Can. J. Math., Vol. XXXIII, No. 4, 1981, pp. 872–884

PROJECTIVE ELEMENTS IN CATEGORIES WITH PERFECT θ-CONTINUOUS MAPS

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0. Introduction. In 1958 Gleason [**6**] proved the following:

THEOREM. *In the category of compact Hausdorff spaces and continuous maps, the projective elements are precisely the extremally disconnected spaces.*

The projective elements in many topological categories with perfect continuous functions as morphisms have been found since that time. For example: In the following categories the projective elements are precisely the extremally disconnected spaces:

(i) The category of Tychonov spaces and perfect continuous functions. [**4**] [**11**].

(ii) The category of regular spaces and perfect continuous functions. [**4**] [**12**].

(iii) The category of Hausdorff spaces and perfect continuous functions. [**10**] [**1**].

(iv) In the category of Hausdorff spaces and continuous k-maps the projective members are precisely the extremally disconnected k-spaces. [**14**].

In 1963 Iliadis [**7**] constructed for every Hausdorff space X the so called Iliadis absolute $E[X]$, which is a maximal pre-image of X under irreducible θ-continuous maps. Moreover he proved that in the category of Hausdorff spaces and perfect irreducible θ-continuous surjections the projective elements are precisely the extremally disconnected spaces.

Our aim is to study the projective members in the following categories:

(v) $\mathscr{H}_{2+}$(resp. $\mathscr{H}_2$): the category of H-closed Urysohn spaces (resp. Hausdorff spaces) and perfect θ-continuous functions. See also [**10**], [**15**].

(vi) $\mathscr{T}_{3\frac{1}{2}}{}^s$(resp. $\mathscr{T}_3{}^s$): the category of those spaces X such that X_s — the semi-regularization of X — is $T_{3\frac{1}{2}}$ (resp. T_3), and perfect θ-continuous functions.

(vii) $\mathscr{T}_{2+}$ (resp. $\mathscr{T}_2$): the category of Urysohn spaces (resp. Hausdorff spaces) and perfect θ-continuous functions.

In the first section we state a collection of preliminary remarks. In the second section we study the image spaces under θ-continuous maps of compact Hausdorff spaces. In the third section we show that in each

Received January 15, 1980 and in revised form June 26, 1980.

of the categories projective objects have to be compact. This leads to the theorem:

The projective elements in $\mathscr{H}_{2+}$, $\mathscr{T}_{3\frac{1}{2}}{}^s$ *and* $\mathscr{T}_3{}^s$ *are precisely the compact extremally disconnected spaces.*

For the category $\mathscr{H}_{2+}$ this theorem is proved by Mioduszewski and Rudolf. (cf. [**10**], [**15**]). We consider both $\mathscr{T}_{3\frac{1}{2}}{}^s$ and $\mathscr{T}_3{}^s$, because of the following characterization of $\mathscr{T}_{3\frac{1}{2}}{}^s$:

X_s is $T_{3\frac{1}{2}}$ if and only if X has an H-closed T_{2+} extension [**13**]. The property for a T_{2+} space X to have such an extension is surprisingly strong. This becomes clear if we consider the category of all T_{2+} spaces for which we show in last section:

The projective elements in $\mathscr{T}_{2+}$ *are precisely the finite spaces.*

As a corollary to this theorem we obtain our main result:

The projective members of $\mathscr{H}_2$ *and* $\mathscr{T}_2$ *are precisely the finite spaces.*

R. G. Woods [**15**] asked for a characterization of the projective elements of these categories in his survey paper on absolutes.

1. Preliminaries. We adopt the following conventions and definitions.

(a) All spaces are assumed to be Hausdorff.

(b) A space is called T_{2+}, or *Urysohn* if:

$$\forall x, y \in X \,\exists\, U_x \,\exists\, U_y \text{ such that } \mathrm{Cl}\, U_x \cap \mathrm{Cl}\, U_y = \emptyset.$$

where U_x and U_y denote open neighborhoods of x and y respectively.

(c) A function $f:X \to Y$ is called *perfect* if:

f is closed,

$\forall y \in Y: f^{-1}(y)$ is compact.

(d) A function $f:X \to Y$ is called θ-*continuous*, if

$$\forall x\, \forall U_{f(x)} \,\exists\, V_x \text{ such that } f[\mathrm{Cl}_X V_x] \subset \mathrm{Cl}_Y U_{f(x)}.$$

Note that if Y is regular, then any θ-continuous function into it is continuous. Moreover, θ-continuous functions may cause unexpected problems, e.g., it is well known that if $f:X \to Y$ is θ-continuous then $f:X \to f[X]$ need not be θ-continuous. In (h) we give the standard example of this fact.

However, if $f:X \to Y$ is θ-continuous and $Y \subset Z$ then also $f:X \to Z$ is θ-continuous.

If $f:X \to Y$ is θ-continuous and if $A \subset X$ then $f|A:A \to Y$ is θ-continuous.

If $f:X \to Y$ is θ-continuous, if Z is a dense subset of Y and if $f[X] \subset Z$, then also $f:X \to Z$ is θ-continuous.

(e) X is called *semi-regular* if the collection of regular open sets (i.e., sets U with $\text{int}(\text{Cl } U) = U$) is a base for the topology of X.

For every space X we define the space X_s, the semi-regularization of X, by taking the collection of regular open sets of X as a base for the space X_s on the same underlying set.

Note that the identity $\text{id}: X_s \to X$ is perfect and θ-continuous; if X is T_2 (resp. T_{2+}) so is X_s.

(f) The *Katětov-extension* $\kappa[X]$ of X is defined by:

$$\kappa[X] = X \cup \{\mathscr{F} \mid \mathscr{F} \text{ open non-fixed ultrafilter on } X\}$$

with topology

X is open in $\kappa[X]$.

For every $\mathscr{F} \in \kappa[X] \backslash X$ the collection $\{\{\mathscr{F}\} \cup F \mid F \in \mathscr{F}\}$ is a local base at $\mathscr{F}$ (see [**8**]).

(g) Note that if X is discrete, then $(\kappa[X])_s = \beta X$ (βX is the Čech-Stone compactification of X).

(h) Consider the countable discrete space ω. Now parts (e) and (g) imply that $\text{id}: \beta\omega \to \kappa[\omega]$ is perfect and θ-continuous, and so the embedding $\text{id}: \omega^* \to \kappa[\omega]$ is θ-continuous, but clearly

$$\text{id}: \omega^* \to \text{id}(\omega^*) = \kappa[\omega] \backslash \omega$$

is not θ-continuous, because $\kappa[\omega] \backslash \omega$ is discrete.

(i) In [**7**] the Iliadis-absolute $E[X]$ of a space X is constructed as follows: $E[X]$ is the subset of the fixed ultrafilters of the Stone space of the Boolean algebra of the regular open sets of X. $E[X]$ is extremally disconnected and $T_{3\frac{1}{2}}$. A function

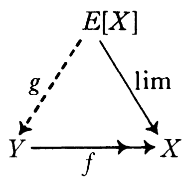
$$\lim_X : E[X] \twoheadrightarrow X$$

is defined by

$$\lim_X(\mathscr{F}) = \bigcap \{\text{Cl}_X(F) \mid F \in \mathscr{F}\}.$$

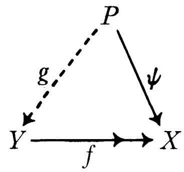
The function $\lim_X : E[X] \twoheadrightarrow X$ turns out to be an irreducible, perfect, θ-continuous surjection. $E[X]$ and $\lim_X : E[X] \twoheadrightarrow X$ have the following property:

If $f: Y \twoheadrightarrow X$ is any irreducible, perfect, θ-continuous surjection, then there exists a $g: E[X] \to Y$ which is θ-continuous, perfect and irreducible such that $f \circ g = \lim_X$. If Y is extremally disconnected then $g: E[X] \twoheadrightarrow Y$ is a θ-homeomorphism.

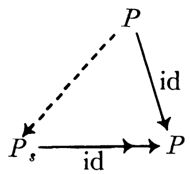


(j) A T_2-space is called *minimal Hausdorff* if, and only if, every strictly weaker topology on the same underlying set fails to be Hausdorff. If X is H-closed then X_s is minimal Hausdorff. If X is moreover a Urysohn space, then X_s is compact (see e.g. [**13**]).

(k) A space P is defined to be *projective* in a category if for each morphism $\psi:P \rightarrow X$ and each surjection $f: Y \rightarrow X$ there exists a morphism $g:P \rightarrow Y$ in the category such that $\psi = f \circ g$.



(l) If P is a projective object in each of the categories with perfect θ-continuous functions then P has to be semi-regular because the map from P_s to P is θ-continuous and perfect. However, if P is not semi-regular then the map from P onto P_s is not closed and hence not perfect and so the diagram cannot be completed.



(m) A semi-regular space which is extremally disconnected is a Tychonov space.

(n) If X is projective in any of the categories $\mathscr{H}_{2+}$, $\mathscr{H}_2$, $\mathscr{T}_{2+}$, $\mathscr{T}_2$, $\mathscr{F}_{3\frac{1}{2}}{}^s$, $\mathscr{T}_3{}^s$ then X is extremally disconnected and Tychonov (see e.g. [**10**]).

2. θ-continuous functions.

2.1 PROPOSITION. *If X is compact and $f:X \rightarrow Y$* is *θ-continuous then:*

(i) *f is perfect.*

(ii) *If Y is T_{2+}, then $f[X]$ is regularly embedded (i.e., for each point y of $Y \backslash f[X]$ there exists a neighborhood O_y of y and an open set $O_{f[X]} \supset f[X]$ which are disjoint, (i.e., $f[X]$ is a θ-closed subset of Y. cf.* [**3**])).

Proof. (i) We show that $f[X]$ is a closed subset of Y. Suppose not. Then $\exists\, y \in Y \backslash f[X]$ such that $\forall\, U_y$ we have $U_y \cap f[X] \neq \emptyset$. Let $\mathscr{U}_y$ be the neighborhood filter of y. Now the collection

$$\{U_y \cap f[X] | U_y \in \mathscr{U}_y\}$$

https://doi.org/10.4153/CJM-1981-068-6 Published online by Cambridge University Press

is a filter. So

$$\{f^{-1}(U_y \cap f[X]) | U_y \in \mathcal{U}_y\}$$

is a centered system on X. Since X is compact, there exists an accumulation point x. (i.e., $\forall V_x$ neighborhood of x: $V_x \cap f^{-1}(U_x \cap f[X]) \neq \emptyset$.)

Now there exist open neighborhoods O_y of y and $O_{f(x)}$ of $f(x)$ which are disjoint. Also

$$O_y \cap \mathrm{Cl}_Y \langle O_{f(x)} \rangle = \emptyset.$$

We can find a neighborhood V_x of x such that

$$f(\mathrm{Cl}_X V_x) \subset \mathrm{Cl}_Y \langle O_{f(x)} \rangle$$

and therefore $f(V_x) \cap O_y = \emptyset$, which implies

$$V_x \cap f^{-1} \langle O_y \cap f[X] \rangle = \emptyset.$$

This is a contradiction.

Since every closed subset B of X is compact and the map $f|B:B \rightarrow Y$ is θ-continuous we obtain that the image of a closed set is closed, and that f is a closed mapping.

A similar argument shows that $f^{-1}(y) \subset X$ is closed and hence compact for every $y \in Y$, because the image of an accumulation point of $f^{-1}(y)$ can never be separated from y.

(ii) Repeat the proof for the filter

$$\{\mathrm{Cl}(U_y) \cap f[X] | U_y \in \mathcal{U}_y\}.$$

2.2 COROLLARY. (i) *If f is a perfect θ-continuous map from a compact space X into a Hausdorff space Y then also the map $f:X \rightarrow Y_s$ is perfect and* θ-continuous.

(ii) *If X is compact and f is a θ-continuous surjection then $f[X]$ is H-closed and $f[X]_s$ is minimal Hausdorff.* (*see* 1(f)).

2.3 PROPOSITION. *Let X be a compact Hausdorff space and let f be a compact and irreducible function of X onto a set Y. Then there exists a minimal Hausdorff topology on Y such that f is θ-continuous.*

Proof. Let $\mathcal{A}$ be the subcollection of all images of closed sets in X. We can use $\mathcal{A}$ as a closed base for a topology on Y. We only need to show that Y with this topology is Hausdorff (i) and that f is θ-continuous (ii).

(i) Y is Hausdorff. Let y and z be two different points of Y. Then $f^{-1}(y)$ and $f^{-1}(z)$ are two disjoint closed subsets of X which can be separated by two disjoint open sets U and V. $X \backslash U$ and $X \backslash V$ together cover X and

$$f\langle X \backslash U \rangle \cup f \langle X \backslash V \rangle = Y.$$

Then $Y\backslash f\langle X\backslash U\rangle$ and $Y\backslash f\langle X\backslash V\rangle$ are two disjoint open neighborhoods of y and z in Y.

(ii) f is θ-continuous. Choose a point $x \in X$ and a neighborhood U of $y = f(x)$ in Y. Then $Y\backslash U$ is closed in Y; and

$$Y\backslash U = \cap \{f(G_i)|G_i \text{ closed in } X, i \in I\}$$

since $y \notin Y\backslash U$, $\exists\, i$ such that $y \notin f(G_i)$; $f^{-1}(y) \cap G_i = \emptyset$; and $x \notin X\backslash G_i$.

We now claim that $f\langle X\backslash G_i\rangle \subset \mathrm{Cl}_Y(U)$. Suppose not. Then there exists a point p in $X\backslash G_i$ such that $f(p) \notin \mathrm{Cl}_Y(U)$. Hence there exists an open set $V = V_{f(p)}$ disjoint from U:

$$Y\backslash V_{f(p)} = \cap \{f(B_j)|B_j \text{ closed in } X \text{ and } j \in J\}$$

and we find a $j \in J$ such that $f(p) \notin f(B_j)$. However, $f(B_j) \cup f(G_i) = Y$ since $U \cap V = \emptyset$ and that $B_j \cup G_i = X$ follows since f is irreducible. However this is a contradiction since p is in neither.

We should consider the above proposition as a way to obtain a θ-continuous quotient of a compact space when the ordinary quotient is not Hausdorff. Our proposition states that it is possible when the involved decomposition is irreducible and compact, and that the decomposition space will be minimal Hausdorff.

3. Compactness of projective objects. In this section we show that in each of the categories $\mathscr{T}_2, \mathscr{T}_{2+}, \mathscr{H}_2, \mathscr{H}_{2+}, \mathscr{T}_3{}^s$ and $\mathscr{T}_{3\frac{1}{2}}{}^s$ the projective objects need to be compact. For the latter two categories we obtain that the projective objects are precisely the compact extremally disconnected compact ones. The technique we use is to show that for a non-compact X there exists a space Y and a perfect θ-continuous map $f:X \to Y$ such that the related map $f_s:X \to Y_s$ need not be closed although the map $\mathrm{id}: Y_s \to Y$ is obviously perfect and θ-continuous.

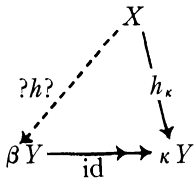
3.1 PROPOSITION. *In each of the categories* $\mathscr{T}_{3\frac{1}{2}}{}^s, \mathscr{T}_3{}^s, \mathscr{T}_{2+}, \mathscr{T}_2, \mathscr{H}_2$ and $\mathscr{H}_{2+}$ *projective objects are always compact.*

Proof. Suppose that X is a projective object in one of the considered categories. From 1.$(l)(m)(n)$ we obtain that projective objects in those categories are extremally disconnected and completely regular and this solves the matter already in the categories $\mathscr{H}_2$ and $\mathscr{H}_{2+}$. For the other four categories we consider two special cases first: (i) X contains no isolated points. (ii) X contains a dense subset D of isolated points.

Case (i): If X is an extremally disconnected space without isolated points then the same properties are shared by βX. It is well known that βX is for such a space a retract of some βY for a discrete space Y.

Because βX has no isolated points we conclude that $X \subset \beta X \subset \beta Y\backslash Y$. Let $h:X \to \beta Y$ be such an embedding. Note that $h[X]$ cannot be a closed subset of βY because X is not compact. Let κY be the Katětov extension of Y, then κY is defined on the same underlying set as βY and hence it is a H-closed Urysohn space with βY as its semi-regularization. So κY fits in all four categories.

Let h_κ be the map from X into κY which corresponds to h, then the corresponding map $h_\kappa : X \to \kappa Y$ is perfect and θ-continuous, and the identity mapping $\mathrm{id}: \beta Y \to \kappa Y$ is perfect θ-continuous one to one and onto.



However, the mapping h itself is not perfect since the image of the closed set X is not closed. We conclude that X cannot be projective.

Case (ii): Suppose that X is an extremally disconnected space which has a dense set D of isolated points. Because of 1.(1)(m) we find that X has to be completely regular and extremally disconnected and therefore $D \subset X \subset \beta D$. Choose a point p from $\beta D \backslash X$ and define X^+ to be $X \cup \{p\}$. Define N^+ to be the convergent sequence $N \cup \{\infty\}$ and let $Y = X^+ \times N^+$. Define a subspace Z of Y by

$$Z = \{(x, n) | x \in D\} \cup \{(x, \infty) | x \in X^+\}$$

and define a space $Z^\#$ from Z by making $\{p\} \times \{\infty\} \cup D \times N$ extra open. The space $Z^\#$ is a Urysohn space and its semi-regularization is Z because the extra open set of $Z^\#$ is dense in Z. Since Z is obviously completely regular we obtain that both $Z^\#$ and Z are in all four considered categories. The mapping $\mathrm{id}: Z \to Z^\#$ is θ-continuous, perfect and onto, the mapping $f^\# : X \to Z^\#$ defined by $f^\#(x) = (x, \infty)$ is also θ-continuous and perfect; however, the mapping $f: X \to Z$ cannot be perfect since the image of X is not closed in Z because the point $p * \infty$ is in its closure. This shows that X cannot be a projective element.

Next let X be an arbitrary non-compact extremally disconnected Tychonov space. Then $X = X_d \cup X_n$ in which X_d is the closure of all isolated points of X, which is clopen, and X_n is the complement of X_d. Now at least one of the two clopen parts is non-compact and we can repeat either the first or the second case argument to show that X cannot be projective. (Note that a clopen subspace of a projective space is projective.)

As we already mentioned in the introduction the following theorem is well known (see e.g. [**10**] and [**15**]), but we present another proof.

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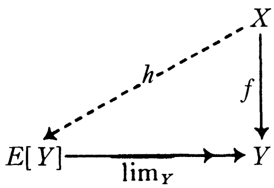
3.2 THEOREM. *The projective objects in* $\mathscr{H}_{2+}$ *are precisely the compact extremally disconnected spaces.*

Proof. A projective object in $\mathscr{H}_{2+}$ needs to be compact and extremally disconnected (cf. 1.(m)).

Conversely, we suppose that X is a compact e.d. space, and that $f:X \rightarrow Y$ and $g:Z \twoheadrightarrow Y$ are two morphisms in the category, and that g is moreover surjective. According to 1.(j) the spaces Z_s and Y_s are compact. So the spaces Z_s and Y_s are regular and the function $f_s:X \rightarrow Y_s$ is continuous and hence perfect, and $g_s:Z_s \twoheadrightarrow Y_s$ is continuous. Since X is projective in the category of regular spaces we obtain a mapping $h:X \rightarrow Z_s$ such that $g_s \circ h = f_s$. Let $h':X \rightarrow Z$ be the induced mapping; then we obtain that h' is θ-continuous and hence perfect (cf. 2.1) and moreover $g \circ h' = f$. We obtain that X is projective.

Remark. The same proof does not work in the cases $\mathscr{T}_{3\frac{1}{2}}{}^s$ and $\mathscr{T}_3{}^s$ since the mappings f_s and g_s need not be closed.

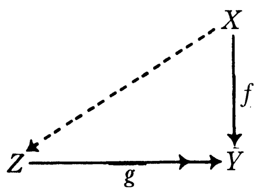
3.3 LEMMA. *In each of the categories* $\mathscr{T}_{3\frac{1}{2}}{}^s, \mathscr{T}_3{}^s, \mathscr{T}_{2+}, \mathscr{T}_2, \mathscr{H}_2$ *we have that X is projective if and only if for each morphism* $f:X \rightarrow Y$ *there exists a morphism* $h:X \rightarrow E[Y]$ *such that* $f = \lim_Y \circ h$.



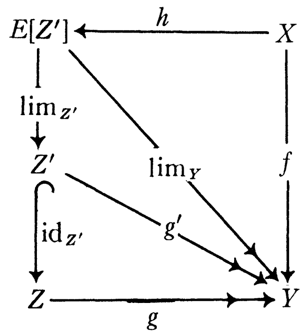
Note that $E[Y]$ is always a member of the category in each of the cases. cf. (1.i).

Proof. $\Rightarrow$. This is obvious from the definition of projectivity.

$\Leftarrow$. We consider the following diagram: It is easy to find a closed subspace Z' of Z such that the mapping $g' = g|Z'$ is θ-continuous perfect and irreducible. We do not assume that the space Z' is necessarily an element of the considered category; it is sufficient to choose it from $\mathscr{T}_2$.



The space $E[Z']$ is an extremally disconnected space which is mapped perfect, irreducible and θ-continuously onto Y by the mapping $g' \circ \lim_{Z'}$. According to Iliadis [**7**] the spaces $E[Z']$ and $E[Y]$ are homeomorphic and $g' \circ \lim_{Z'} = \lim_Y$. From our assumptions it follows that there exists a mapping $h:X \to E[Z']$ such that $\lim_Y \circ h = f$.



Although Z' need not be in our category, the spaces $E[Z']$ and Z both are and the mapping $\mathrm{id}_{Z'} \circ \lim Z'$ is closed, perfect and θ-continuous and so is

$$\mathrm{id}_{Z'} \circ \lim Z' \circ h:X \to Z.$$

This proves the lemma.

3.4 THEOREM. *The projective objects in $\mathcal{T}_3{}^s$ and $\mathcal{T}_{3\frac{1}{2}}{}^s$ are precisely the compact extremally disconnected spaces.*

Proof. According to 3.1 we know that any projective element in those categories needs to be compact and extremally disconnected.

Conversely, according to 3.3 we only have to show that for a compact extremally disconnected space X and for each map $f:X \to Y$ there exists a mapping $h:X \to E[Y]$ such that $\lim_Y \circ h = f$. Let Y_s be the semi-regularization of Y; then $f_s:X \to Y_s$ is perfect and continuous since Y_s is regular and X is compact according to our assumptions. By the regularity of Y_s and the fact that $\lim_Y$ is θ-continuous and perfect we can quickly deduce that $\lim_Y$, regarded as a function from $E[Y]$ to Y_s, is a closed map. As it is obviously also compact, irreducible and continuous (since Y_s is regular), it follows from the uniqueness of the Iliadis-absolute (see 1(i)) that $E[Y] = E[Y_s]$ and $\lim_Y:E[Y] \to Y_s$ is a closed, continuous and compact map. Since X is projective in $\mathcal{T}_3$, we find that there exists a mapping $h:X \to E[Y_s]$ such that

$$\lim_{Y_s} \circ h = f_s$$

and since h is θ-continuous and perfect we obtain that

$$\lim_Y \circ h = f$$

and we have found the required factorization of f over $E[Y]$.

4. Finiteness of projective objects. In this section we show that the only projective objects of the categories $\mathscr{T}_{2+}$, $\mathscr{T}_2$ and $\mathscr{H}_2$ are the finite discrete spaces. In order to do this we have an example which shows that the ordinary circle can be mapped θ-continuously into a space such that the image is discrete and the absolute of the image space contains no compact subsets of large cardinality.

4.1 LEMMA. *Let X be an infinite compact extremally disconnected space. Then X can be mapped onto the circle $\mathfrak{C}$.*

Proof. Since X has a countable discrete subset D, then we can map this set onto the rationals in $\mathfrak{C}$. This map can be extended over X because D is C^*-embedded in X (see [**5**]).

4.2 PROPOSITION. *Let $f:X \to Y$ be a perfect θ-continuous map from a compact extremally disconnected space X into a Urysohn space Y. Suppose that there exists a compact subset B of the Iliadis absolute $E[Y]$ such that* lim $(B) = f[X]$. *Let $g:Z \twoheadrightarrow Y$ be any perfect θ-continuous surjection of a Urysohn space Z onto Y. Then there exists a map $h:X \to Z$ in $\mathscr{T}_{2+}$ such that $g \circ h = f$.*

Proof. In order to prove that there exists an $h:X \to E[Y]$ such that $g \circ h = f$ we consider $E[Y] \times X$. Define

$$H \subset B \times X \subset E[Y] \times X$$

by

$$H = \{(b, x) | b \in B, x \in X, \lim (b) = f(x)\}.$$

We first show that H is closed in $B \times X$.

Let $(b, x) \notin H$. Then lim $(b) \neq f(x)$. We can separate lim (b) and $f(x)$ by two disjoint closed neighborhoods U_b and $U_{f(x)}$ of lim (b) (resp. $f(x)$). In B there exists a neighborhood V_b of b such that lim $(V_b) \subset \mathrm{Cl}\, U_b$, and similarly there is a neighborhood V_x of x in X such that $f(V_x) \subset U_{f(x)}$. Clearly $V_b \times V_x$ is an open neighborhood of (b, x) in $B * X$ which is disjoint from H. We conclude that H is closed and compact.

Next we choose a minimal closed subset H' of H such that the projection $\pi_x:H' \to X$ is perfect and onto and therefore irreducible. The projection π_x has to be a homeomorphism on H' since X is extremally disconnected and hence the extremal pre-image under continuous irreducible maps in compact Hausdorff. (See 1(i).)

We now define $h:X \to E[Y]$ by

$$h(x) = \pi_{E[Y]} \langle \pi_x^{-1}(x) \cap H' \rangle.$$

Finally we can choose an arbitrary closed subset Z' of Z such that the mapping $g|Z':Z' \to Y$ is irreducible. Iliadis [**7**] showed that in that case $E[Z'] = E[Y]$ up to a homeomorphism and we find the required mapping

h as the composition of h' with

$$\lim_z : E[Z'] \to Z' \quad \text{and} \quad \mathrm{id}_z : Z' \to Z$$

and obtain

$$h = \mathrm{id}_z \circ \lim_z \circ h' : X \to Z \quad \text{and} \quad g \circ h = f.$$

4.3 *Remark*. This proposition has also a converse. If there exists an $h:X \to E[Y]$ such that $g \circ h = f$ then the image $h[X]$ is compact because $E[Y]$ is completely regular and hence h is continuous. We choose B to be $h[X]$.

4.4 PROPOSITION. *A space X is projective in $\mathscr{T}_2$ if and only if X is also projective in $\mathscr{H}_2$.*

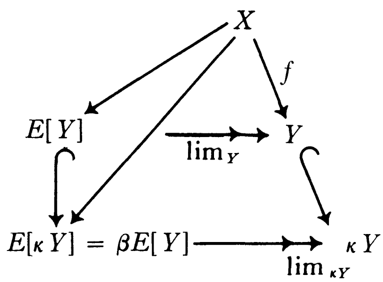
Proof. $\Rightarrow$. If X is projective in $\mathscr{T}_2$ then X is compact Hausdorff and extremally disconnected, and since X is a member of $\mathscr{H}_2$ in that case X has to be projective in $\mathscr{H}_2$.

$\Leftarrow$. If X is projective in $\mathscr{H}_2$ then also in this case X is compact and e.d. Suppose that $f:X \to Y$ is perfect and θ-continuous, then so is $f':X \to \kappa Y$. Iliadis [**7**] proved that $E[\kappa Y] = \beta E[Y]$ and that the inverse image of Y under the map $\lim_{\kappa Y}$ is still $E[Y]$.

Since X is projective in $\mathscr{H}_2$ we obtain a mapping $h:X \to E[\kappa Y]$ such that

$$\lim_{\kappa Y} \circ h = f'$$

and the range of h is in the domain of $\lim_Y$ and therefore $\lim_Y \circ h = f$. Observe that $E[Y]$ is dense in $E[\kappa Y]$, (cf. 1.d). From 3.3 we obtain that X is projective in $\mathscr{T}_2$.



4.5 THEOREM. *The projective objects in the categories $\mathscr{T}_{2+}$, $\mathscr{T}_2$ and $\mathscr{H}_2$ are precisely the finite objects.*

Proof. Assume that X is an infinite projective member of the category $\mathscr{T}_{2+}$. Then X is compact and extremally disconnected. Let $\mathfrak{C}$ be the unit circle which we represent as the reals modulo 1 and let $\mathbf{Z}$ be the set of integers. According to 4.1 there exists a continuous map from the com-

https://doi.org/10.4153/CJM-1981-068-6 Published online by Cambridge University Press

pact extremally disconnected set X onto $\mathfrak{C}$. We define the following topology on $Y = \mathfrak{C} \times \mathbf{Z}$. Every point (c, z) with $z \neq 0$ is open. For every point $(c, 0)$ and every ϵ between 0 and 1 we define a basic open neighborhood

$$B_{\epsilon n}(c) = \{(d, z)|c - \epsilon < d \leqq c, z < -n\}$$
$$\cup \{(d, z)|c + \epsilon > d \geqq c, z > n\} \cup \{(c, 0)\}.$$

In the topology generated in this way the circle $\mathfrak{C}_0 = \mathfrak{C} \times \{0\}$ is a closed and relatively discrete nowhere dense subset. The space Y is obviously semi-regular. The mapping $f: \mathfrak{C} \rightarrow Y$ defined by $f(c) = (c, 0)$ is perfect and θ-continuous, because for every basic neighborhood $B_{\epsilon n}(c)$ of $(c, 0)$ the neighborhood $\langle c - \epsilon, c + \epsilon \rangle$ is mapped into $\mathrm{Cl}_Y \langle B_{\epsilon n}(c) \rangle$. According to 4.1 X can be mapped onto $\mathfrak{C}$ by a perfect θ-continuous map. It follows that X can be mapped onto Y by a perfect θ-continuous map k such that $k(\mathfrak{C}) = \mathfrak{C}_0$, where $\mathfrak{C}_0$ denotes $\mathfrak{C} * \{0\}$. Because Y is Urysohn, $\lim: E[Y] \twoheadrightarrow Y$ is a morphism in $\mathscr{T}_{2+}$. Since X is projective in $\mathscr{T}_{2+}$, there exists a morphism $h: X \twoheadrightarrow E[Y]$ in $\mathscr{T}_{2+}$ such that $\lim_Y \circ h = k$. By 4.3 $h[X]$ is a compact subset B of $E[Y]$, and B is obviously mapped onto $\mathfrak{C}_0$ by $\lim_Y$.

Let Y^+ be the subspace of Y consisting of all (c, z) with $z > 0$, and let Y^- be the subspace of Y consisting of all (c, z) with $z < 0$. Both Y^+ and Y^- are regular open and their union is dense. So they define a partition of $E[Y]$ into two disjoint clopen parts called A^+ (resp. A^-). Let $B^+ = A^+ \cap B$ and $B^- = A^- \cap B$.

If we put Y^* to be the disjoint topological sum of $\mathrm{Cl}_Y \langle Y^+ \rangle$ and $\mathrm{Cl}_Y \langle Y^- \rangle$ then

$$E[Y] = E[Y^*] = A^+ \cup A^-.$$

Let Y_s^+ (resp. Y_s^-) be the semi-regularization of $\mathrm{Cl}_Y \langle Y^+ \rangle$ (resp. $\mathrm{Cl}_Y \langle Y^- \rangle$). Then there exists a copy $\mathfrak{C}^+$ of $\mathfrak{C}_0$ in Y_s^+ and a copy $\mathfrak{C}^-$ in Y_s^-. Both those copies have the topology of the Sorgenfrey circle. Moreover, the spaces Y_s^+ and Y_s^- are both regular and we obtain that the mapping

$$\lim_{Y^*}: E[Y] \rightarrow Y_s^*$$

is a continuous perfect surjection. Now $\lim_{Y^*}[B^+]$ and $\lim_{Y^*}[B^-]$ have to be compact subsets of the Sorgenfrey line and hence both are countable, because the Sorgenfrey line does not contain uncountable compact subsets. This is a contradiction since the union of those images has to cover $\mathfrak{C}_0$. We obtain that no projective member of $\mathscr{T}_{2+}$ is infinite.

The previous part shows that there is no perfect θ-continuous mapping $h: X \rightarrow E[Y]$ with $\lim_Y \circ h = f$ for any compact extremally disconnected space X. Since all objects and morphisms used here are also in the category $\mathcal{T}_2$ we obtain that no projective member of $\mathcal{T}_2$ is infinite, and with the previous proposition we also find that the projective members of $\mathcal{H}_2$ are finite.

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