Canad. Math. Bull. Vol. **28**(4), 1985

# CONVERGENCE AND ANALYTIC CONTINUATION FOR A CLASS OF REGULAR $C$-FRACTIONS

BY

D. MASSON

*Dedication: In memory of Robert Arnold Smith*

ABSTRACT. Regular $C$-fractions $f(\alpha) \equiv 1 + a_1\alpha/1 + a_2\alpha/1 + \ldots$ with $a_n = an^2 + bn + c + V_n$, $|V_n|$ sufficiently small are examined. In the case $V_n = 0$, exact expressions are obtained which reveal a two sheeted Riemann structure for $f(\alpha)$. If $V_n \neq 0$ analytic properties are obtained by means of perturbation theory applied to the associated difference equation. A conjecture that $f(\alpha)$ is the ratio of two entire functions of $1/\sqrt{\alpha}$ for an even larger class of $C$-fractions is proved for the case $a_n = \prod_{i=1}^{N} (n + r_i)^{p_i}$, $r_i \neq -n$, $\sum_{i=1}^{N} p_i = 2$.

1. **Introduction**. The connection between continued fractions

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots \tag{1}$$

and three term recursion relations

$$X_n - b_n X_{n-1} - a_n X_{n-2} = 0 \tag{2}$$

lies at the heart of continued fraction theory [5], [7].

The solution of (2) in terms of a minimal (or subdominant) solution $X_n^{(s)}$ and a dominant solution $X_n^{(d)}$ with the property

$$\lim_{n\to\infty} X_n^{(s)}/X_n^{(d)} = 0 \tag{3}$$

provides a necessary and sufficient condition for the convergence of (1) with:

PINCHERLE'S THEOREM [3]: *Let* $a_n \neq 0$, $n \geq 1$. *Then*

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n} = -X_0^{(s)}/X_{-1}^{(s)}. \tag{4}$$

Although the existence of a minimal solution may be obtained from the asymptotics of (2), instability limits its usefulness in determining $X_n^{(s)}$ because of the build up of

Received by the editors July 20, 1984 and, in revised form January 25, 1985.
Research partially supported by NSERC.
AMS Subject Classification (1980): 30B70, 39A10, 40A15, 33A30.

errors. Algorithms for the practical determination of $X_n^{(s)}$ and its more detailed properties usually rely on the interplay between (1) and (2) (see Gautschi [2] and Henrici in Appendix B of [3]). Thus, new exact minimal solutions and approximation methods are welcome additions to the theory.

If $b_n \neq 0$, $n \geq 0$ then the identity (equivalence transformation)

$$b_0 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n}{b_n} = b_0\left(1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n b_n^{-1} b_{n-1}^{-1}}{1}\right)$$

allows attention to be focused on continued fractions having $b_n = 1$. With the introduction of a parameter $\alpha$ in the partial numerators one obtains the regular corresponding fraction ($C$-fraction)

$$1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{a_n \alpha}{1}, \qquad a_n \neq 0, n \geq 1 \tag{5}$$

and a further connection with analytic function theory [1], [3], [7]. Convergence will now occur for $\alpha$ in certain complex domains and one can discuss the analytic properties of such $C$-fractions. Some typical classical results are:

(1) If $a_n > 0$, $n \geq 1$, then (5) converges to a meromorphic function of $\alpha$ if and only if $a_n \to 0$ (Stieltjes, loc. cit. [7], p. 210).[1]

(2) If $a_n \to a > 0$ then (5) converges to a meromorphic function of $\alpha$ in the cut plane $\alpha \notin (-\infty, -1/4a]$ (Van Vleck, loc. cit. [7], p. 210).[2]

(3) If $a_n > 0$, $n \geq 1$ and $\sum_{n=1}^{\infty} a_n^{-1/2} = \infty$, then (5) converges to a holomorphic function of $\alpha$ in the cut plane $|\arg \alpha| < \pi$ (Stieltjes, see [7], Thm. 28.1).[3]

In case (2) or (3) with $a_n \not\to 0$ one can inquire about a nontrivial analytic continuation of the corresponding $C$-fraction. A recent result for case (2) due to Thron and Waadeland [6] is:

(4) If the convergence of $a_n$ to $a > 0$ is geometric or faster ($|a_n - a| \leq dK^n$, $K < 1$), then (5) has a square root branch cut on $\alpha \leq -1/4a$.

Analytic continuation for case (3) where $a_n \to \infty$ does not appear to have been examined except for the special case $a_n = bn + c$ where in [4] it was shown that

$$1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(n+c)\alpha}{1} = \sqrt{\alpha}\, D_{-c}(1/\sqrt{\alpha})/D_{-c-1}(1/\sqrt{\alpha}), \ |\arg \alpha| < \pi \tag{6}$$

($D_\lambda(z)$ the parabolic cylinder function). One then has convergence in the cut $\alpha$-plane, $|\arg \alpha| < \pi$ to the branch of a meromorphic function of $\beta = 1/\sqrt{\alpha}$.

We believe this to be a general feature of a wide class of $C$-fractions and venture the following conjecture.

---

[1] $a_n \to 0$ implies convergence to a meromorphic function of $\alpha$ but the converse is not necessarily true unless $a_n > 0$.

[2] $a > 0$ is for conveniece without loss of generality.

[3] If $a_n > 0$, $n \geq n_0 > 1$ and $\sum_{n=n_0}^{\infty} a_n^{-1/2} = \infty$, replace "holomorphic" by "meromorphic".

CONJECTURE: *If* $a_n = a \prod_{i=1}^{N} (n + r_i)^{p_i}$ *with* $r_i \neq -n$, $n \geq 1$, $a > 0$ *and* $0 < p \leq 2$ *where* $p = \sum_{i=1}^{N} p_i$, *then* (5) *converges in the cut* $\alpha$*-plane* $|\arg \alpha| < \pi$ *to the branch of a meromorphic function of* $\beta = 1/\sqrt{\alpha}$.

If this is indeed true then one has a class of $C$-fractions associated with a surprisingly simple global analytic structure consisting of only two Riemann sheets. We hope to make the conjecture more plausible by continuing here the investigation in [4], where it was explicitly shown to be true for the case $p = 1$, $N = 1$.

In Section 2 we prove the conjecture for the case $N = 2$, $p_1 = p_2 = 1$ by obtaining exact expressions for (5) and the solutions to its associated difference equation in terms of hypergeometric functions.

In Section 3 we introduce a perturbation method for obtaining minimal solutions and their analytic properties using a Volterra equation and apply it to the case $a_n = an^2 + bn + c + V_n$ with $a \neq 0$ and $|V_n| \leq$ const. $n^{1-\epsilon}$, $\epsilon > 0$.

In Section 4 we indicate how the conjecture can be resolved in terms of the properties of an associated difference equation and sketch a proof of the conjecture for the case $p = 2$. The method of proof yields a simple solution to the analytic continuation of (5) in terms of limit averaging formulae for $X_0^{(s)}$ and $X_{-1}^{(s)}$.

2. **Exact results**. From the general theory of Stieltjes type continued fractions (Theorem 28.1 of [7]) one may conclude that if $a$, $b$, $c$ are real then

$$f(a,b,c,\alpha) \equiv 1 + \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(an^2 + bn + c)\alpha}{1} \tag{7}$$

converges in the cut $\alpha$-plane.

Some special cases are evaluated in Wall [7] where it is shown that

$$\frac{1}{\sqrt{\alpha}} \int_0^\infty \frac{e^{-u/\sqrt{\alpha}}\,du}{\cosh^{b+1} u} = 1/f(1,b,0,\alpha),\ b > -1,\ |\arg \alpha| < \pi$$

$$\frac{1}{2\sqrt{\alpha}} \left[\psi\left(\frac{3}{4} + \frac{1}{4\sqrt{\alpha}}\right) - \psi\left(\frac{1}{4} + \frac{1}{4\sqrt{\alpha}}\right)\right] = 1/f(1,0,0,\alpha),\ |\arg \alpha| < \pi$$

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}\,du}{\sqrt{u}(1 + 2\alpha u)} = 1/f(0,1,0,\alpha),\ |\arg \alpha| < \pi.$$

It is actually possible to evaluate the *general case for complex coefficients* by solving the associated difference equation in terms of hypergeometric functions. For the case $a \neq 0$ (we take $a = 1$ without loss of generality) one has:

THEOREM 1: *If* $n^2 + bn + c \neq 0$, $n \geq 1$ *and* $0 < |\alpha| < \infty$ *then*

$$X_n - X_{n-1} - \alpha(n^2 + bn + c)X_{n-2} = 0 \tag{8}$$

*has linearly independent analytic solutions* $X_n(\pm\beta)$, $\beta = 1/\sqrt{\alpha}$ *where*

$$(9)\quad X_n(\beta) = (2\beta)^{-n} \frac{\Gamma\left(n+2+\frac{b}{2}+\mu\right)\Gamma\left(n+2+\frac{b}{2}-\mu\right)}{\Gamma\left(n+\frac{5}{2}+\frac{b}{2}-\frac{\beta}{2}\right)}$$

$$\times {}_2F_1\left(n+2+\frac{b}{2}+\mu, n+2+\frac{b}{2}-\mu; n+\frac{5}{2}+\frac{b}{2}-\frac{\beta}{2}; \frac{1}{2}\right),\ n \geq -1$$

and $\mu = (b^2 - 4c)^{1/2}/2$.

PROOF: Let $u_n(x) = {}_2F_1(\alpha_n, \beta_n; \gamma_n; x)$ with

$$\alpha_n = n + \frac{b}{2} + \mu, \quad \beta_n = n + \frac{b}{2} - \mu, \quad \gamma_n = n + \frac{1}{2} + \frac{b}{2} - \frac{\beta}{2}.$$

Then

$$X_{n-2}(\beta) = (2\beta)^{-n+2} \frac{\Gamma(\alpha_n)\Gamma(\beta_n)}{\Gamma(\gamma_n)} u_n\left(\frac{1}{2}\right).$$

Using the facts

$$\alpha_n\beta_n = n^2 + bn + c,$$

$$u_n'(x) = \frac{\alpha_n\beta_n}{\gamma_n} u_{n+1}(x),$$

and

$$\gamma_n - (\alpha_n + \beta_n + 1)\left(\frac{1}{2}\right) = -\frac{\beta}{2}$$

one obtains

$$X_n(\beta) - X_{n-1}(\beta) - \beta^{-2}(n^2 + bn + c)X_{n-2}(\beta) = 4(2\beta)^{-n} \frac{\Gamma(\alpha_n)\Gamma(\beta_n)}{\Gamma(\gamma_n)}$$

$$\times \left[x(1-x)u_n''(x) + (\gamma_n - (\alpha_n + \beta_n + 1)x)u_n'(x) - \alpha_n\beta_n u_n(x)\right]_{x=1/2} = 0$$

since $u_n(x)$ satisfies the hypergeometric equation in the square brackets. □

COMMENT: Note that $\beta^n X_n(\beta)$ is an entire function of $\beta$ since $(\Gamma(C))^{-1}\, {}_2F_1(A, B; C; \frac{1}{2})$ is an entire function of $C$.

LEMMA 2: *The large n behaviour of* $X_n(\beta)$ *in* (9) *is given by*

$$(10)\quad X_n(\beta) = (2\beta)^{-n} 2\sqrt{\pi} e^{-n} (2n)^{n+1+b/2+\beta/2}\left(1 + O\left(\frac{1}{n}\right)\right).$$

PROOF: One uses Stirling's asymptotic formula for the three Γ functions which appear in (9) together with the formulae

$${}_2F_1\left(A, B; C; \frac{1}{2}\right) = \left(\frac{1}{2}\right)^{C-A-B} {}_2F_1\left(C-A, C-B; C; \frac{1}{2}\right)$$

https://doi.org/10.4153/CMB-1985-050-5 Published online by Cambridge University Press

and

$$ {}_2F_1\left(A,B;C;\frac{1}{2}\right) = 1 + O\left(\frac{1}{C}\right) \quad \text{for} \quad \operatorname{Re} C \to \infty. $$

Thus the large $n$ behaviour of the hypergeometric function which appears in (9) is given by $\sqrt{2}\, 2^{n+1+b/2+\beta/2}(1 + O(1/n))$ and (10) is obtained. □

LEMMA 3: *If* $|\arg \alpha| < \pi$, $0 < |\alpha| < \infty$ (i.e. $0 < \operatorname{Re} \beta$, $|\beta| < \infty$) *then the difference equation* (8) *has a subdominant solution* $X_n^{(s)}$ *given by*

$$ X_n^{(s)} = X_n(-\beta). \tag{11} $$

PROOF: From (10) one has

$$ X_n(-\beta)/X_n(\beta) = (-1)^n(2n)^{-\beta}\left(1 + O\left(\frac{1}{n}\right)\right) \tag{12} $$

which has limit 0 as $n \to \infty$ if $\operatorname{Re} \beta > 0$. □

THEOREM 4: *If* $|\arg \alpha| < \pi$, $0 < |\alpha| < \infty$ *then the regular C-fraction* (7) *with* $a = 1$ *is*[4]

$$ f(1,b,c,\alpha) = \sqrt{\alpha}\left(1 + b + \frac{1}{\sqrt{\alpha}}\right) \times \frac{{}_2F_1\left(\frac{b}{2} + \mu, \frac{b}{2} - \mu; \frac{1}{2} + \frac{b}{2} + \frac{1}{2\sqrt{\alpha}}; \frac{1}{2}\right)}{{}_2F_1\left(1 + \frac{b}{2} + \mu, 1 + \frac{b}{2} - \mu; \frac{3}{2} + \frac{b}{2} + \frac{1}{2\sqrt{\alpha}}; \frac{1}{2}\right)}. \tag{13} $$

PROOF: From Theorem 1, Lemma 3 and Pincherle's Theorem one has $f(1,b,c,\alpha) = (X_{-1}(-\beta) - X_0(-\beta))//X_{-1}(-\beta)$ with $X_n(-\beta)$ determined from (9). If $c \neq 0$ this yields $-\alpha c X_{-2}(-\beta)/X_{-1}(-\beta)$ which reduces to the right side of (13). If $c = 0$ (i.e. $\mu = \pm b/2$) then $X_{-2}$ does not exist but a limit as $c \to 0$ again yields (13) with the numerator hypergeometric function now equal to one. □

For the case $a = 0$, $b \neq 0$ one has:

THEOREM 5: *If* $n + c \neq 0$, $n \geq 1$ *and* $0 < |\alpha| < \infty$ *then*

$$ X_n - X_{n-1} - \alpha(n + c)X_{n-2} = 0 \tag{14} $$

*has linearly independent analytic solutions* $X_n(\pm\beta)$, $\beta = 1/\sqrt{\alpha}$ *where*

$$ X_n(\beta) = \beta^{-n}\Gamma(2 + n + c)D_{-c-n-2}(-\beta), \quad n \geq -1 \tag{15} $$

---

[4]Perron ([5], Ch. 11, §82) considered the connection between continued fractions with $a_n = (an^2 + bn + c)/(d + en)(d + e(n - 1))$ and hypergeometric functions but his method required $e \neq 0$. In §83 he used a different method (Cesàro's) to obtain $f(1,b,c,\alpha)$ as a ratio of integrals for the case $\alpha, b, c > 0, 0 \leq b^2 - 4c < (1 + 1/\sqrt{\alpha})^2$.

*and* $D_\lambda(\beta)$ *is the parabolic cylinder function. Furthermore* $X_n(-\beta)/X_n(\beta) = (-1)^n \exp(-2\beta\sqrt{n})(1 + O(1/\sqrt{n}))$ *so that if* $|\arg\alpha| < \pi$ (*i.e.* Re $\beta > 0$) *then* (14) *has a subdominant solution* $X_n^{(s)} = X_n(-\beta)$.

PROOF: See [4].

THEOREM 6:

$$f(0,1,c,\alpha) = \sqrt{\alpha}\,\frac{D_{-c}(1/\sqrt{\alpha})}{D_{-c-1}(1/\sqrt{\alpha})}, \quad |\arg\alpha| < \pi. \tag{16}$$

PROOF: One uses Pincherle's Theorem and Theorem 5. See also [4]. □

COMMENT: One again has solutions with the property that $\beta^n X_n(\beta)$ is an entire function of $\beta$. Thus in both Theorems 4 and 6 one has $f(a,b,c,\alpha)$ given by the ratio of two entire functions of $\beta$ and the result:

COROLLARY 7: *If* $a > 0$ *or if* $a = 0$, $b > 0$ *then* $f(a,b,c,\alpha)$ *is the branch of a meromorphic function of* $\beta = 1/\sqrt{\alpha}$ *with branch cut along the negative* $\alpha$*-axis.*

3. **Perturbation Theory**. In analogy with the theory of second order linear differential equations, one may use the associated difference equation to obtain properties of continued fractions which are "small perturbations" of a known continued fraction.

Consider the second order linear difference operator $L_n$ defined by

$$L_n(X_n) = X_n - X_{n-1} - \alpha a_n X_{n-2} \tag{17}$$

and let $X_n^{(1)}$ $X_n^{(2)}$ be linearly independent solutions to

$$L_n(X_n) = 0. \tag{18}$$

If the Wronskian is defined by

$$W(f_n, g_n) = f_n g_{n+1} - g_n f_{n+1} \tag{19}$$

then from (18) one has

$$W(X_n^{(1)}, X_n^{(2)}) = -\alpha a_{n+1} W(X_{n-1}^{(1)}, X_{n-1}^{(2)}). \tag{20}$$

Let $G_{nm}$ (a Green's function for $L_n$) be defined by

$$G_{nm} = \frac{X_n^{(1)}X_m^{(2)} - X_n^{(2)}X_m^{(1)}}{W(X_m^{(1)}, X_m^{(2)})}. \tag{21}$$

Then $G_{nm}$ has the obvious properties

$$L_n(G_{nm}) = 0, \quad G_{nn} = 0, \quad G_{n+1n} = -1 \tag{22}$$

and the property

$$-\alpha a_n G_{n-2n-1} = 1 \tag{23}$$

which follows from (20).

https://doi.org/10.4153/CMB-1985-050-5 Published online by Cambridge University Press

LEMMA 8: *The linear difference equation*

$$L_n(Y_n) = f_n \tag{24}$$

*has a solution*

$$Y_n = X_n^{(1)} + \sum_{m=n+1}^{\infty} G_{nm} f_{m+1} \tag{25}$$

*provided that the summation converges.*

PROOF:

$$L_n(Y_n) = L_n(X_n^{(1)}) + \sum_{m=n+1}^{\infty} L_n(G_{nm}) f_{m+1}$$
$$+ (G_{nn} - G_{n-1n} - \alpha a_n G_{n-2n}) f_{n+1} - \alpha a_n G_{n-2n-1} f_n = f_n$$

where (18), (22) and (23) have been used. □

In order to solve the linear difference equation

$$Y_n - Y_{n-1} - \alpha(a_n + V_n) Y_{n-2} = 0 \tag{26}$$

with boundary condition $Y_n - X_n^{(1)} \to 0$, one may use (25) with $f_n = \alpha V_n Y_{n-2}$. This yields the Volterra sum equation

$$Y_n = X_n^{(1)} + \sum_{m=n+1}^{\infty} \alpha G_{nm} V_{m+1} Y_{m-1} \tag{27}$$

which may be solved by iteration to obtain

$$Y_n = \sum_{r=0}^{\infty} Y_{nr} \tag{28}$$

with $Y_{n0} = X_n^{(1)}$ and

$$Y_{nr} = \sum_{m=n+1}^{\infty} \alpha G_{nm} V_{m+1} Y_{m-1\,r-1}, \quad r = 1, 2, \ldots . \tag{29}$$

The method is justified by showing that the above summations converge if $|V_n|$ is sufficiently small for $n$ large.

Given a subdominant solution to (18) with known analytic properties, this method is capable of not only proving the existance of a subdominant solution to (26) but also determining its analytic properties. This in turn yields information on the analytic continuation of the corresponding $C$-fraction.

For example with $a_n = 1$, $|V_n| \leq dK^n$, $K < 1$ one reproduces the Thron and Waadland result (4) mentioned in Section 1, namely:

THEOREM 9: *Let* $a_n = 1$, $|V_n| \leq dK^n$, $K < 1$. *Then*

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(1 + V_n)\alpha}{1}, \quad 1 + V_n \neq 0, \quad n \geq 1$$

*converges for* $\alpha \notin (-\infty, -\frac{1}{4}]$ *to a function whose analytic continuation is meromorphic in* $z = \sqrt{1 + 4\alpha}$ *in the domain* $|(1 - z)/(1 + z)| < K^{-1}$.

PROOF: Let

$$X_n^{(1)} = \left(\frac{1 - z}{2}\right)^n, \quad X_n^{(2)} = \left(\frac{1 + z}{2}\right)^n.$$

Then

$$\alpha G_{nm} X_{m-1}^{(1)} = -\left(\frac{1 - z}{2}\right)^n \left[1 + \left(\frac{1 - z}{1 + z}\right) + \ldots + \left(\frac{1 - z}{1 + z}\right)^{m-n-1}\right], \quad m \geq n + 1.$$

Hence

$$|\alpha G_{nm} V_{m+1} X_{m-1}^{(1)}| \leq \left|\frac{1 - z}{2}\right|^n (m - n) d K^{m+1} P^{m-n-1}, \quad m \geq n + 1$$

where

$$P = \max\left(1, \left|\frac{1 - z}{1 + z}\right|\right), \quad z \neq -1.$$

If $KP < 1$, this implies the absolute convergence of (29) for $r = 1$, together with the estimate $|Y_{n1}| \leq Q_n |1/2 - z/2|^n$, $Q_n = dK^{n+2}(1 - KP)^2$ and by induction on $r$, $|Y_{nr}| \leq Q_n^r |1/2 - z/2|^n$. Thus (28) converges absolutely if $Q_n < 1$ and one obtains $Y_n$ for $n \geq n_0$ (with $n_0$ determined by $Q_{n_0} < 1$), together with the estimate $|Y_n| \leq |(1 - z)/2|^n / (1 - Q_n)$. This implies that if $n \geq n_0$, $K|(1 - z)/(1 + z)| < 1$, then $Y_n$ is to (26) as $((1 - z)/2)^n$ is to (18). In particular:

(a) $$\underset{n=n_0+2}{\overset{\infty}{\mathrm{K}}} \frac{\alpha(1 + V_n)}{1} = -Y_{n_0+1}/Y_{n_0}, \quad 0 < \left|\frac{1 - z}{1 + z}\right| < 1,$$

(b) $Y_n$, $n \geq n_0$ is analytic in $z$ if $K \left|\frac{1 - z}{1 + z}\right| < 1$,

where (a) follows form Pincherle's Theorem and (b) follows from Weierstrass' Theorem on the analyticity of a uniformly convergent sequence of analytic functions. This establishes the Theorem since it suffices to consider the tail of the continued fraction. □

On applying the method to a perturbation of the continued fraction of Section 2, one obtains the following companion to Theorem 4 and Corollary 7.

THEOREM 10: *Let* $a_n = n^2 + bn + c$, $|V_n| \leq dn^{1-\epsilon}$, $\epsilon > 0$. *Then* $\mathrm{K}_{n=1}^{\infty}((a_n + V_n)\alpha/1)$, $a_n + V_n \neq 0$, $n \geq 1$ *converges for* $|\arg \alpha| < \pi$ *to a function whose analytic continuation is a meromorphic function of* $z = \sqrt{\alpha}$ *in the domain exterior to the circle* $|z| = -\cos(\arg z)/\epsilon$.

PROOF: Let $X_n^{(1)} = X_n(-\beta)$, $X_n^{(2)} = X_n(\beta)$ where $X_n(\beta)$ is given by (9) and $\beta = z^{-1}$. From (10), (12), and (20)

https://doi.org/10.4153/CMB-1985-050-5 Published online by Cambridge University Press

$$\alpha G_{nm}X_{m-1}(-\beta) \sim -\frac{X_n(-\beta)}{2m^2}\left(1-(-1)^{n+m}\left(\frac{m}{n}\right)^{-\beta}\right) \quad \text{for} \quad m,n \to \infty.$$

Thus $|\alpha G_{nm}X_{m-1}(-\beta)V_{m+1}| \leq C|X_n(-\beta)|m^{-1-\epsilon}Q_{nm}$, $n \geq n_0$ with $Q_{nm} = \max(1, (m/n)^{-\mathrm{Re}\,\beta})$ and $C$ independent of $n$ and $m$. From (29) one has for Re $\beta > -\epsilon$, $|Y_{n1}| \leq C|X_n(-\beta)|\sum_{m=n+1}^{\infty} m^{-1-\epsilon}Q_{nm} \leq C|X_n(-\beta)|n^{-\epsilon}/(\epsilon + P)$ where $P = \min(0, \mathrm{Re}\,\beta)$ and by induction $|Y_{nr}| \leq C^r|X_n(-\beta)|n^{-r\epsilon}/\pi_{s=1}^{r}(s\epsilon + P)$. Equation (28) then yields $|Y_n| \leq |X_n(-\beta)| \exp(Cn^{-\epsilon}/(\epsilon + P))$, Re $\beta > -\epsilon$, $n \geq n_0$. This implies that:

(a) $\displaystyle \mathop{\mathrm{K}}_{n=1}^{\infty} \frac{(n^2 + bn + c + V_n)\alpha}{1} = -Y_0/Y_{-1}$, $\quad$ Re $\beta > 0$,

(b) $\beta^n Y_n$, $n \geq -1$ is analytic in $\beta$ for Re $\beta > -\epsilon$,

where (a) follows from Pincherle's Theorem and Lemma 3 and (b) for $n \geq n_0$ follows from Weierstrass' Theorem and Theorem 1. For $n < n_0$ one uses (26) with backward recursion together with the condition $a_n + V_n \neq 0$, $n \geq 1$. □

COMMENT: The restriction Re $\beta > -\epsilon$ is misleading because the estimates involve $|V_{m+1}|$. Singularities are not necessarily present in the disc $|z| \leq -\cos(\arg z)/\epsilon$ (apart from an essential singularity at $z = 0$). It is an oscillating behaviour, such as $(-1)^n$, in $V_n$ which appears to produce singularities.

4. **Conjecture and proof for $p = 2$.** In order to prove the conjecture of Section 1 it is natural to examine the difference equation

$$X_n - X_{n-1} - \alpha \prod_{i=1}^{N} (n + r_i)^{p_i} X_{n-2} = 0. \tag{30}$$

If one puts

$$X_n = (2^p\alpha)^{n/2} \prod_{i=1}^{N} \Gamma^{p_i}((n + r_i + 2)/2)Z_n \tag{31}$$

then (30) becomes

$$Z_n - \beta b_n Z_{n-1} - Z_{n-2} = 0 \tag{32}$$

with $\beta = 1/\sqrt{\alpha}$ and $b_n = 2^{-p/2}\prod_{i=1}^{N} \Gamma^{p_i}((n + r_i + 1)/2)/\Gamma^{p_i}((n + r_i + 2)/2)$. From Stirling's formula one has

$$b_n \sim n^{-p/2}\left(1 + \frac{b^{(1)}}{n} + \frac{b^{(2)}}{n^2} + \dots\right) \tag{33}$$

and one can check that for $n \to \infty$

$$Z_n(\beta) \sim \begin{cases} n^{\beta/2}, & p = 2 \\ \exp(\beta n^{1-p/2}/(2-p)), & 0 < p < 2 \end{cases} \tag{34}$$

https://doi.org/10.4153/CMB-1985-050-5 Published online by Cambridge University Press

are asymptotic solutions to (32). Since (32) has a coefficient $\beta b_n$ with $b_n \to 0$ and the boundary condition (34) is an entire function of $\beta$, one suspects that such a $Z_n(\beta)$ may itself be an entire function of $\beta$.

Given a solution with the above boundary condition one has a second linearly independent solution $(-1)^n Z_n(-\beta)$. It is then clear that for Re $\beta > 0$ one expects

$$Z_n^{(d)} = Z_n(\beta)$$
$$Z_n^{(s)} = (-1)^n Z_n(-\beta)$$

with

$$Z_n^{(s)}/Z_n^{(d)} \sim \begin{cases} (-1)^n n^{-\beta}, & p = 2 \\ (-1)^n \exp(-2\beta n^{1-p/2}/(2-p)), & 0 < p < 2 \end{cases}$$

and corresponding dominant and subdominant solutions to (30) via (31).

The proof of the conjecture thus hinges on demonstrating that the solution $Z_n(\beta)$ satisfying the boundary condition (34) is an entire function of $\beta$. More generally one can ask the question: For what $b_n$ does (32) have a minimal solution which is an entire function of $\beta$? The case $b_n \sim 1/n\,(1 + b^{(1)}/n + \ldots)$ is seen below to be sufficient.

PROOF FOR $p = 2$: From (32) and (33) one obtains a Frobenius expansion

(35) $$Z_n(\beta) \sim n^{\beta/2}(1 + c^{(1)}/n + c^{(2)}/n^2 + \ldots)$$

with recursion relations determining the coefficients $c^{(i)}$ in terms of $\beta$ and $b^{(i)}$. From (35) one obtains

(36) $$\Delta^m Z_n(\beta) \sim n^{\beta/2-m}\left(\prod_{i=0}^{m-1}(\beta/2 - i) + O\left(\frac{1}{n}\right)\right).$$

where $\Delta$ is the difference operator.

Let $A_n$ be the solution to (32) which satisfies the initial condition $A_{-1} = 0$, $A_0 = 2$. One has $A_n$ a polynomial of degree $n$ in $\beta$ with $A_n$ odd (even) for $n$ odd (even). Thus,

(37) $$A_n = a(\beta)Z_n(\beta) + a(-\beta)(-1)^n Z_n(-\beta).$$

From (35) and (32) one has the Wronskian $Z_n(\beta)Z_{n-1}(-\beta) + Z_n(-\beta)Z_{n-1}(\beta) = 2$ so that

(38) $$a(\beta) = Z_{-1}(-\beta).$$

From (37), (36) with $m = 1$ and (35) one obtains

$$A_n + A_{n-1} \sim \left[2a(\beta)n^{\beta/2} - a(-\beta)\frac{\beta}{2}(-1)^n n^{-\beta/2-1}\right]\left(1 + O\left(\frac{1}{n}\right)\right)$$

so that

$$a(\beta) = \lim_{n\to\infty} n^{-\beta/2}(A_n + A_{n-1})/2$$

https://doi.org/10.4153/CMB-1985-050-5 Published online by Cambridge University Press

uniformly for Re $\beta \geq$ Re $\beta_0 > -1$, $|\beta| \leq M < \infty$. Hence, from the Weierstrass Theorem, one has $a(\beta) = Z_{-1}(-\beta)$ analytic for Re $\beta > -1$, $|\beta| < \infty$. By taking the limit of an average of more and more terms one obtains $Z_{-1}(-\beta)$ analytic for $|\beta| < \infty$. In particular (35), (36), (37) and (38) imply

$$(39) \qquad Z_{-1}(-\beta) = \lim_{n\to\infty} \frac{n^{-\beta/2}}{2m} (A_n + 2A_{n-1} + \ldots + 2A_{n-m+1} + A_{n-m})$$

for Re $\beta > -m$, $|\beta| < \infty$.

By a similar argument, $Z_0(-\beta)$ is analytic for $|\beta| < \infty$ and the proof of the conjecture for $p = 2$ follows from Pincherle's Theorem. □

COMMENT: Equation (39) may be expanded in powers of $\beta$ to obtain limit averaging formulae for the power series coefficients of $Z_{-1}(-\beta)$ (and similarly for $Z_0(-\beta)$). One can thus obtain the continued fraction in terms of a ratio of two convergent power series in $\beta$. This solves the analytic continuation problem in principle. The practical utility of the method will have to be determined by numerical experiment and/or error analysis.

ACKNOWLEDGEMENT: The author is thankful to J. Chalk for discussions and to J. Repka for inviting papers dedicated to R. A. Smith.

## REFERENCES

1. G. A. Baker, Jr. and P. Graves-Morris, *Padé approximants Part I: Basic theory*, Vol. 13 in, Encyclopedia of mathematics and its applications (Addison-Wesley, Reading Mass., 1981).

2. W. Gautschi, *Computational aspects of three-term recursion relations*, SIAM Review, **9** (1967), pp. 24–82.

3. W. B. Jones and W. J. Thron, *Continued fractions analytic theory and applications*, Vol. 11 in, Encyclopedia of mathematics and its applications (Addison-Wesley, Reading Mass., 1980).

4. D. Masson, *The rotating harmonic oscillator eigenvalue problem I. Continued fractions and analytic continuation*, J. Math. Phys. **24** (1983), pp. 2074–2088.

5. O. Perron, *Die Lehre von den Kettenbruchen* (Verlag und Druck, Leipzig und Berlin, 1929).

6. W. J. Thron and H. Waadeland, *Analytic continuation of functions defined by means of continued fractions*, Math. Scand. **47** (1980), pp. 72–90.

7. H. S. Wall, *Analytic theory of continued fractions* (D. Van Nostrand, Princeton, N.J., 1948).

UNIVERSITY OF TORONTO
TORONTO, ONTARIO, CANADA