $i\geq 2$ define $Y_i=X_i\cup\{x_1\}$. Define $g_i:Y_i\to Y_i$ by $g_i(y)=g(y)$ and let $\eta_i:Y_i\to Y_i$ be the constant mapping given by $\eta_i(y)=x_1$. Then $\Lambda(Y_i;g_i)$ is a group with zero. To see this, let $\alpha\in\Lambda(Y_i;g_i)$. Then either $\alpha(x_i)=x_1$ (in which case $\alpha=\eta_i$) or $\alpha(x_i)\in X_i$ and so $\alpha(x_i)=g_i^r(x_i)$ with $0\leq r<c_i$ (in which case $\alpha=g_i^r$ and is therefore a bijection). It follows that $\Lambda(Y_i;g_i)\simeq\mathbb{Z}^0_{c_i}$.

Now define $\psi:\Lambda(X;g)\to\mathop{\times}_{i=2}^{m}\Lambda(Y_i;g_i)$ as in the proof of Theorem 12 to obtain an isomorphism $\Lambda(X;g)\simeq\mathop{\times}_{i=2}^{m}\mathbb{Z}^0_{c_i}$. ◇

COROLLARY 1. *If $\Lambda(X;g)$ is inverse but not a group, then*

$$|\Lambda(X;g)|=\prod_{i=2}^{m}(1+c_i).\quad ◇$$

COROLLARY 2. *For every integer $k \geqslant 2$ there exists an Ockham algebra $(L;f)$ of finite boolean type such that $(L;f) \in \mathbf{P}_{k,0}$ and* $\mathrm{End}(L;f)$ *is an inverse semigroup isomorphic to* $\mathbb{Z}_k^0$.

Proof. Let $X_1 = \{x_{1,1}\}$ and $X_2 = \{x_{2,1}, \ldots, x_{2,k}\}$ be disjoint sets of cardinalities 1 and k. Define a permutation g on $X = X_1 \cup X_2$ by setting

$$g(x_{1,1}) = x_{1,1}, \qquad g(x_{2,i}) = x_{2,i+1} \qquad (\mathrm{mod}\ k).$$

Clearly, X_1, X_2 are the orbits of g. By Theorems 13 and 14 we have that $\Lambda(X;g)$ is an inverse semigroup isomorphic to $\mathbb{Z}_k^0$. ◇

Finally, in view of the above results, it is natural to consider the situation that holds when $|\Lambda(X;g)|$ is a prime.

THEOREM 15. *If $(L;f)$ is an endomorphism regular Ockham algebra of finite boolean type and* $\mathrm{End}(L;f)$ *is of prime cardinality p then* $\mathrm{End}(L;f)$ *is isomorphic either to the group $\mathbb{Z}_p$ or to the inverse semigroup $\mathbb{Z}_{p-1}^0$ $(p \geqslant 3)$.*

Proof. As usual, let $(X;g)$ be the dual space of $(L;f)$. If $\Lambda(X;g)$ is a group then, by Theorem 12 and its Corollary 1, we have necessarily $m = 1$ and $c_1 = p$, whence $\Lambda(X;g) \simeq \mathbb{Z}_p$.

Suppose now that $\Lambda(X;g)$ is not a group. Without loss of generality we may suppose that $c_1 \leqslant c_2 \leqslant \ldots \leqslant c_m$. Observe that

(*a*) $p \geqslant 3$.

In fact, suppose that $p = 2$. Then necessarily $g \neq \mathrm{id}_X$; for otherwise $\Lambda(X;g)$ is the full transformation semigroup on X, and this can never have cardinality 2. Since, by hypothesis, $|\Lambda(X;g)| = 2$ it follows that $\Lambda(X;g) = \{\mathrm{id}_X, g\} \simeq \mathbb{Z}_2$, a contradiction.

(*b*) $m \geqslant 2$.

In fact, suppose that $m = 1$. Then, by the Corollary of Theorem 2, we have $c_1 = p$ whence $\Lambda(X;g)$ is a group by Theorem 12, a contradiction.

(*c*) $c_1 < c_2$.

In fact, if $c_1 = c_2$ then $J_1 = \{j; c_j \mid c_1\} = \{j; c_j \mid c_2\} = J_2$, whence we have $\sum_{j \in J_1} c_j = \sum_{j \in J_2} c_j = r$ say, and clearly $r \geqslant 2$. The Corollary of Theorem 2 now gives the contradiction $r^2 \mid p$.

(*d*) $\sum_{j \in J_1} c_j = 1$.

In fact, if $\sum_{j \in J_1} c_j \neq 1$ then by the Corollary of Theorem 2 and the fact that p is prime we have $\prod_{i=2}^{m} (\sum_{j \in J_i} c_j) = 1$, whence $c_1 < c_2 = \ldots = c_m = 1$ which is absurd.

It follows from these observations that $J_1 = \{1\}$ and $c_1 = 1$. Then $\prod_{i=2}^{m} (\sum_{j \in J_i} c_j) = p$ gives $m = 2$ and $\sum_{j \in J_2} c_j = p$. It follows that $J_2 = \{1, 2\}$ and therefore $p = c_1(c_1 + c_2) = 1 + c_2$ whence $c_2 = p - 1$. It now follows by Theorems 13 and 14 that $\Lambda(X;g) \simeq \mathbb{Z}_{p-1}^0$. ◇

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