VII

Effective field theory for low-energy QCD

At the lowest possible energies, the Standard Model involves only photons, electrons, muons, and pions, as these are the lightest particles in the spectrum. As we increase the energy slightly, kaons and etas become active. The light pseudoscalar hadrons would be massless Goldstone bosons in the limit that the $u,\ d,\ s$ quark masses vanished. We give a separate discussion of this portion of the theory because it is an important illustration of effective field theory and because it can be treated with a higher level of rigor than most other topics.

VII–1 QCD at low energies

The $SU(2)$ chiral transformations,

$$\psi_{L,R} \equiv \begin{pmatrix} u \\ d \end{pmatrix}_{L,R} \rightarrow \exp\ (-i\boldsymbol{\theta}_{L,R} \cdot \boldsymbol{\tau})\psi_{L,R}, \tag{1.1}$$

almost give rise to an invariance of the QCD lagrangian for small m_u, m_d, but do not appear to induce a left–right symmetry of the particle spectrum. This is because the axial symmetry is dynamically broken (i.e. hidden) with the pion being the (approximate) Goldstone boson. Vectorial isospin symmetry, i.e., simultaneous $SU(2)$ transformations of ψ_L and ψ_R, remains as an approximate symmetry of the spectrum.

Isospin symmetry is seen from the near equality of masses in the multiplets $(\pi^\pm, \pi^0)$, (K^+, K^0), (p, n), etc. In the language of group theory, we say that $SU(2)_L \times SU(2)_R$ has been dynamically broken to $SU(2)_V$. What is the evidence that such a scenario is correct? Ultimately it comes from the predictions which result, such as those which we detail in the remainder of this chapter.

The effective lagrangian for pions at very low-energy has already been developed in Chap. IV. In particular we recall the formalism of Sect. IV–6 which includes couplings to left-handed (right-handed) currents $\ell_\mu(x)$ ($r_\mu(x)$), and scalar and pseudoscalar densities $s(x)$ and $p(x)$, with the resulting $\mathcal{O}(E^2)$ lagrangian,

$$\mathcal{L}_2 = \frac{F_\pi^2}{4}\,\mathrm{Tr}\,\left(D_\mu U D^\mu U^\dagger\right) + \frac{F_\pi^2}{4}\,\mathrm{Tr}\,\left(\chi U^\dagger + U\chi^\dagger\right)\,,$$
$$U = \exp(i\boldsymbol{\tau}\cdot\boldsymbol{\pi}/F_\pi), \qquad \chi = 2B_0(s+ip), \tag{1.2}$$

where $D_\mu U \equiv \partial_\mu U + i\ell_\mu U - iUr_\mu$ and B_0 is a constant. *QCD* in the absence of sources is recovered with $\ell_\mu = r_\mu = p = 0$ and $s = \mathbf{m}$, where $\mathbf{m}$ is the quark mass matrix.

Vacuum expectation values and masses

With a dynamically broken symmetry, the lagrangian is invariant but the vacuum state does not share this symmetry. A useful measure of this noninvariance in *QCD* is the vacuum expectation value of a scalar bilinear,

$$\langle 0\left|\bar\psi\psi\right|0\rangle = \langle 0\left|\bar\psi_L\psi_R\right|0\rangle + \langle 0\left|\bar\psi_R\psi_L\right|0\rangle. \tag{1.3}$$

Up to small corrections, isospin symmetry implies

$$\langle 0\left|\bar u u\right|0\rangle = \langle 0\left|\bar d d\right|0\rangle. \tag{1.4}$$

Such matrix elements, if nonzero, cannot be invariant under separate left-handed or right-handed $SU(2)$ transformations. Indeed, it is evident from Eq. (1.3) that the vacuum expectation value couples together the left-handed and right-handed sectors.

One way that the vacuum expectation values of Eq. (1.4) affect phenomenology is through the pion mass. If the u and d quarks were massless, the pion would be a true Goldstone boson with $m_\pi = 0$. The part of the *QCD* lagrangian which explicitly violates chiral symmetry is the collection of quark mass terms,

$$\mathcal{H}_{\rm mass} = -\mathcal{L}_{\rm mass} = m_u\bar u u + m_d\bar d d. \tag{1.5}$$

To first order in the symmetry breaking, the pion mass is generated by the expectation value of this hamiltonian,

$$m_\pi^2 = \langle\pi\left|m_u\bar u u + m_d\bar d d\right|\pi\rangle. \tag{1.6}$$

This quantity can be related to the vacuum expectation value by using the chiral lagrangian. Taking both the pion and vacuum matrix elements of Eq. (1.2) and using the notation of Sect. IV–6, we have

$$m_\pi^2 = (m_u + m_d)B_0\,, \qquad \langle 0\left|\bar q q\right|0\rangle = -\frac{\partial\mathcal{L}}{\partial s^0} = -F_\pi^2 B_0 = -\frac{F_\pi^2 m_\pi^2}{m_u + m_d}. \tag{1.7}$$

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Thus since both B_0 and the quark masses are required to be positive, consistency requires that $\langle 0|\bar{q}q|0\rangle$ be nonzero and negative. However, without a separate determination of the quark masses (the origin of which must lie outside chiral symmetry) we do not know either $\langle 0|\bar{q}q|0\rangle$ or $m_u + m_d$ independently.

As an aside, we note that for Goldstone bosons there is a clear answer to the perennial question of whether one should treat symmetry breaking in terms of a linear or quadratic formula in the meson mass. For states of appreciable mass, the two procedures are equivalent to first order in the symmetry breaking since

$$\delta(m^2) \equiv (m_0 + \delta m)^2 - m_0^2 = 2m_0\,\delta m + \cdots . \tag{1.8}$$

However, when the symmetry expansion is about a *massless* limit, the m *vs* m^2 distinction becomes important. Because pions are bosonic fields we require their effective lagrangian to have the properly normalized form,

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \boldsymbol{\pi} \cdot \partial^\mu \boldsymbol{\pi} - m_\pi^2 \boldsymbol{\pi} \cdot \boldsymbol{\pi}\right) + \cdots . \tag{1.9}$$

The prediction for the pion mass must then have the form,

$$m_\pi^2 = (m_u + m_d)B_0 + (m_u + m_d)^2 C_0 + (m_u - m_d)^2 D_0 + \cdots . \tag{1.10}$$

In principle, Nature could decide in favor of either $m_\pi^2 \propto m_q$ or $m_\pi^2 \propto m_q^2$ depending on whether the renormalized parameter B_0 vanishes or not. However, the choice $B_0 = 0$ is not 'natural' in that there is no symmetry constraint to force this value. Since one generally expects a nonzero value for B_0, the squared pion mass is linear in the symmetry-breaking parameter m_q. There is every indication that $B_0 \neq 0$ in *QCD*.

Quark mass ratios

The addition of an extra quark adds to the number of possible hadrons. If the strange quark mass is not too large, there are additional low-mass particles associated with the breaking of chiral symmetry. Including the quark mass terms,

$$\mathcal{L}_{\text{mass}} = \bar{\psi}_L \mathbf{m} \psi_R + \bar{\psi}_R \mathbf{m} \psi_L , \qquad \mathbf{m} = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}, \tag{1.11}$$

the *QCD* lagrangian has an approximate $SU(3)_L \times SU(3)_R$ global symmetry. If the u, d, s quarks were massless, the dynamical breaking of $SU(3)_L \times SU(3)_R$ to vector $SU(3)$ would produce eight Goldstone bosons, one for each generator of $SU(3)$. These would be the three pions $\pi^\pm$, π^0, four kaons $K^\pm$, K^0, $\bar{K}^0$, and one neutral particle η_8 with the quantum numbers of the eighth component of the octet.

Due to nonzero quark masses, these mesons are not actually massless, but should be light if the quark masses are not 'too large'.

What should the K, η_8 masses be? Unfortunately, *QCD* is unable to answer this question, even if we were able to solve the theory precisely. This is because the quark masses are free parameters in *QCD*, and thus must be determined from experiment. This means that the π, K, and η_8 masses can be used to determine the quark masses rather than vice versa. The discussion is somewhat more subtle than this simple statement would indicate. Quark masses need to be renormalized, and hence to specify their values one has to specify the renormalization prescription and the scale at which they are renormalized. Under changes of scale, the mass values change, i.e., they 'run.' However, quark mass *ratios* are rather simpler. The *QCD* renormalization is flavor-independent, at least to lowest order in the masses. In this situation, mass ratios are independent of the renormalization. There can be some residual scheme dependence through higher-order dependence of the renormalization constants on the quark masses. However, to first order, we can be confident that the mass ratio determined by the π, K, η_8 masses is the *same* ratio as found from the mass parameters of the *QCD* lagrangian.

The content of chiral $SU(3)$ is contained in an effective lagrangian expressed in terms of $U = \exp[i(\boldsymbol{\lambda} \cdot \boldsymbol{\varphi})/F]$ and having the same form as Eq. (1.2). The matrix field $\boldsymbol{\lambda} \cdot \boldsymbol{\varphi}$ contained in U has the explicit representation,

$$\frac{1}{\sqrt{2}} \sum_{a=1}^{8} \lambda^a \varphi^a = \begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix}, \tag{1.12}$$

as expressed in terms of the pseudoscalar meson fields. If we choose the parameters in Eq. (1.2) to correspond to *QCD* without external sources, viz.,

$$s = \mathbf{m}\,, \qquad p = 0\,, \qquad D_\mu U = \partial_\mu U, \tag{1.13}$$

the meson masses obtained by expanding to order $\boldsymbol{\varphi}^2$ are

$$\begin{aligned} m_\pi^2 &= B_0(m_u + m_d)\,, \quad & m_{K^\pm}^2 &= B_0(m_s + m_u), \\ m_{K^0}^2 &= B_0(m_s + m_d)\,, \quad & m_{\eta_8}^2 &= \frac{1}{3} B_0(4m_s + m_u + m_d). \end{aligned} \tag{1.14}$$

Defining $m_K^2 = \frac{1}{2}(m_{K^\pm}^2 + m_{K^0}^2)$, we obtain from Eq. (1.14) the mass relations,

$$\frac{\hat{m}}{m_s} = \frac{m_\pi^2}{2m_K^2 - m_\pi^2} \simeq \frac{1}{26}, \tag{1.15a}$$

$$m_{\eta_8}^2 = \frac{1}{3}\left(4m_K^2 - m_\pi^2\right), \tag{1.15b}$$

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where $\hat{m} \equiv (m_u + m_d)/2$. Eq. (1.15a) demonstrates the extreme lightness of the u, d quark masses. Most estimates of the strange quark mass place it at around $m_s(2\ \text{GeV}) \sim 100$ MeV [RPP 12], so that $\hat{m} \sim 4$ MeV, i.e., significantly smaller than the scale of *QCD*, Λ_{QCD}. Of course, the existence of very light quarks in the Standard Model is no more (or less) a mystery than is the existence of very heavy quarks. Both are determined by the Yukawa couplings of fermions to the Higgs boson, which are unconstrained (and not understood) input parameters of the theory. In any case, the small values of the u, d masses are responsible in *QCD* for the light pion, and for the usefulness of chiral symmetry techniques.

The mass relation of Eq. (1.15b) is the Gell-Mann–Okubo formula as applied to the octet of Goldstone bosons [GeOR 68]. It predicts $m_{\eta_8} = 566$ MeV, not far from the mass of the $\eta(549)$. The small difference between these mass values can be accounted for by second-order effects in the mass expansion. In particular, mixing of the η_8 with an $SU(3)$ singlet pseudoscalar produces a mass shift of order $(m_s - \hat{m})^2$. The difference between the predicted and physical masses is then an estimate of accuracy of the lowest-order predictions.

The use of the full pseudoscalar octet allows us to be sensitive to isospin breaking due to quark mass differences in a way not possible using only pions. This is because, to first order, the $\Delta I = 2$ mass difference $m_{\pi^\pm} - m_{\pi^0}$ is independent of the $\Delta I = 1$ mass difference $m_d - m_u$. In contrast, the kaons experience a mass splitting of first order in $m_d - m_u$. In particular, the quark mass contribution to the kaon mass difference is

$$\left(m_{K^0}^2 - m_{K^+}^2\right)_{\text{qk-mass}} = (m_d - m_u)\, B_0 = \left[\frac{m_d - m_u}{m_s - \hat{m}}\right]\left(m_K^2 - m_\pi^2\right). \tag{1.16}$$

In addition, there are electromagnetic contributions of the form

$$\left(m_{K^0}^2 - m_{K^+}^2\right)_{\text{em}} = m_{\pi^0}^2 - m_{\pi^+}^2. \tag{1.17}$$

This result, called Dashen's theorem [Da 69], follows in an effective lagrangian framework from (i) the vanishing of the electromagnetic self-energies of neutral mesons at lowest order in the energy expansion, and (ii) the fact that K^+ and π^+ fall in the same U-spin multiplet and hence are treated identically by the electromagnetic interaction, itself a U-spin singlet.[1] By isolating the quark mass and electromagnetic contributions to the kaon mass difference, we can write a sum rule,

$$\begin{aligned}\left[\frac{m_d - m_u}{m_s - \hat{m}}\right]\left(m_K^2 - m_\pi^2\right) &= \left[\frac{m_d - m_u}{m_d + m_u}\right] m_\pi^2 \\ &= \left(m_{K^0}^2 - m_{K^+}^2\right) - \left(m_{\pi^0}^2 - m_{\pi^+}^2\right),\end{aligned} \tag{1.18}$$

[1] Recall that U-spin is the $SU(2)$ subgroup of $SU(3)$ under which the d and s quarks are transformed.

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which yields

$$\frac{m_d - m_u}{m_s - \hat{m}} = 0.023, \qquad \frac{m_d - m_u}{m_d + m_u} = 0.29. \tag{1.19}$$

The u quark is seen to be lighter than the d quark, with $m_u/m_d \simeq 0.55$. The reason why this large deviation from unity does not play a major role in low-energy physics is that *both m_u and m_d are small compared to the confinement scale of QCD*. This is, in fact, the origin of isospin symmetry, which in terms of quark mass is simply the statement that neither m_u nor m_d plays a major physical role, aside from the crucial fact that $m_\pi \neq 0$. Why these two masses lie so close to zero is a question which the Standard Model does not answer.

Pion leptonic decay, radiative corrections, and F_π

Throughout our previous discussion of chiral lagrangians, the pion decay constant F_π has played an important role. It is defined by the relation

$$\left\langle 0 \left| A_\mu^j(0) \right| \pi^k(\mathbf{p}) \right\rangle = i F_\pi p_\mu \delta^{jk}, \tag{1.20}$$

where the axial-vector current A_μ^j is expressible in terms of the quark fields $\psi \equiv \begin{pmatrix} u \\ d \end{pmatrix}$ as

$$A_\mu^j = \bar{\psi} \gamma_\mu \gamma_5 \frac{\tau^j}{2} \psi. \tag{1.21}$$

This amplitude gives us the opportunity to display the way that the electroweak interactions are matched on to the low-energy strong interactions, and so we treat this topic in some detail.

The pion matrix element is probed experimentally in the decays $\pi \to \bar{e}\nu_e$ and $\pi \to \bar{\mu}\nu_\mu$, which are induced by the weak hamiltonian,

$$\mathcal{H}_{\rm w} = \frac{G_F}{\sqrt{2}} V_{\rm ud} \bar{\psi}_d \gamma_\lambda (1+\gamma_5) \psi_u \left[\bar{\psi}_{\nu_e} \gamma^\lambda (1+\gamma_5) \psi_e + \bar{\psi}_{\nu_\mu} \gamma^\lambda (1+\gamma_5) \psi_\mu \right]. \tag{1.22}$$

The decay $\pi^+ \to \mu^+ \nu_\mu$ has invariant amplitude,

$$\begin{aligned} \mathcal{M}_{\pi^+ \to \mu^+ \nu_\mu} &= \frac{G_F}{\sqrt{2}} V_{\rm ud} \sqrt{2} F_\pi p_\lambda \bar{u}_\nu \gamma^\lambda (1+\gamma_5) v_\mu \\ &= -G_F V_{\rm ud} F_\pi m_\mu \bar{u}_\nu (1-\gamma_5) v_\mu, \end{aligned} \tag{1.23}$$

where the Dirac equation has been used to obtain the second line. An analogous expression holds for $\pi^+ \to e^+ \nu_e$. We see here the well-known *helicity suppression* phenomenon. That is, the weak interaction current contains the left-handed chiral projection operator $(1+\gamma_5)$, which in the *massless* limit produces only left-handed

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particles and right-handed antiparticles. However, such a configuration is forbidden in the decay of a spin-zero particle to massless $\bar{\mu}\nu_\mu$ or $\bar{e}\nu_e$ because the leptons would be required to have combined angular momentum $J_z = 1$ along the decay axis. Thus the amplitudes for $\pi \to \bar{\mu}\nu_\mu,\ \bar{e}\nu_e$ must vanish in the limit $m_\mu = m_e = 0$. Since the neutrino is always left-handed, the $\mu^+,\ e^+$ in pion decay must have right-handed helicity to conserve angular momentum. It is helicity flip which introduces the factors of m_μ, m_e. The decay rate is found to be

$$\Gamma_{\pi^+\to\mu^+\nu_\mu} = \frac{G_F^2}{4\pi} F_\pi^2 m_\mu^2 m_\pi |V_{\rm ud}|^2 \left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 . \tag{1.24}$$

However, before using this expression to extract the pion decay constant, one must include radiative corrections. We shall do this in some detail because it illustrates the way to match electroweak loops onto hadronic calculations. Since a complete analysis would be overly lengthy, we present a simplified argument which stresses the underlying physics.

In Chap. V we found that the radiative correction to the muon lifetime is ultraviolet finite even in the approximation of a strictly local weak interaction. However, this is *not* the case for semileptonic transitions, as can be easily demonstrated. Consider the photon loop diagrams shown in Fig. VII–1. We divide the photon integration into hard and soft components. The former, which determine the ultraviolet properties of the diagrams, have short wavelengths $\lambda \ll R$, where R is a typical hadronic size, and are sensitive to the weak interaction at the quark level. In Landau gauge (i.e. ξ=0), the ultraviolet divergences arising from the wavefunction renormalization and vertex renormalization diagrams depicted in Fig. VII–1 vanish. For example, the vertex term is

$$\begin{aligned} I_{\rm vertex}^{\rm (u.v.)} &\sim \frac{iG_F}{\sqrt{2}} e^2 Q_4 Q_3 \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}\right) \\ &\quad \times \bar{u}_4 \gamma^\mu \frac{\not{k}}{k^2}\gamma_\lambda(1+\gamma_5)\frac{\not{k}}{k^2}\gamma^\nu u_3 \bar{u}_2 \gamma^\lambda (1+\gamma_5) u_1 \\ &\sim \frac{iG_F}{\sqrt{2}} e^2 Q_4 Q_3 \int \frac{d^4k}{(2\pi)^4} \bar{u}_4 \left[\frac{2\not{k}\gamma_\lambda\not{k}}{k^6} + \frac{\gamma_\lambda}{k^4}\right](1+\gamma_5) u_3 \bar{u}_2 \gamma^\lambda (1+\gamma_5) u_1, \end{aligned} \tag{1.25}$$

where $Q_i e$ is the electric charge of the $i^{\rm th}$ particle. Using

$$\int \frac{d^4k}{(2\pi)^4}\frac{k_\mu k_\nu}{k^6} = \frac{g_{\mu\nu}}{4}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^4} \sim \frac{i g_{\mu\nu}}{32\pi^2}\ln\Lambda, \tag{1.26}$$

we find that $I_{\rm vertex}^{\rm (u.v.)} = 0$ as claimed. It is clear, employing a Fierz transformation, that photon exchange between particles 4,1 and 2,3 is also ultraviolet-finite.

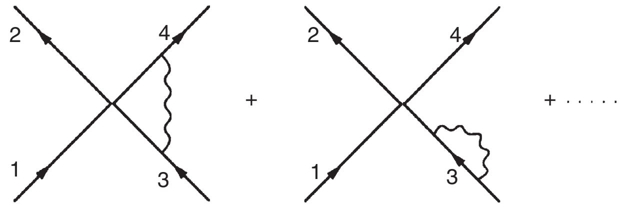

Fig. VII–1 Photonic radiative corrections to the weak quark–quark interaction.

This result simply represents the nonrenormalization of the vertex of a conserved current found in Chap. V. The only ultraviolet divergences then arise from final-state and initial-state interactions, i.e., photon exchange between particles 2,4, and 1,3, for which

$$\begin{aligned} I_{\rm fsi}^{(\rm u.v.)} &\sim \frac{-iG_F}{\sqrt{2}} e^2 Q_4 Q_2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} \left(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2} \right) \\ &\quad \times \bar{u}_4 \gamma^\mu \frac{\not{k}}{k^2} \gamma^\lambda (1+\gamma_5) u_3 \bar{u}_2 \gamma^\nu \frac{\not{k}}{k^2} \gamma_\lambda (1+\gamma_5) u_3 + (2,4 \to 1,3) \\ &\sim -\frac{G_F}{\sqrt{2}} \frac{e^2}{32\pi^2} Q_4 Q_2 \ln\Lambda [\bar{u}_4 \gamma_\mu \gamma_\alpha \gamma_\lambda (1+\gamma_5) u_3 \bar{u}_2 \gamma^\mu \gamma^\alpha \gamma^\lambda (1+\gamma_5) u_1 \\ &\quad - 4\bar{u}_4 \gamma_\lambda (1+\gamma_5) u_3 \bar{u}_2 \gamma^\lambda (1+\gamma_5) u_1] + (2,4 \to 1,3). \end{aligned} \tag{1.27}$$

Using the identity in Eq. (C–2.5) for reducing the product of three gamma matrices, Eq. (1.27) becomes

$$I_{\rm fsi}^{(\rm u.v.)} = -\mathcal{M}^{(0)} \times \frac{3\alpha}{2\pi} (Q_4 Q_2 + Q_3 Q_1) \ln(\Lambda/\mu_L), \tag{1.28}$$

where $\mathcal{M}^{(0)}$ is the lowest-order vertex. However, the full calculation of the radiative corrections must include the propagator for the W boson as well. When the contact weak interaction is replaced by the W-exchange diagram and is added to that with the photon-exchange replaced by Z-exchange, one obtains a finite result at the ultraviolet end with $\Lambda = m_Z$. The integral is cut off at the lower end at some point $\mu_L \sim m_\ell$ below which the full hadronic structure must be considered. In the case of muon decay we have

$$Q_e Q_{\nu_\mu} + Q_{\nu_e} Q_\mu = 0. \tag{1.29}$$

Thus, as found in Chap. V, there is no divergence. On the other hand, for beta decay we obtain

$$Q_e Q_u + Q_{\nu_e} Q_{d,s} = -\frac{2}{3}. \tag{1.30}$$

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We observe that there exists an important difference between the beta-decay effective weak coupling (G_β) and the muon-decay coupling (G_μ)

$$G_\beta = G_\mu \left(1 + \frac{\alpha}{\pi} \ln \frac{M_Z}{\mu_L}\right). \tag{1.31}$$

This hard-photon correction must be added to the soft-photon component, which can be found be evaluating the radiative corrections to a structureless ('point') pion with a high-energy cut-off μ_H. These were calculated long ago with the result [Be 58, KiS 59],

$$\frac{\Gamma_{\pi^+\to\mu^+\nu_\mu}}{\Gamma^{(0)}_{\pi^+\to\mu^+\nu_\mu}} = 1 + \frac{\alpha}{2\pi}\left(B(x) + 3\ln\frac{\mu_H}{m_\pi} - 6\ln\frac{\mu_H}{m_\mu}\right), \tag{1.32}$$

where

$$B(x) = 4\left[\frac{x^2+1}{x^2-1}\ln x - 1\right]\left[\ln(x^2-1) - 2\ln x - \frac{3}{4}\right] + 4\frac{x^2+1}{x^2-1}L(1-x^{-2}) - \ln x - \frac{3}{4} + \frac{10x^2-7}{(x^2-1)^2}\ln x + \frac{15x^2-21}{4(x^2-1)}, \tag{1.33}$$

with $L(z) = \int_0^z \frac{dt}{t}\ln(1-t)$ being the Spence function and $x = m_\pi/m_\mu$. Adding the hard- and soft-photon contributions with $\mu_H = \mu_L \simeq m_\rho$, we find the full radiative correction,

$$\Gamma_{\pi^+\to\mu^+\nu_\mu} \simeq \Gamma^{(0)}\left[1 + \frac{\alpha}{2\pi}\left(B(x) + 3\ln\frac{M_Z}{m_\pi} + \ln\frac{M_Z}{m_\rho} - 6\ln\frac{m_\rho}{m_\mu}\right)\right]. \tag{1.34}$$

Taking $V_{\rm ud}$ from Sect. XII–4 and $\Gamma^{(\rm expt)}_{\pi^+\to\mu^+\nu_\mu} = 3.841 \times 10^7\ {\rm s}^{-1}$, we find

$$F_\pi = 92.2 \pm 0.2\ \text{MeV}, \tag{1.35}$$

where we have appended an uncertainty associated with possible radiative effects $\mathcal{O}(\alpha/2\pi)$ that are not included in Eq. (1.34). For chiral symmetry applications in this book we shall generally employ the value

$$F_\pi \simeq 92\ \text{MeV}. \tag{1.36}$$

A clear indication of the importance of radiative corrections can be seen in the ratio

$$R = \frac{\Gamma_{\pi^+\to e^+\nu_e}}{\Gamma_{\pi^+\to\mu^+\nu_\mu}}, \tag{1.37}$$

which is strongly suppressed by the helicity mechanism discussed earlier. Application of the lowest-order formula given in Eq. (1.24) leads to a prediction

$$R^{(0)} = \frac{m_e^2}{m_\mu^2}\left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 = 1.283 \times 10^{-4}, \tag{1.38}$$

in disagreement with the measured value

$$R_{\text{expt}} = (1.230 \pm 0.004) \times 10^{-4}. \tag{1.39}$$

However, when the full radiative correction given in Eq. (1.34) is employed, the theoretical prediction is modified to become

$$R_{\text{thy}} = R^{(0)}\left(1 - 3\frac{\alpha}{\pi}\ln\frac{m_\mu}{m_e} + \cdots\right) = (1.2353 \pm 0.0001) \times 10^{-4}, \tag{1.40}$$

which is consistent with the experimental value.

VII–2 Chiral perturbation theory to one loop

Let us summarize the development thus far. Interactions of the Goldstone bosons can be expressed in terms of an effective lagrangian having the correct symmetry properties. To lowest order in the energy expansion, i.e., to order E^2, it suffices to use the minimal lagrangian of Eq. (1.2) at tree level. In the $SU(2)$ theory, this involves just the known constants F_π and m_π. At the next order, one encounters both the general $\mathcal{O}(E^4)$ lagrangian, given below, and also one-loop diagrams [ApB 81, GaL 84, 85*a*]. The $\mathcal{O}(E^4)$ lagrangian introduces new parameters, which must be determined from experiment. It is also necessary to give a prescription which allows one to handle the loop calculations. The general method is described in this section.

The program is called *chiral perturbation theory*. If one works to order E^4 in the energy expansion, there are typically three ingredients:

(1) the general lagrangian $\mathcal{L}_2$ (of order E^2) which is to be used both in loop diagrams and at tree level,
(2) the general lagrangian $\mathcal{L}_4$ (of order E^4) which is to be used only at tree level,
(3) the renormalization program which describes how to make physical predictions at one-loop level.

The general $\mathcal{O}(E^2)$ lagrangian has already been given in Eq. (1.2). Now we shall turn to the construction of the chiral $SU(n)$ lagrangian to order E^4.

The order E^4 lagrangian

The $\mathcal{O}(E^4)$ lagrangian can involve either four-derivative operators or two-derivative operators together with one factor of the quark mass term, $\chi \sim 2m_q B_0$ (which itself is of order m_π^2 or m_K^2) or products of two quark mass factors. There are four possible chiral-invariant terms with four separate derivatives,

$$\begin{aligned}&\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger\right), \quad \mathrm{Tr}\left(D_\mu U D_\nu U^\dagger D^\mu U D^\nu U^\dagger\right), \\ &\mathrm{Tr}\left(D_\mu U D_\nu U^\dagger\right) \cdot \mathrm{Tr}\left(D^\mu U D^\nu U^\dagger\right), \quad [\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right)]^2.\end{aligned} \tag{2.1}$$

Other structures, such as

$$\left[\mathrm{Tr}\left(\lambda^a U^\dagger D_\mu U\right)\mathrm{Tr}\left(\lambda^a U^\dagger D^\mu U\right)\right]^2, \tag{2.2}$$

can be expressed in terms of these by using $SU(n)$ matrix identities.

For the case of $SU(3)$, the operators in Eq. (2.1) are not linearly independent. The identities quoted in Eq. (II–2.17) can be used to show that

$$\begin{aligned}\mathrm{Tr}\left(D_\mu U D_\nu U^\dagger D^\mu U D^\nu U^\dagger\right) &= \frac{1}{2}\left[\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right)\right]^2 \\ &+ \mathrm{Tr}\left(D_\mu U D_\nu U^\dagger\right) \cdot \mathrm{Tr}\left(D^\mu U D^\nu U^\dagger\right) - 2\,\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger\right),\end{aligned} \tag{2.3}$$

leaving only three independent operators in this class. In $SU(2)$, a further identity,

$$2\,\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger\right) = \left[\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right)\right]^2, \tag{2.4}$$

leaves us with only two independent $\mathcal{O}(E^4)$ terms.

Another conceivable class of operators could have at least two derivatives acting on a single chiral matrix, such as

$$\mathrm{Tr}\left(D_\mu U D^\mu U^\dagger\right) \cdot \mathrm{Tr}\left(U^\dagger D_\nu D^\nu U\right). \tag{2.5}$$

However, since the E^4 lagrangian is to be used only at tree level, all states to which it is applied obey the equation of motion,

$$D^\mu\left(U^\dagger D_\mu U\right) + \frac{1}{2}\left(\chi^\dagger U - U^\dagger \chi\right) = 0. \tag{2.6}$$

This can be used to eliminate all the double-derivative operators in favor of those involving four single derivatives or with factors of χ.

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The remaining operators are reasonably straightforward to determine, and the *most general* $\mathcal{O}(E^4)$ $SU(3)$ chiral lagrangian is,[2]

$$\begin{aligned}
\mathcal{L}_4 &= \sum_{i=1}^{10} L_i O_i \\
&= L_1 \left[\mathrm{Tr}\, \left(D_\mu U D^\mu U^\dagger \right) \right]^2 + L_2 \, \mathrm{Tr}\, \left(D_\mu U D_\nu U^\dagger \right) \cdot \mathrm{Tr}\, \left(D^\mu U D^\nu U^\dagger \right) \\
&\quad + L_3 \, \mathrm{Tr}\, \left(D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger \right) \\
&\quad + L_4 \, \mathrm{Tr}\, \left(D_\mu U D^\mu U^\dagger \right) \mathrm{Tr}\, \left(\chi U^\dagger + U \chi^\dagger \right) \\
&\quad + L_5 \, \mathrm{Tr}\, \left(D_\mu U D^\mu U^\dagger \left(\chi U^\dagger + U \chi^\dagger \right) \right) + L_6 \left[\mathrm{Tr}\, \left(\chi U^\dagger + U \chi^\dagger \right) \right]^2 \\
&\quad + L_7 \left[\mathrm{Tr}\, \left(\chi^\dagger U - U \chi^\dagger \right) \right]^2 + L_8 \, \mathrm{Tr}\, \left(\chi U^\dagger \chi U^\dagger + U \chi^\dagger U \chi^\dagger \right) \\
&\quad + i L_9 \, \mathrm{Tr}\, \left(L_{\mu\nu} D^\mu U D^\nu U^\dagger + R_{\mu\nu} D^\mu U^\dagger D^\nu U \right) + L_{10} \, \mathrm{Tr}\, \left(L_{\mu\nu} U R^{\mu\nu} U^\dagger \right),
\end{aligned} \tag{2.7}$$

where $L_{\mu\nu}$, $R_{\mu\nu}$ are the field-strength tensors of external sources given in Eq. (IV–6.10). This is a central result of the effective lagrangian approach to the study of low-energy strong interactions. Much of the discussion in the chapters to follow will concern the above operators and involve a phenomenological determination of the $\{L_i\}$. In chiral $SU(2)$, three operators become redundant.

For completeness, we note that there may also exist two combinations of the external fields,

$$\mathcal{L}_{\mathrm{ext}} = \beta_1 \, \mathrm{Tr}\, \left(L_{\mu\nu} L^{\mu\nu} + R_{\mu\nu} R^{\mu\nu} \right) + \beta_2 \, \mathrm{Tr}\, \left(\chi^\dagger \chi \right),$$

which are chirally invariant without involving the matrix U. These do not generate any couplings to the Goldstone bosons and hence are not of great phenomenological interest. However, if one were to use the effective lagrangian to describe correlation functions of the external sources, these two operators can generate contact terms.

Finally, we summarize in Table VII–1 a set of values for the low-energy constants $\{L_i\}$ as obtained phenomenologically via a global fit to a range of low-energy data [BiJ 12]. (In this extraction certain assumptions are made also about the size of $\mathcal{O}(p^6)$ chiral coefficients, since they also contribute to observables.) These constants provide a characterization of the low-energy dynamics of *QCD*.

The renormalization program

The renormalization procedure is as follows. The lagrangian, $\mathcal{L}_2$, when expanded in terms of the meson fields, specifies a set of interaction vertices. These can be

[2] We are using the operator basis and notation first set down by Gasser and Leutwyler [GaL 85*a*].

Table VII–1. *Renormalized coefficients in the chiral lagrangian $\mathcal{L}_4$ given in units of 10^{-3} and evaluated at renormalization point $\mu = m_\rho$ [BiJ 12].*

Coefficient	Value	Origin
L_1^r	1.12 ± 0.20	$\pi\pi$ scattering
L_2^r	2.23 ± 0.40	and
L_3^r	-3.98 ± 0.50	$K_{\ell 4}$ decay
L_4^r	1.50 ± 1.01	F_K/F_π
L_5^r	1.21 ± 0.08	F_K/F_π
L_6^r	1.17 ± 0.95	F_K/F_π
L_7^r	-0.36 ± 0.18	Meson masses
L_8^r	0.62 ± 0.16	F_K/F_π
L_9^r	7.0 ± 0.2	Rare pion
L_{10}^r	-5.6 ± 0.2	decays

used to calculate tree-level and one-loop diagrams for any transition of interest. This result is added to the contribution which comes from the vertices contained in the $\mathcal{O}(E^4)$ lagrangian $\mathcal{L}_4$, treated at tree level only. At this stage, the result contains both bare parameters and divergent loop integrals. One needs to determine the parameters from experiment. The first step involves mass and wavefunction renormalization, as well as renormalization of F_π. In addition, the parameters entering from $\mathcal{L}_4$ need to be determined from data. If the lagrangian is indeed the most general one possible, relations between observables will be *finite* when expressed in terms of physical quantities. All the divergences will be absorbed into defining a set of renormalized parameters. This fundamental result is demonstrated explicitly in App. B–2.

There exists always an ambiguity of what finite constants should be absorbed into the renormalized parameters L_i^r. This ambiguity does not affect the relationship between observables, but only influences the numerical values quoted for the low-energy constants. Similarly, the regularization procedure for handling divergent integrals is arbitrary.[3] We use dimensional regularization and the renormalization prescription,

$$L_i^r = L_i - \frac{\gamma_i}{32\pi^2}\left[\frac{2}{d-4} - \ln(4\pi) + \gamma - 1\right], \qquad (2.8)$$

[3] Care must be taken that the regularization procedure does not destroy the chiral symmetry. Dimensional regularization does not cause any problems. When using other regularization schemes, one sometimes needs to append an extra contact interaction to maintain chiral invariance [GeJLW 71]. The problem arises due to the presence of derivative couplings, which imply that the interaction Hamiltonian is not simply the negative of the interaction lagrangian. The contact interaction vanishes in dimensional regularization.

where the constants γ_i are numbers given in Table B–1. When working to $\mathcal{O}(E^4)$ the following procedure is applied. One first computes the relevant vertices from $\mathcal{L}_2$ and $\mathcal{L}_4$. There are too many possible vertices to make a table of Feynman rules practical. In practice, the needed amplitudes are calculated for each application. The vertices from $\mathcal{L}_2$ are then used in loop diagrams, including mass and wave-function renormalizations. The results may be expressed in terms of the renormalized parameters of Eq. (2.8). If these low-energy constants can be determined from other processes, one has obtained a well-defined result. Including loops does add important physics to the result. The low-energy portion of the loop integrals describes the propagation and rescattering of low-energy Goldstone bosons, as required by the unitarity of the S matrix. One-loop diagrams add the unitarity corrections to the lowest-order amplitudes and in addition contain mass contributions and other effects from low energy.

The effective lagrangian may be used in the context either of chiral $SU(2)$ or of chiral $SU(3)$. Because $SU(2)$ is a subgroup of $SU(3)$, the general $SU(3)$ lagrangian of Eq. (2.7) is also valid for chiral $SU(2)$. However, the $SU(2)$ version has fewer low-energy constants, so that only certain combinations of the L_i^r will appear in pionic processes. If one is dealing with reactions involving only pions at low energy, the kaons and the eta are heavy particles and may be integrated out, such that only pionic effects need to be explicitly considered. This procedure produces a shift in the values of the low-energy renormalized constants L_i^r such that the L_i^r of a purely $SU(2)$ chiral lagrangian and an $SU(3)$ one will differ by a finite calculable amount. In this book, we shall use the $SU(3)$ values as our basic parameter set. The $SU(2)$ coefficients can be found by first performing calculations in the $SU(3)$ limit and then treating m_K^2, m_η^2 as large. Equivalently, all may be calculated at the same time using the background field method [GaL 85a]. The results are

$$
\begin{aligned}
&2L_1^{(2)r} + L_3^{(2)r} = 2L_1^r + L_3^r - \frac{1}{4}\ell_K, && L_2^{(2)r} = L_2^r - \frac{1}{4}\ell_K,\\
&2L_4^{(2)r} + L_5^{(2)r} = 2L_4^r + L_5^r - \frac{3}{2}\ell_K, && L_9^{(2)r} = L_9^r - \ell_K,\\
&2L_6^{(2)r} + L_8^{(2)r} = 2L_6^r + L_8^r - \frac{3}{4}\ell_K - \frac{1}{12}\ell_\eta, && L_{10}^{(2)r} = L_{10}^r + \ell_K,\\
&L_4^{(2)r} - L_6^{(2)r} - 9L_7^{(2)r} - 3L_8^{(2)r} = L_4^r - L_6^r - 9L_7^r - 3L_8^r + \frac{3}{2}\ell_k\\
&\qquad\qquad + \frac{F_\pi^2}{24m_\eta^2} + \frac{5}{1152\pi^2}\ln\frac{m_\eta^2}{\mu^2}, \qquad (2.9)
\end{aligned}
$$

where we use the superscript (2) to indicate constants in the $SU(2)$ theory and define $\ell_i \equiv \left[\ln(m_i^2/\mu^2) + 1\right]/384\pi^2$. In practice, these shifts are much smaller than the magnitude of the low-energy constants, so that we always simply quote the $SU(3)$ value.

Let us now calculate the mass and wavefunction renormalization constants to $\mathcal{O}(E^4)$ in chiral $SU(2)$. Setting $\chi = (m_u + m_d)B_0 \equiv m_0^2$, we may expand the basic lagrangian as

$$\mathcal{L}_2 = \frac{1}{2}\left[\partial^\mu \boldsymbol{\varphi} \cdot \partial_\mu \boldsymbol{\varphi} - m_0^2 \boldsymbol{\varphi} \cdot \boldsymbol{\varphi}\right] + \frac{m_0^2}{24F_0^2}(\boldsymbol{\varphi} \cdot \boldsymbol{\varphi})^2 + \frac{1}{6F_0^2}\left[(\boldsymbol{\varphi} \cdot \partial^\mu \boldsymbol{\varphi})\left(\boldsymbol{\varphi} \cdot \partial_\mu \boldsymbol{\varphi}\right) - (\boldsymbol{\varphi} \cdot \boldsymbol{\varphi})\left(\partial^\mu \boldsymbol{\varphi} \cdot \partial_\mu \boldsymbol{\varphi}\right)\right] + \mathcal{O}(\boldsymbol{\varphi}^6), \qquad (2.10)$$

$$\mathcal{L}_4 = \frac{m_0^2}{F_0^2}\left[16L_4^{(2)} + 8L_5^{(2)}\right]\frac{1}{2}\partial_\mu \boldsymbol{\varphi} \cdot \partial^\mu \boldsymbol{\varphi} - \frac{m_0^2}{F_0^2}\left[32L_6^{(2)} + 16_8^{(2)}\right]\frac{1}{2}m_0^2 \boldsymbol{\varphi} \cdot \boldsymbol{\varphi} + \mathcal{O}(\boldsymbol{\varphi}^4),$$

where F_0 denotes the value of F_π prior to loop corrections. When this lagrangian is used in the calculation of the propagator, the terms of $\mathcal{O}(\boldsymbol{\varphi}^4)$ in $\mathcal{L}_2$ will contribute to the self-energy via one-loop diagrams, which involve the following d-dimensional integrals,

$$\begin{aligned}
\delta_{jk} I(m^2) &= i\Delta_{Fjk}(0) = \langle 0|T\boldsymbol{\varphi}_j(x)\boldsymbol{\varphi}_k(x)|0\rangle, \\
I(m^2) &= \mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\,\frac{i}{k^2 - m^2} = \frac{\mu^{4-d}}{(4\pi)^{d/2}}\Gamma\left(1 - \frac{d}{2}\right)\left(m^2\right)^{\frac{d}{2}-1}, \\
\delta_{jk} I_{\mu\nu}(m^2) &= -\partial_\mu\partial_\nu i\Delta_{Fjk}(0) = \langle 0|T\partial_\mu\boldsymbol{\varphi}_j(x)\partial_\nu\boldsymbol{\varphi}_k(x)|0\rangle, \\
I_{\mu\nu}(m^2) &= \mu^{4-d}\int \frac{d^dk}{(2\pi)^d}k_\mu k_\nu \frac{i}{k^2 - m^2} = g_{\mu\nu}\frac{m^2}{d}I(m^2). \qquad (2.11)
\end{aligned}$$

These contributions can be read off from $\mathcal{L}_2$ by considering all possible contractions among the $\mathcal{O}(\boldsymbol{\varphi}^4)$ terms, and result in the quadratic effective lagrangian,

$$\begin{aligned}
\mathcal{L}_{\text{eff}} &= \frac{1}{2}\partial^\mu \boldsymbol{\varphi} \cdot \partial_\mu \boldsymbol{\varphi} - \frac{1}{2}m_0^2 \boldsymbol{\varphi} \cdot \boldsymbol{\varphi} + \frac{5m_\pi^2}{12F_\pi^2}I(m_\pi^2)\boldsymbol{\varphi} \cdot \boldsymbol{\varphi} \\
&\quad + \frac{1}{6F_\pi^2}\left(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}\right)I(m_\pi^2)\left(\delta_{ij}\partial^\mu \boldsymbol{\varphi}_k \partial_\mu \boldsymbol{\varphi}_l + \delta_{k\ell}m_\pi^2 \boldsymbol{\varphi}_i \boldsymbol{\varphi}_j\right) \\
&\quad + \frac{1}{2}\partial_\mu \boldsymbol{\varphi} \cdot \partial^\mu \boldsymbol{\varphi}\frac{m_\pi^2}{F_\pi^2}\left[16L_4^{(2)} + 8L_5^{(2)}\right] - \frac{1}{2}m_\pi^2 \boldsymbol{\varphi} \cdot \boldsymbol{\varphi}\frac{m_\pi^2}{F_\pi^2}\left[32L_6^{(2)} + 16L_8^{(2)}\right] \\
&= \frac{1}{2}\partial^\mu \boldsymbol{\varphi} \cdot \partial_\mu \boldsymbol{\varphi}\left[1 + \left(16L_4^{(2)} + 8L_5^{(2)}\right)\frac{m_\pi^2}{F_\pi^2} - \frac{2}{3F_\pi^2}I(m_\pi^2)\right] \\
&\quad - \frac{1}{2}m_0^2 \boldsymbol{\varphi} \cdot \boldsymbol{\varphi}\left[1 + \left(32L_6^{(2)} + 16L_8^{(2)}\right)\frac{m_\pi^2}{F_\pi^2} - \frac{1}{6F_\pi^2}I(m_\pi^2)\right]. \qquad (2.12)
\end{aligned}$$

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To one-loop order, there are no other contributions to the self energy. Observe that we have changed $m_0,\ F_0$ into $m_\pi,\ F_\pi$ in all of the $\mathcal{O}(E^4)$ corrections, as the difference between the two is of yet higher order in the energy expansion. If we expand in powers of $d-4$ and define the renormalized pion field as $\boldsymbol{\varphi}_r = Z_\pi^{-1/2}\boldsymbol{\varphi}$ with

$$Z_\pi = 1 - \frac{8m_\pi^2}{F_\pi^2}\left(2L_4^{(2)} + L_5^{(2)}\right) + \frac{m_\pi^2}{24\pi^2 F_\pi^2}\left[\frac{2}{d-4} + \gamma - 1 - \ln 4\pi + \ln\frac{m_\pi^2}{\mu^2}\right], \tag{2.13}$$

then the lagrangian assumes the canonical form

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\partial_\mu\boldsymbol{\varphi}_r\cdot\partial^\mu\boldsymbol{\varphi}_r - \frac{1}{2}m_\pi^2\boldsymbol{\varphi}_r\cdot\boldsymbol{\varphi}_r. \tag{2.14}$$

Note that, using the definitions of the renormalized parameters, the physical pion mass is identified as

$$m_\pi^2 = m_0^2\left[1 - \frac{8m_\pi^2}{F_\pi^2}\left[2L_4^{(2)r} + L_5^{(2)r} - 4L_6^{(2)r} - 2L_8^{(2)r}\right] + \frac{m_\pi^2}{32\pi^2 F_\pi^2}\ln\frac{m_\pi^2}{\mu^2}\right]. \tag{2.15}$$

The quantity $\mathcal{L}_{\text{eff}}$ in Eq. (2.14) is the quadratic portion of the one-loop effective lagrangian. Since loop effects have already been accounted for, it is to be used at tree level. This is a simple application of the background field renormalization discussed in App. B–2.

VII–3 The nature of chiral predictions

In order to understand how predictions are made in effective field theory as well as the range of validity of the energy expansion, let us work out several examples. At first, these will seem to be rather obscure processes, but they are the simplest hadronic reactions of *QCD*. As the bosonic interactions of the Goldstone bosons of the theory, they are the cleanest processes for demonstrating the dynamical content of the symmetries and anomalies of *QCD*.

The pion form factor

The electromagnetic form factor of charged pions is required by Lorentz invariance and gauge invariance to have the form[4]

$$\langle\pi^+(\mathbf{p}_2)\left|J^\mu_{\text{em}}\right|\pi^+(\mathbf{p}_1)\rangle = G_\pi(q^2)\,(p_1+p_2)^\mu\,, \tag{3.1}$$

[4] The neutral pion form factor is required to vanish by charge conjugation invariance.

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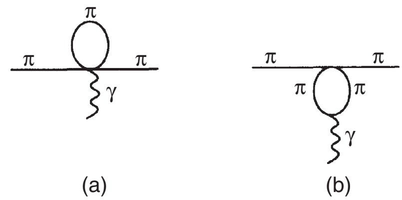

Fig. VII–2 Radiative corrections to the pion form factor.

where $q^\mu = (p_1 - p_2)^\mu$ and $G_\pi(0) = 1$. The electromagnetic current may be identified from the effective lagrangian of Eq. (1.2) by setting $\ell^\mu = r^\mu = eQA^\mu$, $\chi = 2B_0\mathbf{m}$, where Q is the quark charge matrix and $\mathbf{m}$ is the quark mass matrix. To $\mathcal{O}(E^4)$, we then find

$$J^\mu_{\text{em}} = -\frac{\partial\mathcal{L}}{\partial(eA_\mu)} = (\boldsymbol{\varphi}\times\partial^\mu\boldsymbol{\varphi})_3\left[1 - \frac{1}{3F^2}\boldsymbol{\varphi}\cdot\boldsymbol{\varphi} + \mathcal{O}(\boldsymbol{\varphi}^4)\right]$$
$$+ (\boldsymbol{\varphi}\times\partial^\mu\boldsymbol{\varphi})_3\left[16L_4^{(2)} + 8L_5^{(2)}\right]\frac{m_\pi^2}{F^2} + \frac{4L_9^{(2)}}{F^2}\partial^\nu\left(\partial^\mu\boldsymbol{\varphi}\times\partial_\nu\boldsymbol{\varphi}\right)_3 + \cdots . \tag{3.2}$$

The renormalization of this current involves the Feynman diagrams in Fig. VII–2. That of Fig. VII–2(a) is simply found using the integral previously defined in Eq. (2.11),

$$J^\mu_{\text{em}}\Big|_{(2a)} = -\frac{5}{3F_\pi^2}(\boldsymbol{\varphi}\times\partial^\mu\boldsymbol{\varphi})_3\, I(m_\pi^2). \tag{3.3}$$

Evaluation of Fig. VII–2(b) is somewhat more complicated. Using the elastic $\pi^+\pi^-$ scattering amplitude given by $\mathcal{L}_2$,

$$\langle\pi^+(\mathbf{k}_1)\pi^-(\mathbf{k}_2)|\pi^+(\mathbf{p}_1)\pi^-(\mathbf{p}_2)\rangle$$
$$= \frac{i}{3F_0^2}\left(2m_0^2 + p_1^2 + p_2^2 + k_1^2 + k_2^2 - 3\,(p_1 - k_1)^2\right), \tag{3.4}$$

we compute the vertex amplitude to be

$$\langle J^\mu_{\text{em}}\rangle_{(2b)} = -\frac{i}{3F_\pi^2}\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{\left(k+\frac{1}{2}q\right)^2 - m_\pi^2 + i\epsilon}\,\frac{1}{\left(k-\frac{1}{2}q\right)^2 - m_\pi^2 + i\epsilon}$$
$$\times\left[4m_\pi^2 + \left(k+\frac{q}{2}\right)^2 + \left(k-\frac{q}{2}\right)^2 - 3\left(k + \frac{(p_1+p_2)}{2}\right)^2\right]2k^\mu. \tag{3.5}$$

Upon integration, most terms drop out because of antisymmetry under $k^\mu \to -k^\mu$, and we find

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$$\langle J^\mu_{\rm em}\rangle_{(2b)} = \frac{2i}{F_\pi^2}\int \frac{d^4k}{(2\pi)^4}\,\frac{k^\mu k\cdot(p_1+p_2)}{\left(\left(k+\frac{1}{2}q\right)^2 - m_\pi^2\right)\left(\left(k-\frac{1}{2}q\right)^2 - m_\pi^2\right)}. \tag{3.6}$$

We can evaluate this integral using dimensional regularization,

$$\begin{aligned}\langle J^\mu_{\rm em}\rangle_{(2b)} = -\frac{2}{F_\pi^2}\,\frac{\mu^{4-d}}{(4\pi)^{d/2}}\int dx\Bigg[-\frac{1}{2}\,\frac{(p_1+p_2)^\mu\,\Gamma\left(1-\frac{d}{2}\right)}{\left(m_\pi^2 - q^2x\,(1-x)\right)^{1-d/2}} \\ + q^\mu q\cdot(p_1+p_2)\left(x+\frac{1}{2}\right)^2\Gamma\left(2-\frac{d}{2}\right)\left(m_\pi^2 - q^2x\,(1-x)\right)^{d/2-2}\Bigg],\end{aligned} \tag{3.7}$$

where as usual μ is an arbitrary scale introduced in order to maintain the proper dimensions. On-shell, we can disregard the term in q^μ since $q\cdot(p_1+p_2) = m_\pi^2 - m_\pi^2 = 0$. For the remaining piece, we expand about $d = 4$ to obtain

$$\begin{aligned}\langle J^\mu_{\rm em}\rangle_{(2b)} = \frac{1}{(4\pi F_\pi)^2}\,(p_1+p_2)^\mu\int_0^1 dx\ \left(m_\pi^2 - q^2x(1-x)\right) \\ \times\left[\left(\frac{2}{d-4}+\gamma-1-\ln 4\pi\right)+\ln\frac{m_\pi^2 - q^2x(1-x)}{\mu^2}\right],\end{aligned} \tag{3.8}$$

and the x-integration then yields

$$\begin{aligned}\langle J^\mu_{\rm em}\rangle_{(2b)} = \frac{1}{(4\pi F_\pi)^2}\,(p_1+p_2)^\mu\left\{\left(m_\pi^2-\frac{1}{6}q^2\right)\left[\frac{2}{d-4}+\gamma-1-\ln 4\pi\right.\right. \\ \left.\left.+\ln\frac{m_\pi^2}{\mu^2}\right]+\frac{1}{6}\left(q^2-4m_\pi^2\right)H\left(\frac{q^2}{m_\pi^2}\right)-\frac{1}{18}q^2\right\},\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}H(a) &\equiv -\int_0^1 dx\,\ln\left(1-ax(1-x)\right) \\ &= \begin{cases} 2-2\sqrt{\frac{4}{a}-1}\,{\rm ctn}^{-1}\sqrt{\frac{4}{a}-1} & (0<a<4) \\ 2+\sqrt{1-\frac{4}{a}}\left[\ln\frac{\sqrt{1-\frac{4}{a}}-1}{\sqrt{1-\frac{4}{a}}+1}+i\pi\theta(a-4)\right] & (\text{otherwise}).\end{cases}\end{aligned} \tag{3.10}$$

Now we add everything together. The tree-level amplitude is modified by wave-function renormalization,

$$\begin{aligned}Z_\pi G_\pi^{(\rm tree)}(q^2) = \Bigg[1-\frac{8m_\pi^2}{F_\pi^2}\left(2L_4^{(2)}+L_5^{(2)}\right) \\ +\frac{m_\pi^2}{24\pi^2F_\pi^2}\left\{\frac{2}{d-4}+\gamma-1-\ln 4\pi+\ln\frac{m_\pi^2}{\mu^2}\right\}\Bigg]\end{aligned}$$

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$$\times\left[1+\frac{8m_\pi^2}{F_\pi^2}\left(2L_4^{(2)}+L_5^{(2)}\right)+2q^2\frac{L_9^{(2)}}{F_\pi^2}\right]$$
$$=\left[1+\frac{m_\pi^2}{24\pi^2F_\pi^2}\left(\frac{2}{d-4}+\gamma-1-\ln 4\pi+\ln\frac{m_\pi^2}{\mu^2}\right)+\frac{2L_9^{(2)}}{F_\pi^2}q^2\right], \tag{3.11}$$

while Figs. VII–2(a,b) contribute as

$$G_\pi(q^2)\Big|_{(2a)}=-\frac{5m_\pi^2}{48\pi^2F_\pi^2}\left\{\frac{2}{d-4}+\gamma-1-\ln 4\pi+\ln\frac{m_\pi^2}{\mu^2}\right\},$$
$$G_\pi(q^2)\Big|_{(2b)}=\frac{1}{16\pi^2F_\pi^2}\left\{\left(m_\pi^2-\frac{1}{6}q^2\right)\left[\frac{2}{d-4}+\gamma-1-\ln 4\pi+\ln\frac{m_\pi^2}{\mu^2}\right]\right.$$
$$\left.+\frac{1}{6}\left(q^2-4m_\pi^2\right)H\left(\frac{q^2}{m_\pi^2}\right)-\frac{1}{18}q^2\right\}, \tag{3.12}$$

respectively. Summing Eqs. (3.11), (3.12) we see that all terms independent of q^2 cancel, $L_9^{(2)}$ becomes $L_9^{(2)r}$ and the final result is

$$G_\pi(q^2)=1+\frac{2L_9^{(2)r}}{F_\pi^2}q^2+\frac{1}{96\pi^2F_\pi^2}\left[\left(q^2-4m_\pi^2\right)H\left(\frac{q^2}{m_\pi^2}\right)-q^2\ln\frac{m_\pi^2}{\mu^2}-\frac{q^2}{3}\right]. \tag{3.13}$$

The divergences have been absorbed in $L_9^{(2)r}$, while the imaginary part required by unitarity is contained in $H(q^2/m_\pi^2)$. Note that the loops also induce a non-power-law behavior in $G_\pi(q^2)$. However, numerically this turns out to be small and is unobservable in practice. A simple linear approximation,

$$G_\pi(q^2)=1+q^2\left[\frac{2L_9^{(2)r}}{F_\pi^2}-\frac{1}{96\pi^2F_\pi^2}\left(\ln\frac{m_\pi^2}{\mu^2}+1\right)\right]+\cdots, \tag{3.14}$$

is obtained by Taylor expanding about $q^2=0$. The corresponding result for chiral $SU(3)$ is

$$G_\pi(q^2)=1+q^2\left[\frac{2L_9^r}{F_\pi^2}-\frac{1}{96\pi^2F_\pi^2}\left(\ln\frac{m_\pi^2}{\mu^2}+\frac{1}{2}\ln\frac{m_K^2}{\mu^2}+\frac{3}{2}\right)\right]+\cdots. \tag{3.15}$$

The pion form factor is generally parameterized in terms of a charge radius,

$$G_\pi(q^2)=1+\frac{1}{6}\langle r_\pi^2\rangle q^2+\cdots. \tag{3.16}$$

Thus, for any given value of the energy scale μ, the parameter L_9^r can be determined from the experimental charge radius. From the present experimental value $\langle r_\pi^2\rangle=(0.45\pm0.01)\ \text{fm}^2$, we obtain $L_9^r(\mu=m_\rho)=(7.0\pm0.2)\times10^{-3}$.

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The scale μ enters the calculation in such a way that, had we used a different scale μ' but kept the physical result invariant, we would have had

$$L_9^r(\mu') = \begin{cases} L_9^r(\mu) - \dfrac{1}{192\pi^2}\ln\left(\dfrac{\mu'^2}{\mu^2}\right) & (SU(2)) \\ L_9^r(\mu) - \dfrac{1}{128\pi^2}\ln\left(\dfrac{\mu'^2}{\mu^2}\right) & (SU(3)). \end{cases} \tag{3.17}$$

In fact, if we look back to the origin of $\ln\mu$ in the transition from Eq. (3.7) to Eq. (3.8) using

$$\mu^{4-d}\frac{2}{d-4} = \frac{2}{d-4} - \ln\mu^2 + \mathcal{O}(d-4), \tag{3.18}$$

we see that the scale dependence is always tied to the coefficient of the divergence.[5] The general result is then

$$L_i^r(\mu') = L_i^r(\mu) - \frac{\gamma_i}{32\pi^2}\ln\left(\frac{\mu'^2}{\mu^2}\right), \tag{3.19}$$

where $\{\gamma_i\}$ are the constants of Table B–1 of App. B, used in the renormalization condition of Eq. (2.8).

This calculation also nicely illustrates the range of validity of the energy expansion. The pion form factor is well described by a monopole form,

$$G_\pi(q^2) \simeq \frac{1}{1-q^2/m^2} = 1 + \frac{q^2}{m^2} + \cdots, \tag{3.20}$$

with $m \simeq m_{\rho(770)}$. The energy expansion is then in powers of q^2/m^2. At the other extreme, the pion form factor can also be treated in *QCD* when q^2 is very large [BrL 80].

Rare processes

The calculation above is clearly non-predictive as it contains a free parameter, L_9^r, which must be determined phenomenologically. However, predictions do arise when more reactions are considered because relations exist between amplitudes as a consequence of the underlying chiral symmetry. In particular, there exists a set of reactions which are described in terms of two low-energy constants. These pionic reactions are shown in Table VII–2. With the additional input of F_K/F_π, the kaonic reactions shown there are also predicted. Each case contains hadronic form factors

[5] We have chosen to keep the low-energy constants $\{L_i\}$ dimensionless in the extension to d dimensions. In [GaL 85*a*], the constants have dimension μ^{d-4}. However, the resulting physics is identical in the limit $d \to 4$.

Table VII–2. *The radiative complex of pion and kaon transitions.*

Pions	Kaons
$\gamma \to \pi^+\pi^-$	$\gamma \to K^-K^+$
$\gamma\pi^+ \to \gamma\pi^+$	$\gamma K^+ \to \gamma K^+$
$\pi^+ \to e^+\nu_e\gamma$	$K^+ \to e^+\nu_e\gamma$
$\pi^+ \to \pi^0 e^+\nu_e$	$K \to \pi e^+\nu_e$
$\pi^+ \to e^+\nu_e e^+e^-$	$K^+ \to e^+\nu_e e^+e^-$
	$K^+ \to \pi^0 e^+\nu_e\gamma$

which need to be calculated. This section briefly describes the procedure for relating such reactions in chiral perturbation theory. All calculations follow the pattern described above, so that we shall only quote the results [GaL 85*a*, DonH 89].

In the processes involving photons ($\pi^+ \to e^+\nu_e\gamma, \pi^+ \to e^+\nu_e e^+e^-$ and $\gamma\pi^+ \to \gamma\pi^+$), there are always Born diagrams where the photon couples to hadrons through the known $\pi\pi\gamma$ coupling. These are shown in Fig. VII–3. In addition, there can be direct contact interactions associated with the structure of the pions. These introduce new form factors. For the decays $\pi^+ \to e^+\nu_e\gamma$, $e^+\nu_e e^+e^-$, the matrix elements are

$$\begin{aligned}\mathcal{M}_{\pi^+\to e^+\nu_e\gamma} &= -\frac{eG_F}{\sqrt{2}}\cos\theta_1 M_{\mu\nu}(p,q)\epsilon^{\mu *}(q)\bar{u}(p_\nu)\gamma^\nu(1+\gamma_5)v(p_e),\\ \mathcal{M}_{\pi^+\to e^+\nu_e e^+e^-} &= \frac{e^2G_F}{\sqrt{2}}\cos\theta_1 M_{\mu\nu}(p,q)\frac{1}{q^2}\\ &\quad\times \bar{u}(p_2)\gamma^\mu v(p_1)\bar{u}(p_\nu)\gamma^\nu(1+\gamma_5)v(p_e),\end{aligned} \tag{3.21}$$

where the hadronic part of the quantity $M_{\mu\nu}$ has the general structure

$$\begin{aligned}M_{\mu\nu}(p,q) &= \int d^4x\, e^{iq\cdot x}\langle 0\left|T\left(J_\mu^{\rm em}(x)J_\nu^{1-i2}(0)\right)\right|\pi^+(\mathbf{p})\rangle\\ &= -\sqrt{2}\,F_\pi\frac{(p-q)_\nu}{(p-q)^2-m_\pi^2}\langle\pi^+(\mathbf{p}-\mathbf{q})\left|J_\mu^{\rm em}\right|\pi^+(\mathbf{p})\rangle + \sqrt{2}\,F_\pi g_{\mu\nu}\\ &\quad - h_A\left((p-q)_\mu q_\nu - g_{\mu\nu}q\cdot(p-q)\right) - r_A(q_\mu q_\nu - g_{\mu\nu}q^2)\\ &\quad + ih_V\epsilon_{\mu\nu\alpha\beta}q^\alpha p^\beta.\end{aligned} \tag{3.22}$$

The first line represents the tree diagram and in subsequent lines the subscripts V and A indicate whether the vector or axial-vector portions of the weak currents are involved. The form factor r_A in Eq. (3.22) can only contribute with virtual photons, i.e., as in $\pi^+ \to e^+\nu_e e^+e^-$.

The $\gamma\pi^+ \to \gamma\pi^+$ reaction is analyzed in terms of the pion's electric and magnetic polarizabilities, α_E and β_M, which describe the response of the pion to electric

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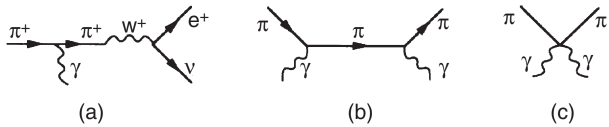

Fig. VII–3 Tree diagrams for (a) $\pi^+ \to e^+\nu_e\gamma$, $\pi^+ \to e^+\nu_e e^+ e^-$, and (b)–(c) $\gamma\pi^+ \to \gamma\pi^+$.

and magnetic fields. In the static limit, electromagnetic fields induce the electric and magnetic dipole moments,

$$\mathbf{p}_E = 4\pi\alpha_E \mathbf{E}, \qquad \boldsymbol{\mu}_M = 4\pi\beta_M \mathbf{H}, \tag{3.23}$$

which correspond to an interaction energy

$$V = -2\pi \left(\alpha_E \mathbf{E}^2 + \beta_M \mathbf{H}^2\right). \tag{3.24}$$

These forms emerge in the non-relativistic limit of the general Compton amplitude

$$\begin{aligned} -iT_{\mu\nu}(p, p', q_1) &= -i \int d^4x\, e^{iq_1\cdot x} \langle \pi^+(\mathbf{p}') \left| T\left(J_\mu^{\rm em}(x) J_\nu^{\rm em}(0)\right) \right| \pi^+(\mathbf{p})\rangle \\ &= \frac{\left(2p' + q_2\right)_\nu (2p - q_1)_\mu}{(p - q_1)^2 - m_\pi^2} + \frac{\left(2p' + q_1\right)_\mu (2p - q_2)_\nu}{(p - q_2)^2 - m_\pi^2} - 2g_{\mu\nu} \\ &\quad + \sigma \left(q_{2\mu} q_{1\nu} - g_{\mu\nu} q_1 \cdot q_2\right) + \cdots, \end{aligned} \tag{3.25}$$

where σ is a coefficient proportional to the polarizability and q_1^μ, q_2^μ are the photon momenta, taken as outgoing, with $p = p' + q_1 + q_2$. The first three pieces are the Born and seagull diagrams. The last contains the extra term which emerges from higher-order chiral lagrangians, and the ellipses indicate the presence of other possible gauge-invariant structures, which we shall not need.

The chiral predictions are obtained in the same manner as used for the pion form factor. The results at $q^2 \simeq 0$ are

$$\begin{aligned} h_V &= \left.\frac{N_c}{12\sqrt{2}\pi^2 F_\pi}\right|_{N_c=3} = 0.027\, m_\pi^{-1}, \quad \frac{h_A}{h_V} = 32\pi^2 \left(L_9^{(2)r} + L_{10}^{(2)r}\right), \\ \frac{r_A}{h_V} &= 32\pi^2 \left[L_9^{(2)r} - \frac{1}{192\pi^2}\left(\ln\frac{m_\pi^2}{\mu^2} + 1\right)\right], \quad \alpha_E + \beta_M = 0, \\ \alpha_E &= \frac{\alpha}{2m_\pi}\sigma = \frac{4\alpha}{m_\pi F_\pi^2}\left[L_9^{(2)r} + L_{10}^{(2)r}\right] - \frac{\alpha}{m_\pi}\frac{1}{16\pi^2 F_\pi^2}\left(1 + F(t/m_\pi^2)\right), \end{aligned} \tag{3.26}$$

where $t = (q_1 + q_2)^2$ and

$$F(x) \equiv -\frac{4}{x}\sinh^2\left(\sqrt{x}/2\right) \xrightarrow[1\gg x]{} -1 - \frac{x}{12} + \cdots. \tag{3.27}$$

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Table VII–3. *Chiral predictions and data in the radiative complex of transitions.*

Reaction	Quantity	Theory	Experiment
$\gamma \to \pi^+\pi^-$	$\langle r_\pi^2 \rangle$ (fm^2)	0.45^a	0.45 ± 0.01
$\gamma \to K^+K^-$	$\langle r_K^2 \rangle$ (fm^2)	0.45	0.31 ± 0.03
$\pi^+ \to e^+\nu_e\gamma$	$h_V (m_\pi^{-1})$	0.027	0.0254 ± 0.0017
	h_A/h_V	0.441^a	0.441 ± 0.004
$K^+ \to e^+\nu_e\gamma$	$(h_V + h_A)(m_K^{-1})$	0.136	0.133 ± 0.008
$\pi^+ \to e^+\nu_e e^+ e^-$	r_A/h_V	2.6	2.2 ± 0.3
$\gamma\pi^+ \to \gamma\pi^+$	$(\alpha_E + \beta_M)(10^{-4}$ fm)	0	0.17 ± 0.02
	$(\alpha_E - \beta_M)\,(10^{-4}$ fm)	5.6	13.6 ± 2.8
$K \to \pi e^+\nu_e$	$\xi = f_-(0)/f_+(0)$	-0.13	-0.17 ± 0.02
	λ_+ (fm^2)	0.067	0.0605 ± 0.001
	λ_0 (fm^2)	0.040	0.0400 ± 0.002

[a]Used as input.

The prediction for h_V is especially interesting since h_V is related by an isospin rotation to the amplitude for $\pi^0 \to \gamma\gamma$ (cf. Prob. VII–2). As we will show in Sect. VII–6, this is absolutely predicted from the axial anomaly. The presence of L_{10}^r implies that one of the above measurements must be used to determine it. We use the precisely known value for h_A/h_V to yield

$$L_{10}^r(\mu = m_\rho) = -(5.6 \pm 0.2) \times 10^{-3} \,. \tag{3.28}$$

The results are compared with experiment in Table VII–3.

We see that, with one exception, the chiral predictions are in agreement with experiment. That exception, the electric polarizability in $\gamma\pi^+ \to \gamma\pi^+$, comes from two difficult experiments. One uses a pion beam on a heavy Z atom [An *et al*. 85] and the coulomb exchange in $\pi^+ A \to \gamma\pi^+ A$ is used to provide the extra photon (this is called the Primakoff effect). The tree diagram must be carefully subtracted off. The second experiment involves the use of a high-energy photon beam and the $p(\gamma, \gamma\pi^+)n$ reaction and extrapolation to the virtual pion pole [Ah *et al*. 05]. In this case there exist a large number of background processes which must be subtracted. We note however that a recent experiment [Fr 12] using the Primakoff effect, not yet included in the averages above, obtains $\alpha_E - \beta_M = 3.8 \pm 1.4 \pm 1.6$. Before being concerned with the possible discrepancy with the chiral prediction, it would be preferable to have the experimental situation clarified. We have also listed the known results on kaonic processes predicted by the same constants. The analyses for $\gamma \to K^+K^-$ and $K \to e^+\nu_e\gamma$ are identical to the above results.

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Pion–pion scattering

The elastic scattering of two Goldstone bosons is the purest manifestation of the chiral effective field theory of *QCD*. It is a classic topic with a long history. We use it here as an example of the convergence of the perturbative effective field theory expansion.

The scattering of pions can be classified as S-wave, P-wave, D-wave, etc., with low partial waves dominating at low-energy. The amplitudes also can be decomposed in overall isospin, $I = 0,\ 1,\ 2$. Because the pions are spinless bosons, Bose symmetry requires that the even partial waves carry $I = 0,\ 2$ and odd angular momentum requires $I = 1$. At low energies, the partial wave amplitude can be expanded in terms of a scattering length a_ℓ^I and slope b_ℓ^I, defined by

$$\mathrm{Re}\, T_\ell^I = \left(\frac{q^2}{m_\pi^2}\right)^\ell \left(a_\ell^I + b_\ell^I \frac{q^2}{m_\pi^2} + \cdots\right), \tag{3.29}$$

where $q^2 \equiv \left(s - 4m_\pi^2\right)/4$. Since the chiral expansion is similarly a power series in the energy, a_ℓ^I and b_ℓ^I provide a useful set of quantities to study. In practice, they are extracted from data by using dispersion relations and crossing symmetry to extrapolate some of the higher-energy data down to threshold. The only accurate very low-energy data are those from $K \to \pi\pi e \bar{\nu}_e$. The experimental values are given in Table VII–4.

At lowest order in the energy expansion, the amplitude for $\pi\pi$ scattering [We 66] can be obtained from $\mathcal{L}_2$ with the result

$$A(s,t,u) = \frac{s - m_\pi^2}{F_\pi^2}. \tag{3.30}$$

Table VII–4. *The pion scattering lengths and slopes.*

	Experimental	Lowest order[a]	First two orders[a]
a_0^0	0.220 ± 0.005	0.16	0.20
b_0^0	0.25 ± 0.03	0.18	0.26
a_0^2	-0.044 ± 0.001	-0.045	-0.041
b_2^2	-0.082 ± 0.008	-0.089	-0.070
a_1^1	0.038 ± 0.002	0.030	0.036
b_1^1	—	0	0.043
a_2^0	$(17 \pm 3) \times 10^{-4}$	0	20×10^{-4}
a_2^2	$(1.3 \pm 3) \times 10^{-4}$	0	3.5×10^{-4}

[a]Predictions of chiral symmetry.

This produces the scattering lengths and slopes

$$a_0^0 = \frac{7m_\pi^2}{32\pi F_\pi^2}, \quad a_0^2 = -\frac{m_\pi^2}{16\pi F_\pi^2}, \quad a_1^1 = \frac{m_\pi^2}{24\pi F_\pi^2},$$
$$b_0^0 = \frac{m_\pi^2}{4\pi F_\pi^2}, \quad b_0^2 = -\frac{m_\pi^2}{8\pi F_\pi^2}, \tag{3.31}$$

with the numerical values shown in the table. It is remarkable that the lowest-energy form of a scattering process may be determined entirely from symmetry considerations. At next order in the expansion, one considers loop diagrams and the E^4 lagrangian. The convergence towards the experimental values can be seen in the table. Also shown are the best theoretical results which combine dispersive constraints with chiral perturbation theory [CoGL 01].

The nature of the chiral expansion also becomes evident within this process. The lowest-order predictions are real and grow monotonically. As such, they must eventually violate the unitarity constraint at some point. The worst case is the $I = \ell = 0$ amplitude

$$T_0^0 = \frac{1}{32\pi F_\pi^2}(2s - m_\pi^2), \tag{3.32}$$

which violates the simplest consequence of unitarity,

$$\frac{s - 4m_\pi^2}{s} \left| T_\ell^I \right|^2 < 1, \tag{3.33}$$

below $\sqrt{s} = 700$ MeV. In addition, there are no imaginary terms, which must be present due to the unitarity constraint,

$$\text{Im } T_\ell^I = \left(\frac{s - 4m_\pi^2}{s} \right)^{1/2} \left| T_\ell^I \right|^2 . \tag{3.34}$$

These drawbacks are remedied order by order in the energy expansion. Note that since $\left| T_\ell^I \right|$ starts at order E^2, Im T_ℓ^I starts at order E^4. When one works to order E^4, loop diagrams generate an imaginary piece given by Eq. (3.34) with the lowest-order predictions for T_ℓ^I inserted on the right-hand side. This process proceeds order by order in the energy expansion.

We have displayed in Table VII–4 how the $\mathcal{O}(E^4)$ predictions modify the scattering lengths. Aside from the renormalization of m_π and F_π, the corrections depend only on the low-energy constants $\left(2L_1^r + L_3^r\right)$ and L_2^r. Let us also give a pictorial representation of the result. One may see the order E^4 improvement and the nature of the chiral expansion by considering the $I = 1$, $\ell = 1$ channel, where some of the higher-energy data are shown in Fig. VII–4. The resonance structure visible is the $\rho(770)$. The chiral prediction is

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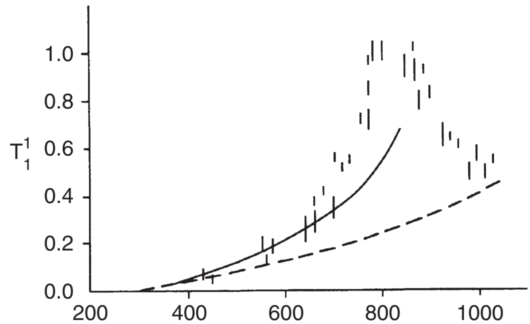

Fig. VII–4 Scattering in the $I = 1, \ell = 1$ channel.

$$T_1^1 = \frac{s - 4m^2}{96\pi^2 F^2}\left[1 + 4\left(L_2^{(2)r} - 2L_1^{(2)r} - L_3^{(2)r}\right)\frac{s}{F^2}\right], \tag{3.35}$$

with loops having a negligible effect. The lowest-order result is given by the dashed line. It clearly does not reflect the presence of the $\rho(770)$ resonance. The solid line represents the result at order E^4 and starts to reproduce the low-energy tail of the $\rho(770)$. It is, of course, impossible to represent a full Breit–Wigner shape by two terms in an energy expansion; all orders are required. The chiral predictions at $\mathcal{O}(E^4)$ may reproduce the first two terms, with the resulting expansion being in powers of q^2/m_ρ^2.

VII–4 The physics behind the *QCD* chiral lagrangian

For the most part, we have been using chiral lagrangians as our primary tool for making predictions based on the symmetry structure of *QCD*. In this section, we pause to examine which features of *QCD* are important in determining the structure of chiral lagrangians. The general strategy can perhaps be appreciated by a comparison of low-energy and high-energy *QCD* methodology. At high energies, due to the asymptotic freedom of *QCD*, hard scattering processes can be calculated in a power-series expansion in the strong coupling constant. However, some dependence on 'soft' physics remains in the form of structure functions, fragmentation functions, etc. These are not calculable perturbatively and must be determined phenomenologically from the data. At high energy, then, the predictions of *QCD* are relations among amplitudes parameterized in terms of various phenomenological structure functions and the strong coupling constant. At very low energies, because of the symmetries of *QCD*, low-energy scatterings and decays can be calculated in a power series expansion in the energy. However, some dependence on 'harder' physics remains in the form of the constants $\{L_i^r\}$. These are not calculable from the symmetry structure and must be determined phenomenologically from the data.

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At low energy, then, the predictions of *QCD* take the form of relations among amplitudes whose structure is based on symmetry constraints but which are parameterized in terms of empirical constants. Nevertheless, *QCD* should in principle also predict the very structure functions and low-energy constants which are employed by these techniques. The trouble at present is that we do not have techniques of comparable rigor with which to calculate these quantities. Nevertheless, by using models plus phenomenological insight we can learn a bit about the physics which leads to the chiral lagrangian.

The low-energy constants F_π and m_π which occur at order E^2 do not reveal much about the structure of the theory. All theories with a slightly broken chiral $SU(2)$ symmetry will have an identical structure at order E^2. The pion decay constant F_π will be sensitive to the mass scale of the underlying theory, while the pion mass m_π will be determined by the amount of symmetry breaking. However, approached phenomenologically, these are basically free parameters and do not differentiate between competing theories.

The situation is different at order E^4. Here, the chiral lagrangian contains many terms, and the pattern of coefficients is a signature of the underlying theory. The linear sigma model without fermions provides us with an example of how one can compare a theory with the real world. In Sects. IV–2,4 we calculated the tree-level terms in $\mathcal{L}_4$ which would be present in the linear sigma model, and obtained a result expressible as

$$2L_1 + L_3 = 2L_4 + L_5 = 8L_6 + 4L_8 = \frac{F_\pi^2}{4m_S^2} = \frac{1}{8\lambda}, \quad L_{2,7,9,10} = 0. \tag{4.1}$$

This pattern is quite different from the structure obtained phenomenologically. It appears that the linear sigma model is not a good representation of the real world.

Unfortunately, it is harder to theoretically infer the $\{L_i\}$ directly from *QCD*. However, a look at phenomenology indicates that we should consider the effects of vector mesons, in particular the $\rho(770)$. This is the most clear in the pion form factor that shows a dramatic ρ resonance in the timelike region. Indeed, the whole form factor can be well understood in a simple model as being a Breit–Wigner shape due to the ρ resonance

$$G_\pi(q^2) = -\frac{m_\rho^2}{q^2 - m_\rho^2 + i m_\rho \Gamma_\rho(q)\theta(q^2 - 4m_\pi^2)}, \tag{4.2}$$

where the normalization is chosen to enforce the condition $G_\pi(0) = 1$. This works even in the timelike region. Comparison with the chiral lagrangian approach implies that this model would predict

$$L_9 = \frac{F_\pi^2}{2m_\rho^2} = 7.2 \times 10^{-3}, \tag{4.3}$$

in good agreement with the value obtained earlier, $L_9 = (7.0 \pm 0.2) \times 10^{-3}$.

This analysis can be extended to L_{10}. This enters into the $W^+\pi^+\gamma$ vertex, which occurs in $\pi^+ \to e^+\nu_e\gamma$. Here, both vector and axial-vector mesons can generate corrections to the basic couplings. Explicit calculation yields [EcGPR 89]

$$L_{10} = \frac{F_{a_1}^2}{4m_{a_1}^2} - \frac{F_\rho^2}{4m_\rho^2} = -5.8 \times 10^{-3}. \tag{4.4}$$

Here, a_1 refers to the lightest axial-vector meson $a_1(1260)$ (cf. Sect. V–3), and F_{a_1} and F_ρ are the couplings of a_1 and ρ to the W^+ and the photons respectively. Again, the result is close to the empirical value cited in Table VII–1, viz. $L_{10}^r = (-5.6 \pm 0.2) \times 10^{-3}$.

The phenomenological low-energy constants are scale-dependent, and their analysis includes loop effects, while those in Eqs. (4.3), (4.4) are constants, to be used at tree level. Nevertheless, there is some sense in comparing them. The effect of loops in processes involving L_9, L_{10} is small, and the scale dependence only makes a minor change, $L_9^r(\mu = 300 \text{ MeV}) = 7.7 \times 10^{-3}$ vs. $L_9^r(\mu = 1 \text{ GeV}) = 6.5 \times 10^{-3}$. Presumably the appropriate scale is near $\mu = m_{\rho(770)}$. The $\rho(770)$ provides a much more important effect here than any other input.

Finally, it also turns out that the use of vector meson exchange leads to a good description of $\pi\pi$ scattering [DoRV 89, EcGPR 89]. This is not too surprising in light of the need for the chiral lagrangians to reproduce the tail of the $\rho(770)$, as described in Sect. VII–3. As a consequence of crossing symmetry, the $\rho(770)$ must also influence the other scattering channels. To a large extent, the chiral coefficients L_1, L_2, L_3 are dominated by the effect of $\rho(770)$ exchange. We see from these examples that phenomenology indicates that the exchange of light vector particles is the most important physics effect behind the chiral coefficients which we have been discussing.

The idea that vector mesons play an important dynamical role is not new. It predates the Standard Model, originating with Sakurai [Sa 69], in a form called *vector dominance*. The vector dominance idea has never been derived from the Standard Model, but nevertheless enjoys considerable phenomenological support. Put most broadly, vector dominance states that the main dynamical effect at energies less than about 1 GeV is associated with the exchange of vector mesons. The use of a chiral lagrangian with parameters described by ρ-exchange is compatible with this idea and puts it on a firmer footing. These considerations suggest that for chiral lagrangians the prime ingredient of *QCD* is the spectrum of the theory. The linear sigma model has a quite different spectrum, with a light scalar and no ρ, and hence does not agree with the data. *QCD*, however, seems to predict that deviations from the lowest-order chiral relation must be in such a form as to reproduce the low-energy tails of the light resonances, in particular the ρ. At present, we cannot rigorously prove this connection. However, it remains a useful picture in estimating various effects of chiral lagrangians.

VII–5 The Wess–Zumino–Witten anomaly action

At this stage one must also include the effect of the axial anomaly. The anomaly influences not only processes involving photons, such as $\pi^0 \to \gamma\gamma$, but also purely hadronic processes. For example the reaction $K\bar{K} \to \pi^+\pi^-\pi^0$, allowed by *QCD*, is not present in any of the chiral lagrangians appearing in previous sections. Its absence is easy to understand because the hadronic part of the lagrangian, with external fields set equal to zero, has the discrete symmetry $\varphi^i \to -\varphi^i$ (i.e. $U \leftrightarrow U^\dagger$) which forbids the transition of an even number of mesons to an odd number. However this is *not* a symmetry of *QCD*. More importantly, there are a set of low-energy relations, the Wess–Zumino consistency conditions [WeZ 71], which must be satisfied in the presence of the anomaly and which involve hadronic reactions. The effect of the anomaly was first analyzed by Wess and Zumino who noted that the result could not be expressed as a single local effective lagrangian, and gave a Taylor expansion representation for it.[6] Witten [Wi 83*a*] subsequently gave an elegant representation of the Wess–Zumino contribution as an integral over a five-dimensional space whose boundary is physical four-dimensional spacetime.

Since the considerations leading to the Wess–Zumino–Witten action can be rather formal, it is best to adopt a direct calculational approach. Fortunately, we are able to employ the familiar sigma model (with fermions) because it contains the same anomaly structure as *QCD*. That is, it is the presence of fermions having the same quantum numbers as quarks which ensures that the anomaly will occur. The absence of gluons in the sigma model is not a problem since, according to the Adler–Bardeen theorem [AdB 69], the inclusion of gluons would not modify the result. Since the sigma model involves coupling between mesons and fermions, we can also observe directly the influence of the anomaly on the Goldstone bosons. Although somewhat technically difficult, our approach will clearly illustrate the connection with treatments of the anomaly based on perturbative calculations.

Consider as a starting point the lagrangian, Eq. (IV–1.11), of the linear sigma model

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi - gv \left(\bar{\psi}_L U \psi_R + \bar{\psi}_R U^\dagger \psi_L \right) + \cdots . \tag{5.1}$$

[6] For a textbook treatment, see [Ge 84].

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We have displayed neither the term containing $\mathrm{Tr}\,(\partial_\mu U \partial^\mu U^\dagger)$ nor any term containing the scalar field S. Such contributions are not essential to our study of the anomaly and will be dropped hereafter. In order to simulate the light quarks of *QCD*, we shall endow each fermion with a color quantum number (letting the number N_c of colors be arbitrary) and assume there are three fermion flavors, each of constituent mass $M = gv$. Although the original linear sigma model has a flavor-$SU(2)$ chiral symmetry, Eq. (5.1) is equally well defined for flavor $SU(3)$.

Our analysis begins by imposing on Eq. (5.1) the change of variable

$$\psi_L'' \equiv \xi^\dagger \psi_L, \quad \psi_R'' \equiv \xi \psi_R\,, \quad \xi\xi = U, \tag{5.2}$$

like that described in App. B–4. This yields

$$\begin{aligned} \mathcal{L} &= \bar{\psi}''(i\not{\mathcal{D}} - M)\psi'', \qquad \mathcal{D}_\mu \equiv \partial_\mu + i\overline{V}_\mu + i\overline{A}_\mu\gamma_5, \\ \overline{V}_\mu &= -\frac{i}{2}\left(\xi^\dagger\partial_\mu\xi + \xi\partial_\mu\xi^\dagger\right)\,, \quad \overline{A}_\mu = -\frac{i}{2}\left(\xi^\dagger\partial_\mu\xi - \xi\partial_\mu\xi^\dagger\right). \end{aligned} \tag{5.3}$$

For this change of variable the jacobian is not unity, and thus we must write the effective action as

$$\begin{aligned} e^{i\Gamma(U)} &= \int [d\psi][d\bar{\psi}]\, e^{i\int d^4x\left(\bar{\psi} i\not{\partial}\psi - M(\bar{\psi}_L U\psi_R + \bar{\psi}_R U^\dagger \psi_L)\right)} \\ &= \int [d\psi''][d\bar{\psi}'']\mathcal{J}\, e^{i\int d^4x\ \bar{\psi}''(i\not{\mathcal{D}} - M)\psi''} \\ &= e^{\ln\mathcal{J}}\ e^{\mathrm{tr}\,\ln(i\not{\mathcal{D}} - M)}. \end{aligned} \tag{5.4}$$

For large M, it can be shown that the $\mathrm{tr}\,\ln(i\not{\mathcal{D}} - M)$ factor does not produce any terms at order E^4 that contain the $\epsilon^{\mu\nu\alpha\beta}$ dependence characteristic of the anomaly.[7] Hence, the effect of the anomaly must lie in the jacobian $\mathcal{J}$, and it is this we must calculate.

It is possible to determine the jacobian by integrating a sequence of infinitesimal transformations. Thus we introduce the extension $\xi \to \xi_\tau$,

$$\xi_\tau \equiv e^{i\frac{\tau\vec{\lambda}\cdot\vec{\varphi}}{2F_\pi}} \equiv \exp i\tau\overline{\varphi}, \tag{5.5}$$

where τ is a continuous parameter and $\xi = \xi_{\tau=1}$. Transformations induced by the infinitesimal parameter $\delta\tau$ will give rise to the infinitesimal quantities $\xi_{\delta\tau}$ and $\delta\mathcal{J}$,

[7] This can be verified by expanding as

$$\mathrm{tr}\,\ln(i\not{D} - M) = \mathrm{tr}\,\ln(-M(1 - i\not{D}/M)) = \mathrm{tr}\,\ln(-M) - \mathrm{tr}\,(i\not{D})^2/2M^2 + \cdots.$$

The first term can be regularized as in the text and directly calculated using the techniques described in App. B. The remaining terms vanish for large M.

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$$\psi = \psi_L + \psi_R \to \psi' = \left[\xi^\dagger_{\delta\tau}\frac{1+\gamma_5}{2} + \xi_{\delta\tau}\frac{1-\gamma_5}{2}\right]\psi,$$
$$\int [d\psi][d\bar{\psi}] = \int [d\psi'][d\bar{\psi}']\, e^{\ln \delta\mathcal{J}}. \tag{5.6}$$

From Eqs. (III–3.44), (III–3.47), we find $\delta\mathcal{J}$ to be

$$\delta\mathcal{J} = e^{-2i\delta\tau\,\mathrm{tr}\,(\overline{\varphi}\gamma_5)}, \tag{5.7}$$

or

$$\left.\frac{d\ln\mathcal{J}}{d\tau}\right|_{\tau=0} = -2i\,\mathrm{tr}\,(\overline{\varphi}\,\gamma_5)\,. \tag{5.8}$$

This result should be familiar from our discussion of the axial anomaly in Sect. III–3. There remain two steps, first to calculate the *regularized* representation of tr $(\overline{\varphi}\,\gamma_5)$, and then to integrate with respect to τ.

To regularize the trace, we employ the limiting procedure

$$\mathrm{tr}\,(\overline{\varphi}\gamma_5) = \lim_{\epsilon\to 0}\mathrm{tr}\,(\overline{\varphi}\gamma_5\,\exp[-\epsilon \not{D}_\tau \not{D}_\tau]) \quad (\mathcal{D}^\mu_\tau \equiv \partial^\mu + i\overline{V}^\mu_\tau + i\overline{A}^\mu_\tau\gamma_5), \tag{5.9}$$

with $\overline{A}^\mu_\tau$ and $\overline{V}^\mu_\tau$ as in Eq. (5.3), except now constructed from ξ_τ and $\xi^\dagger_\tau$. For arbitrary τ, we make use of the identities

$$\overline{V}^{\mu\nu}_\tau = \partial^\mu\overline{V}^\nu_\tau - \partial^\nu\overline{V}^\mu_\tau + i\left[\overline{V}^\mu_\tau, \overline{V}^\nu_\tau\right] + i\left[\overline{A}^\mu_\tau, \overline{A}^\nu_\tau\right] = 0,$$
$$\overline{A}^{\mu\nu}_\tau = \partial^\mu\overline{A}^\nu_\tau - \partial^\nu\overline{A}^\mu_\tau + i\left[\overline{V}^\mu_\tau, \overline{A}^\nu_\tau\right] + i\left[\overline{A}^\mu_\tau, \overline{V}^\nu_\tau\right] = 0, \tag{5.10}$$

to express $\not{D}_\tau\not{D}_\tau$ in the form

$$\not{D}_\tau\not{D}_\tau = d_\mu d^\mu + \sigma,$$
$$d_\mu = \partial_\mu + i\overline{V}_{\tau\mu} + \sigma_{\mu\nu}\overline{A}^\nu_\tau\gamma_5 = \partial_\mu + \Gamma_{\tau\mu},$$
$$\sigma = -2\overline{A}_{\tau\mu}\overline{A}^\mu_\tau + i\left[\left(\partial_\mu + i\overline{V}_{\tau\mu}\right), \overline{A}^\mu_\tau\right]\gamma_5. \tag{5.11}$$

From the heat-kernel expansion of App. B, we have[8]

$$\mathrm{tr}\,(\overline{\varphi}\gamma_5) \equiv \lim_{\epsilon\to 0} i\int d^4x\;\mathrm{Tr}\left(\frac{\overline{\varphi}\gamma_5}{(4\pi\epsilon)^2}\sum_n \epsilon^n a_n\right)$$
$$= \frac{i}{16\pi^2}\lim_{\epsilon\to 0}\int d^4x\;\mathrm{Tr}\left(\overline{\varphi}\gamma_5\left[\frac{a_1}{\epsilon} + a_2 + \cdots\right]\right). \tag{5.12}$$

Carrying out the ‘Tr’ operation, which involves some application of Dirac algebra, yields

[8] Note the distinction between ‘tr’ and ‘Tr’, as in Eq. (III–3.48).

$$\mathrm{Tr}\,(\gamma_5\overline{\varphi}a_2) = 2iN_c\,\mathrm{Tr}\left(\frac{8}{3}\epsilon_{\mu\nu\alpha\beta}\overline{\varphi}\overline{A}_\tau^\mu\overline{A}_\tau^\nu\overline{A}_\tau^\alpha\overline{A}_\tau^\beta\right) + \cdots, \tag{5.13}$$

where the ellipses denote contributions not involving $\epsilon_{\mu\nu\alpha\beta}$ and the factor N_c comes from the sum over each fermion color. Combining the above ingredients, we have for the regulated action

$$\begin{aligned}\Gamma(\overline{\varphi}) &= -i\ln\mathcal{J} + \cdots\\ &= \frac{N_c}{4\pi^2}\int_0^1 d\tau\int d^4x\ \mathrm{Tr}\left(\frac{8\overline{\varphi}}{3}\epsilon_{\mu\nu\alpha\beta}\overline{A}_\tau^\mu\overline{A}_\tau^\nu\overline{A}_\tau^\alpha\overline{A}_\tau^\beta\right) + \cdots,\end{aligned} \tag{5.14}$$

where we recall that $\overline{\varphi} \equiv \vec{\boldsymbol{\lambda}}\cdot\vec{\boldsymbol{\varphi}}/(2F_\pi)$. This result expresses the effect of the anomaly on the Goldstone bosons.

Unfortunately, there is no simple way to integrate the entire expression of Eq. (5.14) in closed form. In principle, we could represent each of the axial-vector currents therein (e.g. $\overline{A}_\tau^\mu$) as a Taylor series expanded about $\tau = 0$ and perform the integrations to obtain a series of local lagrangians. Alternatively, however, one can simply express Eq. (5.14) as an integral over a five-dimensional space provided we identify τ with a fifth-coordinate x_5 (defined to be timelike). In this case, we use

$$\begin{aligned}\xi_\tau A_\tau^\mu\xi_\tau^\dagger &= -\frac{i}{2}U_\tau\partial^\mu U_\tau^\dagger \equiv -\frac{i}{2}L^\mu,\\ \overline{\varphi} &= \frac{1}{2}U_\tau\frac{\partial}{\partial\tau}U_\tau^\dagger \equiv \frac{i}{2}L^5,\end{aligned} \tag{5.15}$$

plus the cyclic property of the trace to write

$$\Gamma_{WZW}(U) = \frac{iN_c}{240\pi^2}\int d^5x\,\epsilon_{ijklm}\,\mathrm{tr}\left(L^iL^jL^kL^lL^m\right), \tag{5.16}$$

where $i,\ldots,m = 5,0,1,2,3$ with $\epsilon^{50123} = +1$. This is Witten's form for the Wess–Zumino anomaly function. The $\tau = 1$ boundary is our physical spacetime, and the fifth coordinate is just an integration variable. Since each term in the Taylor expansion can be integrated, the result depends only on the remaining four spacetime variables. Observe that $\Gamma_{WZW}(U)$ vanishes for U in $SU(2)$ due to the properties of Pauli matrices. For chiral $SU(3)$, the process $K^+K^- \to \pi^+\pi^-\pi^0$ is the simplest one described by this action and after expanding Γ_{WZW}, it is given by the lagrangian,

$$\mathcal{L} = \frac{N_c}{240\pi^2F_\pi^5}\epsilon^{\mu\nu\alpha\beta}\,\mathrm{Tr}\,(\varphi\partial_\mu\varphi\partial_\nu\varphi\partial_\alpha\varphi\partial_\beta\varphi), \tag{5.17}$$

with $\varphi \equiv \boldsymbol{\lambda}\cdot\boldsymbol{\varphi}$.

The above discussion has concerned the impact of the anomaly on the Goldstone modes. We must also determine its proper form in the presence of photons or $W^{\pm}$ fields. For this purpose, we can obtain the maximal information by generalizing the fermion couplings to include arbitrary left-handed or right-handed currents ℓ_μ, r_μ,

$$\mathcal{L} = \bar{\psi} i \not{D} \psi - M \left(\bar{\psi}_L U \psi_R + \bar{\psi}_R U^\dagger \psi_L \right),$$
$$D_\mu = \partial_\mu + i \ell_\mu \frac{1+\gamma_5}{2} + i r_\mu \frac{1-\gamma_5}{2}. \tag{5.18}$$

The calculation of the jacobian then involves the operator

$$\mathcal{D}_\mu = \partial_\mu + i \overline{\ell}_\mu \frac{1+\gamma_5}{2} + i \overline{r}_\mu \frac{1-\gamma_5}{2},$$
$$\overline{\ell}_\mu = \xi_\tau^\dagger \ell_\mu \xi_\tau - i \xi_\tau^\dagger \partial_\mu \xi_\tau, \qquad \overline{r}_\mu = \xi_\tau r_\mu \xi_\tau^\dagger - i \xi_\tau \partial_\mu \xi_\tau^\dagger, \tag{5.19}$$

which generalizes Eq. (5.3). It is somewhat painful to work out the full answer directly, but fortunately we may invoke Bardeen's result of Eq. (III–3.64) for the general anomaly. Using the identities

$$\begin{aligned} \overline{\ell}_{\mu\nu} &= \xi_\tau^\dagger \ell_{\mu\nu} \xi_\tau, & \overline{v}_{\mu\nu} &= \xi_\tau^\dagger \ell_{\mu\nu} \xi_\tau + \xi_\tau r_{\mu\nu} \xi_\tau^\dagger, \\ \overline{r}_{\mu\nu} &= \xi_\tau r_{\mu\nu} \xi_\tau^\dagger, & \overline{a}_\mu &= \left(\overline{\ell}_\mu - \overline{r}_\mu \right)/2, \end{aligned} \tag{5.20}$$

where $\ell_{\mu\nu}, r_{\mu\nu}$ are given in Eq. (III–3.65), we obtain

$$\begin{aligned} \Gamma_{WZW} = -\frac{N_c}{4\pi^2} \int_0^1 d\tau \int d^4x \, \epsilon^{\mu\nu\alpha\beta} \, \mathrm{Tr} \Big[\overline{\varphi} \Big(&-\frac{8}{3} \overline{a}_\mu \overline{a}_\nu \overline{a}_\alpha \overline{a}_\beta \\ &+ \frac{1}{12} (\overline{\ell}_{\mu\nu} \overline{\ell}_{\alpha\beta} + \overline{r}_{\mu\nu} \overline{r}_{\alpha\beta}) + \frac{1}{24} (\overline{\ell}_{\mu\nu} \overline{r}_{\alpha\beta} + \overline{r}_{\mu\nu} \overline{\ell}_{\alpha\beta}) \\ &- \frac{2i}{3} (\overline{a}_\mu \overline{a}_\nu \overline{v}_{\alpha\beta} + \overline{a}_\mu \overline{v}_{\nu\alpha} \overline{a}_\beta + \overline{v}_{\mu\nu} \overline{a}_\alpha \overline{a}_\beta) \Big) \Big]. \end{aligned} \tag{5.21}$$

Note that the first term corresponds to our previous calculation of Eq. (5.14). The WZW anomaly action contains the full influence of the anomalous low energy couplings of mesons to themselves and to gauge fields. By construction, it is gauge invariant. The τ integration can be explicitly performed for all terms but the first in Eq. (5.21). However, in the general nonabelian case the result is extremely lengthy [PaR 85]. For the simpler but still interesting example of coupling to a photon field A_μ, the result is

$$
\begin{aligned}
\Gamma_{WZW}\left(U, A_\mu\right) &= \Gamma_{WZW}(U) \\
&+ \frac{N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\int d^4x \left[eA_\mu \operatorname{Tr}\left(Q\left(R_\nu R_\alpha R_\beta + L_\nu L_\alpha L_\beta\right)\right)\right. \\
&\left. - ie^2 F_{\mu\nu}A_\alpha \operatorname{Tr}\left(Q^2\left(L_\beta + R_\beta\right) + \frac{1}{2}\left(QU^\dagger QUR_\beta + QU\,QU^\dagger L_\beta\right)\right)\right],
\end{aligned}
\tag{5.22}
$$

where $R_\mu \equiv (\partial_\mu U^\dagger)U,\ L_\mu \equiv U\partial_\mu U^\dagger$.[9]

We have seen here that whereas the anomalous divergence of the axial current represents the response to an infinitesimal anomaly transformation, the WZW lagrangian represents the integration of a series of infinitesimal transformations. In our analysis of the sigma model, the anomaly has forced the occurrence of certain couplings, among them $\pi^0 \to \gamma\gamma$, $\gamma \to 3\pi$ and $K\bar{K} \to 3\pi$. As noted earlier, although these results are based on an instructional model, the result has the same anomaly structure as *QCD* because the answer must depend on symmetry properties alone. Indeed, such conclusions were originally deduced from anomalous Ward identities [WeZ 71] without any reference to an underlying model. We regard such predictions as among the most profound consequences of the Standard Model.

VII–6 The axial anomaly and $\pi^0 \to \gamma\gamma$

The description of pions and photons presented thus far does not include the decay $\pi^0 \to \gamma\gamma$. This process is important in *QCD*, because to understand it one must include the anomaly in the axial current. The $\pi^0 \to \gamma\gamma$ amplitude has the general structure

$$
\mathcal{M}_{\pi^0\to\gamma\gamma} = -iA_{\gamma\gamma}\epsilon^{\mu\nu\alpha\beta}\epsilon^*_\mu k_\nu \epsilon'^*_\alpha k'_\beta,
\tag{6.1}
$$

as required by Lorentz invariance, parity conservation, and gauge invariance, and leads to the decay rate

$$
\Gamma_{\pi^0\to\gamma\gamma} = \frac{m^3_{\pi^0}}{64\pi}\left|A_{\gamma\gamma}\right|^2.
\tag{6.2}
$$

From the experimental value, $\Gamma = 7.74 \pm 0.37$ eV, we find

$$
A_{\gamma\gamma} = 0.0252 \pm 0.0006\,\mathrm{GeV}^{-1}.
$$

We can obtain the lagrangian for $\pi^0 \to \gamma\gamma$ from the WZW action of the previous section, restricting the chiral matrices to $SU(2)$.

[9] Witten's original result did not conserve parity, and this was subsequently corrected [PaR 85, KaRS 84].

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$$\mathcal{L}_A = \frac{N_c}{48\pi^2}\epsilon^{\mu\nu\alpha\beta}\left[eA_\mu \operatorname{Tr}\left(QL_\nu L_\alpha L_\beta + QR_\nu R_\alpha R_\beta\right) - ie^2 F_{\mu\nu}A_\alpha T_\beta\right], \quad (6.3)$$

with

$$L_\mu \equiv U\partial_\mu U^\dagger, \qquad R_\mu \equiv \partial_\mu U^\dagger U,$$
$$T_\beta = \operatorname{Tr}\left(Q^2 L_\beta + Q^2 R_\beta + \frac{1}{2}QUQU^\dagger L_\beta + \frac{1}{2}QU^\dagger QUR_\beta\right), \quad (6.4)$$

where A_α is the photon field, $F_{\mu\nu}$ is the photon field strength, and $N_c = 3$ is the number of colors. A crucial aspect of this expression is that it has a known coefficient. In this respect, it is unlike other terms in the effective lagrangian, which have free parameters that need to be determined phenomenologically. This is because it is a prediction of the anomaly structure of *QCD*. A corollary of this is that $\mathcal{L}_A$ must not be renormalized by radiative corrections. This was proven at the quark–gluon level by Adler and Bardeen [AdB 69].

The $\pi^0 \to \gamma\gamma$ amplitude is found by expanding $\mathcal{L}_A$ to first order in the pion field, yielding

$$\mathcal{L}_A = \frac{e^2 N_c}{48\pi^2 F_\pi} 3\operatorname{Tr}\left(Q^2\tau_3\right)\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}A_\alpha\partial_\beta\pi^0 = \frac{\alpha N_c}{24\pi F_\pi}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}\pi^0, \quad (6.5)$$

where we have integrated by parts in the second line. This produces a $\pi^0 \to \gamma\gamma$ matrix element of the form

$$A_{\gamma\gamma} = \frac{\alpha N_c}{3\pi F_\pi} \underset{\mathrm{N}_c=3}{\longrightarrow} 0.0251\,\mathrm{GeV}^{-1}, \quad (6.6)$$

in excellent agreement with the experimental value. This is widely recognized as an important test of *QCD*, both as a measurement of the number of colors and also as a reflection of the symmetries and anomalies of the theory. It is a remarkable result.

What would have happened if the axial anomaly were not present? The decay $\pi^0 \to \gamma\gamma$ could still occur, but it would be suppressed. The $\pi^0 \to \gamma\gamma$ transition must be at least of order E^4, as it must involve the dimension-four operator $F\tilde{F}$. The anomaly occurs at this order. However, non-anomalous lagrangians leading to this transition can be constructed at order E^6. This result was first derived by Sutherland and Veltman using a soft-pion technique [Su 67, Ve 67].

At what level would we expect corrections to the anomaly prediction for $\pi^0 \to \gamma\gamma$? It has been checked that $m_\pi^2 \ln m_\pi^2$ corrections, which in principle can occur when meson loops are present, do not in fact modify the lowest-order result when it is expressed in terms of the physical decay constant F_π. However, there are still corrections of order m_π^2/Λ^2, where Λ is the scale in the energy expansion, which amount to modifications of order 2% [GoBH 02].

Problems

(1) **Radiative corrections and $\pi_{\ell 2}$ decay**

To bring $\Gamma_{\pi\to e\nu_e}/\Gamma_{\pi\to\mu\nu_\mu}$ into agreement with experiment requires a radiative correction whose dominant contribution is the so-called *seagull* component $(\sqrt{2}F_\pi g_{\mu\nu})$ of $M_{\mu\nu}$ (cf. Eq. (3.22)).

(a) Verify that gauge invariance requires

$$iq^\mu M_{\mu\nu}(p,q) = \langle 0|A_\nu^{1-i2}|\pi^+(\mathbf{p})\rangle,$$

and show that the seagull term is required in this regard to cancel the pion pole contribution.

(b) Use the seagull term in Feynman gauge to calculate the radiative correction to $\pi_{\ell 2}$ decay. Introduce a photon cut-off via

$$\frac{1}{k^2} \to \frac{1}{k^2}\frac{-\Lambda^2}{k^2-\Lambda^2}$$

so that

$$\mathcal{M}^{\rm rad} = ie^2\frac{G_F}{\sqrt{2}}V_{\rm ud}\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\frac{-\Lambda^2}{k^2-\Lambda^2}\sqrt{2}F_\pi g^{\mu\lambda} \\ \times \bar{u}(p_\nu)\gamma_\lambda(1+\gamma_5)\frac{1}{-\not{p}_e+\not{k}-m_e}\gamma_\mu v(p_e),$$

and show that

$$\mathcal{M}^{(0)} \to \mathcal{M}^{(0)}\left(1-\frac{3\alpha}{2\pi}\ln\frac{\Lambda}{m_\ell}\right),$$

where

$$\mathcal{M}^{(0)} = i\frac{G_F}{\sqrt{2}}V_{\rm ud}\sqrt{2}F_\pi p_\lambda\bar{u}(p_\nu)\gamma_\lambda(1+\gamma_5)v(p_e) \\ = -i\frac{G_F}{\sqrt{2}}V_{\rm ud}\sqrt{2}F_\pi m_\ell\bar{u}(p_\nu)(1-\gamma_5)v(p_e)$$

is the lowest order amplitude for the $\pi_{\ell 2}$ process. This then is the origin of the lepton-mass-dependent radiative correction.

(2) **Radiative pion decay and the anomaly**

Writing the π^0 decay amplitude as

$$\mathcal{M}_{\pi^0\to\gamma\gamma} = -ie^2\epsilon_1^{\mu*}\epsilon_2^{\nu*}\int d^4x e^{iq_1\cdot x}\langle 0|T(V_\mu^{\rm em}(x)V_\nu^{\rm em}(0))|\pi^0(\mathbf{p})\rangle \\ \equiv -i\epsilon_1^{\mu*}\epsilon_2^{\nu*}\epsilon_{\mu\nu\alpha\beta}q_1^\alpha p^\beta A_{\gamma\gamma},$$

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and the vector current amplitude in radiative π^+ decay as

$$\mathcal{M}^{(V)}_{\pi^+\to\ell^+\nu_\ell\gamma} = -ie\epsilon_1^{\mu *}\int d^4x e^{iq_1\cdot x}\langle 0|T(V_\mu^{\rm em}(x)V_\nu^{1-i2}(0))|\pi^+(\mathbf{p})\rangle$$
$$\equiv -ie\epsilon_1^{\mu *}\epsilon_2^{\nu}\epsilon_{\mu\nu\alpha\beta}q_1^\alpha p^\beta h_V,$$

demonstrate that isotopic spin invariance requires $\sqrt{2}h_V = A_{\gamma\gamma}$.

(3) **Unitarity and the pion form factor**

(a) Verify that the pion form factor given in Eq. (3.13) obeys the strictures of unitarity, i.e.,

$$2\,\mathrm{Im}\,G_\pi(q^2)(p_1-p_2)_\mu = \int \frac{d^3q_1d^3q_2}{(2\pi)^6 2q_1^0 2q_2^0}$$
$$\times (2\pi)^4\delta^4(p_1+p_2-q_1-q_2)(q_1-q_2)_\mu\langle\pi^+(\mathbf{q}_1)\pi^-(\mathbf{q}_2)|\pi^+(\mathbf{p}_1)\pi^-(\mathbf{2}_2)\rangle$$

where the matrix element is the two-derivative (tree-level) pion–pion scattering amplitude given in Eq. (3.4).

(b) How does this result change if the K^+K^- intermediate state is added to $\pi^+\pi^-$?

(4) **Other worlds**

Describe changes in the *macroscopic* world if the quark masses were slightly different in the following ways:

(a) $m_u = m_d = 0$,

(b) $m_u > m_d$,

(c) $m_u = 0,\ m_d = m_s$.