THE CONSTRUCTION OF REPRESENTATIONS OF LIE ALGEBRAS OF CHARACTERISTIC ZERO

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1. Introduction. In this paper a procedure is given whereby, from a representation of an ideal contained in the radical, explicit representations of a Lie algebra by matrices can be constructed in an algebraically closed field of characteristic zero. The construction is sufficiently general to permit one arbitrary eigenvalue to be assigned to the representation of each basis element of the radical not in the ideal. The theorem of Ado is proved as an application of the construction. While Ado's theorem has several proofs (**1**; **3**; **5**; **6**), the present one has a value in its explicitness and in the fact that the degree of the representation can be given.

Throughout this paper L is a Lie algebra over an algebraically closed field F of characteristic zero and $H(L)$ the Poincaré–Birkhoff–Witt associative imbedding algebra of L (**2**; **10**).

2. The construction of a representation of the radical R of L. Applying the theorem of Levi (**9**) L can be decomposed into a linearly direct sum $V \dotplus R$, where V is a semi-simple subalgebra of L. Let T be an ideal of L such that*

$$R \supset T \supseteq T_1 = (L \circ L) \cap R.$$

We will now show that from any representation Q, of finite degree, of T a representation Q', of finite degree, of R can be constructed. It is sufficient to suppose R/T is one dimensional, that is,

$$R = Fa \dotplus T,$$

where the basis of T extended by a gives a basis of R.

THEOREM 2.1. *Within $H(L)$, let $F[a]$ be the ring obtained by the adjunction of the basis element a to the field F, then*†

$$L \circ F[a] \subseteq F[a]T_1.$$

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*$\circ$ means multiplication in the Lie algebra L.

†Juxtaposition means multiplication in the associative imbedding algebra $H(L)$ consequently $x \circ y = xy - yx$.

Proof. We have

$$\begin{aligned} h \circ a^2 &= (h \circ a) \circ a + a(h \circ a) \qquad (h \in L) \\ &= t_1^{(2)} + at_1^{(1)} \qquad (t_1^{(2)}, t_1^{(1)} \in T_1). \end{aligned}$$

By an induction on n

$$h \circ a^n = t_1^{(n)} + \binom{n}{1} a t_1^{(n-1)} + \ldots + \binom{n}{n-1} a^{n-1} t_1^{(1)},$$

giving

$$L \circ F[a] \subseteq F[a]T_1.$$

Let the minimal polynomial for $\tilde{a}$, the regular representation of a, be

$$f(x) = k_0 + k_1 x + \ldots + k_{n-1}x^{n-1} + x^n \qquad (k_i \in F,\ i = 0, 1, \ldots, n-1)$$

and let its zeros be

$$\alpha_1, \alpha_2, \ldots, \alpha_n.$$

Let us now define the polynomials $f^{(s)}(x)$ recursively, as follows:

$$\begin{aligned} f^{(0)}(x) &= e, \text{ the unit element}, \\ f^{(1)}(x) &= x, \\ f^{(2)}(x) &= f^{(1)}(x) f^{(1)}(x - \alpha_1) f^{(1)}(x - \alpha_2) \ldots f^{(1)}(x - \alpha_n), \\ &\ldots \\ f^{(s)}(x) &= f^{(s-1)}(x) f^{(s-1)}(x - \alpha_1) \ldots f^{(s-1)}(x - \alpha_n). \end{aligned}$$

Notice that $f^{(2)}(x) = xf(x)$.

THEOREM 2.2. $L \circ f^{(s)}(a - k) \subseteq f^{(s-1)}(a - k)F[a]T_1 \qquad (k \in F).$

Proof. We have

$$\begin{aligned} h \circ (a - k) &= t_1, \qquad (h \in L, t_1 \in T_1) \\ h \circ (a - k)^2 &= t_1 \tilde{a} + \binom{2}{1}(a - k)t_1, \end{aligned}$$

and, by induction,

$$\ldots$$

$$h \circ (a - k)^{n+1} = t_1 \tilde{a}^n + \binom{n+1}{1}(a - k)t_1 \tilde{a}^{n-1} + \ldots + \binom{n+1}{n}(a - k)^n t_1.$$

Multiplying each of the rows by $k_0, k_1, \ldots, k_{n-1}$, respectively, and adding

$$h \circ ((a - k)f(a - k)) = t_1 f(\tilde{a}) + (a - k)(t_1^{(n-1)} + p_1(a)t_1^{(n-2)} + \ldots + p_{n-1}(a)t_1)$$

where $t_1^{(i)} \in T_1$ and $p_i(a) \in F[a]$, $i = 1, 2, \ldots, n-1$, that is,

$$h \circ f^{(2)}(a - k) = 0 + f^{(1)}(a - k)(t_1^{(n-1)} + \ldots + p_{n-1}(a)t_1).$$

Thus $L \circ f^{(2)}(a-k) \subseteq f^{(1)}(a-k)F[a]T_1$. Observing that

$$h \circ f^{(s+1)}(a-k) = h \circ (f^{(s)}(a-k)f^{(s)}(a-k-\alpha_1)\dots f^{(s)}(a-k-\alpha_n))$$
$$= \sum_{i=0}^{n} (f^{(s)}(a-k)\dots(h \circ f^{(s)}(a-k-\alpha_i))\dots f^{(s)}(a-k-\alpha_n)) \ (\alpha_0 = 0),$$

the theorem follows by an induction on s.

Our aim in developing the properties of $f^{(s)}(a-k)$ is to construct from a representation module M of T, corresponding to a representation Q of T, a representation module N of R. To complete these preliminaries let us note some properties of M.

By a theorem of Jacobson (**7**, Theorem 2 and the first remark) it follows that Q subduces on $T_1 = L \circ L \cap R$ a representation with the property $(Q(t_1))^n = 0$ (n a positive integer) for all $t_1 \in T_1$, that is, $T_1{}^n M = 0$. Let $T_2 \neq 0$ be any ideal where $T \geqslant T_2 \supseteq T_1$ and let n_2 be the least positive integer such that $T_2{}^{n_2}M = 0$. We can then form a sequence of modules

$$0 = M_0 \subset M_1 \subset M_2 \subset \dots \subset M_q = M, \tag{1}$$

invariant under T and such that $T_2 M_i \subseteq M_{i-1}$. Let M_1 consist of all elements u of M such that $T_2 u = 0$. If $M_1 \neq M$, let M_2 consist of all elements u of M such that $T_2 u \subseteq M_1$. Continuing thus, that is, if $M_i \neq M$ defining M_{i+1} as all elements u of M such that $T_2 u \subseteq M_i$, we must finally exhaust M, at M_q say, since $M_{i+1} \supset M_i$.*

To show that $TM_i \subseteq M_i$ ($i = 1, 2, \dots, q$), we have

$$t(t_2 u) - t_2(tu) = (t \circ t_2)u = t_1 u \qquad (t \in T, t_2 \in T_2, t_1 \in T_1 \subseteq T_2, u \in M_1)$$

$0 - t_2(tu) = 0$ giving $tu \in M_1$. Hence $TM_1 \subseteq M_1$. By an induction on i we have $TM_i \subseteq M_i$.

Let the basis $u_1, u_2, \dots, u_s$ for M be determined by taking for M_i a basis $u_1. u_2, \dots u_{h_i}$ extended by $u_{h_i+1}, u_{h_i+2}, \dots, u_{h_{i+1}}$ to give a basis for M_{i+1} ($i = 1, 2, \dots, q-1; h_q = s$).

We can now construct from M a representation module N' consisting of the direct sum of the formal power products $a^n M$, $n = 0, 1, 2, \dots$, with the following rules of computation

$$\sum_{n=s}^{s'} a^n g_n + \sum_{n=s}^{s'} a^n g'_n = \sum_{n=s}^{s'} a^n (g_n + g'_n) \qquad (g_n, g'_n \in M)$$
$$k \sum_{n=s}^{s'} a^n g_n = \sum_{n=s}^{s'} a^n (k g_n) \qquad (k \in F)$$
$$r \sum_{n=s}^{s'} a^n g_n = \sum_{n=s}^{s'} \sum_{j=0}^{n+1} a^j (t_{jn} g_n) \qquad (r \in R)$$

*That the elements of M have such properties follows from $T_2 M \subseteq M$ and hence

$$M \supseteq T_2 M \supseteq T_2^2 M \supseteq \dots T_2^{n_2} \supseteq M = 0.$$

where

$$ra^n = \sum_{j=0}^{n+1} a^j t_{jn}$$

is obtained by the Birkhoff multiplication procedure **(2)** and $t_{jn} \in T$. Clearly N' is generated by the linearly independent elements:

$$u_1, u_2, \ldots, u_s, au_1, au_2, \ldots, au_s, a^2u_1, a^2u_2, \ldots, a^2u_s \ldots .$$

On these elements let us now impose the relations

$$f^{(i)}(a-k)u_j = 0, \quad (j = h_{i-1}+1, h_{i-1}+2, \ldots, h_i; \; i = 1, 2, \ldots, q).$$

As a consequence

$$f^{(i)}(a-k)u_j = 0, \; h_i \geqslant j \geqslant 1, \text{ giving } f^{(i)}(a-k)M_i = 0.$$

THEOREM 2.3. *Let N be the module N' with the relations $f^{(i)}(a-k)M_i = 0$ $(i = 1, 2, \ldots, q)$. Then N is a representation module of R and assigns a representation Q', of finite degree, to R which is faithful if Q is faithful and $Q'(a)$ has one arbitrary eigenvalue.*

Proof. For N to be a representation module of R we must show that the relations of $f^{(i)}(a-k)M_i = 0$ $(i = 1, 2, \ldots, q)$ are invariant under R, that is,

$$r(f^{(i)}(a-k)u) = 0 \qquad (r \in R, \; u \in M_i).$$

We have for $t \in T$,

$$\begin{aligned} t(f^{(i)}(a-k)u) &= (t \circ f^{(i)}(a-k))u + f^{(i)}(a-k)(tu) \\ &= (t \circ f^{(i)}(a-k))u + f^{(i)}(a-k)u' \qquad (u' \in M_i). \\ &= (t \circ f^{(i)}(a-k))u + 0 \end{aligned}$$

As a consequence of Theorem 2.2

$$(t \circ f^{(i)}(a-k))u \in f^{(i-1)}(a-k)F[a](T_1u) \subseteq f^{(i-1)}(a-k)F[a]M_{i-1}.$$

Thus $(t \circ f^{(i)}(a-k))u \in F[a]f^{(i-1)}(a-k)M_{i-1} = 0$. It follows that for all $r \in R$, $r(f^{(i)}(a-k)u) = 0$ $(u \in M_i, \; i = 1, 2, \ldots, q)$. To show that Q', the representation assigned to R by N is faithful if Q is faithful, let us take the basis elements of N to be

$$u_1, \ldots, u_s, au_{h_1+1}, \ldots au_s, a^2u_{h_1+1}, \ldots a^2u_s, \ldots, a^{n_q-1}u_{h_{q-1}}, \ldots, a^{n_q-1}u_s$$

where

$$n_i = (n+1)^{i-1}$$

is the degree of $f^{(i)}(x)$ $(i = 1, 2, 3, \ldots, s)$.* Applying $t \in T$ to each of these elements gives the matrix $Q'(t)$ assigned to t by N, namely,

*The number of basis elements is

$$h_1 + (h_2 - h_1)n_2 + (h_3 - h_2)n_3 + \ldots + (s - h_{q-1})n_q < sn_q.$$

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$$Q'(t) = \begin{bmatrix} Q(t) & & & & \\ & {}^{h_1}_{h_1}Q(t) & & & * \\ & & {}^{h_2}_{h_2}Q(t) & & \\ & & & \ddots & \\ & & & & {}^{h_{q-1}}_{h_{q-1}}Q(t) \end{bmatrix}$$

where ${}^{i}_{i}Q(t)$ is the matrix $Q(t)$ with i rows from the top and i columns from the left deleted.

Applying a to the basis elements gives $Q'(a)$, namely

$$Q'(a) = \begin{bmatrix} k & & & & & & \\ & \ddots & & & & * & \\ & & k & & & & \\ & & & 0 & & & \\ & & & \vdots & \ddots & & \\ & & & 0 & \cdots & 0 & \\ & & & 1 & & & \ddots \\ & & & & \ddots & & & \ddots \end{bmatrix}$$

To show that Q' is faithful if Q is faithful notice that

$$Q'(r) = Q'(k'a + t) = k'Q'(a) + Q'(t) = 0 \ (k' \in F,\ t \in T)$$

gives $k'I = 0$ where I is the unit matrix of dimension $s - h_1$. Thus if $s - h_1 \neq 0$ $k' = 0$ and $Q'(t) = 0$ gives $Q(t) = 0$. If Q is faithful $t = 0$ and so $r = 0$ and Q' is faithful.

If $s = h_1$, $T_2M = 0$ and as Q is faithful $T_2 = 0$ contrary to $T_2 \neq 0$.

An examination of $Q'(a)$ shows that k, any element of F, is an eigenvalue.

3. The construction of a representation of L from a representation of its radical. In order to extend the representation Q' of R to L itself, we introduce the concept of a matrix of invariance. $C(a)$ is a matrix of invariance of the representation Q of T if $C(a)Q(t) - Q(t)C(a) = [C(a), Q(t)] = Q(a \circ t)$ for $a \in L$ and all $t \in T$. In module terms, a is a linear transformation of M, the representation module assigning Q to T, with the property

$$a(tu) - t(au) = (a \circ t)u, \qquad (u \in M).$$

When such matrices exist for all elements of L, a choice can be made so that they almost form a representation, in fact, for $a, b \in L$,

$$[C(a), C(b)] = C(a \circ b) + k(a, b)I,$$

where the $k(a, b) \in F$ form a factor set and $C(t) = Q(t)$, $(t \in T)$ (**8**, Theorem 1.1 and its corollary). To designate this property we shall say that Q is invariant under L.

LEMMA 3.1. *If Q is invariant under L so is Q'.*

Proof. We first define a linear transformation of $w \in N$ by an element v in L but not in R by setting

$$vw = v \sum_{i=1}^{s} p_i(a)u_i = \sum_{i=1}^{s} (v \circ p_i(a))u_i + \sum_{i=1}^{s} p_i(a)(vu_i)$$

where $p_i(a)$ $(i = 1, 2, \ldots, s)$ is a polynomial in a and the product vu_i $(i = 1, 2, \ldots, s)$ is defined by the invariance of Q. For our purpose it is necessary to show that $vM_i \subseteq M_i$, where M_i is any one of the modules of the sequence (1). For $u \in M_1$, $t_2 \in T_2$

$$\begin{aligned} t_2(vu) &= (t_2 \circ v)u + v(t_2u) \\ &= t_1u + v0 \qquad (t_1 \in T_1 \subseteq T_2) \\ &= 0. \end{aligned}$$

Thus $vu \in M_1$ and $vM_1 \subseteq M_1$. By an induction on i

$$vM_i \subseteq M_i \qquad (i = 1, 2, \ldots, q).$$

That the linear transformation of N defined for v preserves the defining relations of N can now be shown, since for $u \in M_i$ $(i = 1, 2, \ldots, q)$

$$v(f^{(i)}(a - k)u_j) = (v \circ f^{(i)}(a - k))u_j + f^{(i)}(a - k)(vu_j) \qquad (j = h_{i-1} + 1, h_{i-1} + 2, \ldots, h_i).$$

Thus

$$\begin{aligned} v(f^{(i)}(a - k)u_j) &\in f^{(i-1)}(a - k)F[a](T_1u_j) + f^{(i)}(a - k)M_i \\ &\subseteq F[a]f^{(i-1)}(a - k)M_{i-1} + 0 = 0, \end{aligned}$$

as required. To show the invariance of Q' observe that

$$(v \circ t)u_i = v(tu_i) - t(vu_i) \qquad (t \in T;\ i = 1, 2, \ldots, s)$$

by the invariance of Q under L and

$$(v \circ a)u_i = v(au_i) - a(vu_i)$$

by definition. It follows that

$$(v \circ r)w = v(rw) - r(vw) \qquad (r \in R)$$

where

$$w = \sum_{i=1}^{s} k_i u_i \in M_1 \qquad (k_i \in F).$$

Let $w_n \in N$ be an element of order n if

$$w_n = \sum_{i=1}^{s} p_i(a) u_i$$

and the maximum degree of the polynomials $p_1(a), p_2(a), \ldots, p_s(a)$ is n. By an induction on the order n of w_n it follows that Q' is invariant under L.

LEMMA 3.2. *Recall that $L = V \dot{+} R$. Let Q', a representation of R, be invariant under L. Let the Lie algebra generated by the matrices of invariance $C(v)$ for all $v \in V$ be V^*. Then the set of elements Z of V^* which annihilate $Q'(r)$ for all $r \in R$ form an ideal and*

$$V^* = W \dot{+} Z$$

where W is a Lie subalgebra.†

Proof. That Z is an ideal follows by an application of the Jacobi identity. Similarly it can be shown that the mapping

$$v \rightarrow C(v) \bmod Z$$

is a homomorphism of V onto the algebra of residue classes V^*/Z. As V is semi-simple it follows that V^*/Z is semi-simple.

Applying the theorem of Levi

$$V^* = W_1 \dot{+} R(V^*)$$

where $R(V^*)$ is the radical of V^* and W_1 a semi-simple subalgebra. As V^*/Z is semi-simple $Z \supseteq R(V)$. Also, as $W_1 \cap Z$ is an ideal of W_1 there exists a complementary ideal W and‡

$$W_1 = W \dot{+} W_1 \cap Z,$$
$$V^* = W \dot{+} W_1 \cap Z \dot{+} R(V) = W \dot{+} Z.$$

THEOREM 3.1. *If Q is invariant under L then Q', the representation of the radical R assigned by the representation module N of Theorem* 2.3, *can be extended to the whole Lie algebra $L = V \dot{+} R$ without a change of degree.*

Proof. By Lemma 3.1, Q' is invariant under L and from Lemma 3.2, for $v \in V$, there exists $C'(v) \in W$ such that

$$C(v) = C'(v) \bmod Z.$$

Replacing $C(v)$ by $C'(v)$ as the matrix of invariance of v and setting $C'(r) = Q'(r)$ for $r \in R$, we have for each $l \in L$ a corresponding matrix $C(l)$. That this correspondence is a representation follows from observing that W is a subalgebra, $C(a)$ a matrix of invariance and Q a representation.

4. Remarks. In constructing the representation of L we assumed the dimension of R/T to be one. If this dimension were n then it is easily seen

†For a less specialized proof see **(6**, Theorem 1.1**)**.

‡For a discussion of this property see **(4**, Theorem vii, p. 83**)**.

that the representation Q' of R would require n constructions analogous to the one given for constructing N and Q' would assign one arbitrary eigenvalue to each basis element of R not in T.

We have now shown, therefore, how a representation Q, of an ideal T contained in the radical R of a Lie algebra L over an algebraically closed field F, of characteristic zero, can be used to construct an explicit representation Q' of the radical, which is faithful if Q is faithful. The construction is of such a nature that it permits us to assign an arbitrary eigenvalue to the representation of each of the basis elements of R not in T. If Q is invariant under L, the constructed representation Q' can be extended to the whole Lie algebra L without a change of degree. If this representation, A say, is not faithful we can construct one which is, if Q is faithful. For let P be the regular representation of L, then form the representation $U = A \dotplus P$. Consider the elements of x in L for which $U(x) = A(x) \dotplus P(x) = 0$. $P(x) = 0$ places x in the centre of L which is nilpotent and so is solvable, so $x \in R$. Since A is faithful over R, $A(x) = 0$ gives $x = 0$. Thus U is faithful.

5. The theorem of Ado. Birkhoff **(2)** has shown that every nilpotent Lie algebra has a faithful representation of finite degree, namely the regular representation of the imbedding algebra modulo a certain invariant subalgebra S. For a suitable choice of basis this representation in terms of matrices is properly triangular.

LEMMA 5.1. *Let L have a non-zero radical R, the maximal nilpotent ideal T whose faithful representation by the Birkhoff procedure is Q. Then Q is invariant under V, that is, there exist matrices of invariance $C(v)$, $v \in V$, such that $[C(v), Q(t)] = Q(v \circ t)$, $(t \in T)$.*

Proof. We have that $H(T)$, the Poincaré–Birkhoff–Witt imbedding algebra of T, is a subalgebra of $H(L)$. Also, to $v \in V$ there corresponds a linear transformation, the derivation $D(v)$, of $H(L)$ defined by

$$D(v)X = vX - Xv = v \circ X \qquad (X \in H(L)).$$

Since

$$D(v)t = vt - tv = v \circ t \in T$$

for all t in T, we have $D(v)T \subseteq T$ and $D(v)(H(T)) \subseteq H(T)$. Hence $D(v)$ subduces a derivation on $H(T)$. Moreover, $D(v)$ leaves invariant the subalgebra S (as defined in **(2)**) of $H(T)$. Hence $D(v)$ subduces a derivation $C(v)$ of the hypercomplex system $H(T)/S$. The representation Q is the regular representation of $H(T)/S$.

Since $C(v)$ is a derivation of $H(T)/S$, for $t \in T$, $u \in H(T)/S$, we have

$$\begin{aligned} C(v)(tu) &= (C(v)t)u + t(C(v)u), \\ &= (v \circ t)u + t(C(v)u), \\ (v \circ t)u &= C(v)(tu) - t(C(v)u) \end{aligned}$$

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or

$$Q(v \circ t) = C(v)Q(t) - Q(t)C(v) = [C(v), Q(t)],$$

in matrix terms, so Q is invariant under V.

THEOREM 5.1 (Ado). *Any Lie algebra L over an algebraically closed field of characteristic zero has a faithful representation by matrices of finite dimensions.*

Proof. If the radical R of L is zero, L can faithfully be represented by its regular representation; so suppose $R \neq 0$, then, by Levi $L = V \dot{+} R$. In L let T be the maximal nilpotent ideal and Q its faithful representation by the Birkhoff procedure such that $Q(t)$, $t \in T$, is properly triangular.

Since $T_1 = L \circ L \cap R$ is nilpotent, $T \supseteq T_1$. Hence Q subduces on T_1 a properly triangular representation. Hence $(Q(t_1))^n = 0$ for all $t_1 \in T_1$ (n a positive integer). If $T_1 = 0$, $R \circ R = 0$ and R is nilpotent. Hence $T = R$ and $Q' = Q$ is a faithful representation of R. If $T \subset R$ we can construct the module N of Theorem 2.3 in one or more steps depending on the dimension of R/T and so assign a faithful representation Q' to R.

Since Q is invariant under L by Lemma 5.1, Q' is invariant under L by Lemma 3.1 and we can construct the representation A of L. If A is not faithful we form the direct sum, U, of A and the regular representation P of L and U is faithful.

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