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CERTAIN INTEGRAL EQUALITIES WHICH IMPLY EQUIMEASURABILITY OF FUNCTIONS

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1. Introduction. 1.1. Two complex measurable functions f and g on complex measure spaces (X, η) and (Y, ν) are *equimeasurable*, abbreviated $f \sim g$, if

$$\eta(f^{-1}(E)) = \nu(g^{-1}(E))$$

for every Borel set $E \subseteq \mathbf{C}$. If Φ is a continuous complex function on $\mathbf{C}$, then we make the following standing hypothesis (H1) which relates Φ, f, and g:

(H1) *For all* $\alpha, \beta \in \mathbf{C}$, *we have*
  a) $\Phi(\alpha + \beta f) \in L^1(X, \eta)$,
  b) $\Phi(\alpha + \beta g) \in L^1(Y, \nu)$, *and*
  c) $\int_X \Phi(\alpha + \beta f) d\eta = \int_Y \Phi(\alpha + \beta g) d\nu$.

In [**5**], Walter Rudin proved that if

$$\Phi(z) = |z|^p, \quad p > 0, p \neq 2, 4, 6, \ldots,$$

then (H1) implies $f \sim g$. With this as motivation, we address the question: *For which* Φ *does* (H1) *imply* $f \sim g$?

It turns out that we can answer the question completely in the case that f and g are bounded:

THEOREM I. *Assume f and g are bounded functions. Then* (H1) *implies $f \sim g$ if and only if Φ is not polyharmonic.*

A function Φ is *polyharmonic* if it has continuous partial derivatives of all orders and is annihilated by some power of the Laplacian $\Delta = (\partial/\partial x)^2 + (\partial/\partial y)^2$.

The problem is more sensitive when f and g are not assumed bounded. In particular, the conclusion of Theorem I fails. We are able, however, to prove that (H1) implies $f \sim g$ under a variety of additional hypotheses on Φ.

THEOREM II. *If $\Phi \not\equiv 0$ and Φ vanishes at infinity, then* (H1) *implies $f \sim g$.*

This is a direct consequence of a lemma contained in [**5**] and will be used in the proof of most of our other results. In the following theorem, Φ is a *radial*

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and *real-valued* function in $C(\mathbf{C})$, and ϕ is the associated function on $[0, \infty)$ satisfying

$$\Phi(z) = \phi(|z|).$$

THEOREM III. *Any one of the following hypotheses on ϕ ensures that* (H1) *implies $f \sim g$:*

(i) *$\phi(t)$ is not a polynomial in t^2, $\phi(t)$ is non-decreasing and k-times differentiable for large t, and $\phi^{(k)}(t)$ tends to a finite limit as $t \to \infty$.*

(ii) *$\phi(t) \geqq 0$ and $\phi(t)$ is concave for large t.*

(iii) *$\phi(t)$ is increasing and $\phi(\sqrt{t})$ is strictly concave for large t.*

(iv) *$\phi(t)$ is convex for large t and the left-hand derivative of ϕ is bounded as $t \to \infty$.*

(v) *$\phi(t) = e^t$.*

Taking $k \geqq p$, we see that case (i) includes the aforementioned result of Rudin. Functions of type (iii) are of importance in [**4**].

We devote Section 2 to a reformulation of the equimeasurability problem in terms of measures on $\mathbf{C}$. We prove Theorem I in Section 3 and show that it fails in the unbounded case with a counterexample in Section 4. In Section 5 we prove Theorems II and III.

Returning to the equimeasurability setting, we prove in Section 6 that certain linear mappings between function algebras are in fact homomorphisms, extending some results of Forelli, Schneider, and Rudin. We conclude the paper with a discussion of generalizations to other dimensions.

In its reformulated version, given in Section 2, our problem is reminiscent of the *Pompeiu problem* and related questions. The reader is referred to Zalcman [**9**] for a very interesting survey of results and a reference list.

I wish to thank Walter Rudin for his advice and encouragement in this work. The material presented here forms a portion of my thesis.

2. Reformulation.

2.1. *Notation.* T will denote the unit circle in $\mathbf{C}$ with normalized one-dimensional Lebesgue measure σ. $C(\mathbf{C})$ is the space of continuous complex functions on $\mathbf{C}$, $\mathbf{C}_0(C)$ is the space of those which vanish at infinity, and $C^\infty(\mathbf{C})$ is the space of those having continuous partial derivatives of all orders.

A function $\Phi \in C(\mathbf{C})$ is *radial* if $\Phi(z_1) = \Phi(z_2)$ whenever $|z_1| = |z_2|$. In this case, Φ has associated with it a continuous function ϕ on $[0, \infty)$ defined by $\Phi(z) = \phi(|z|)$. We say that Φ (or ϕ) is an *even polynomial* of degree $2p$ if $\phi(t)$ is a polynomial of degree p in t^2.

For $\Phi \in C(\mathbf{C})$ and $\alpha, \beta \in \mathbf{C}$, define $\Phi_{\alpha,\beta} \in C(\mathbf{C})$ by

$$\Phi_{\alpha,\beta}(z) = \Phi(\alpha + \beta z), \quad \text{for all } z \in \mathbf{C},$$

and write $[\Phi]$ for the linear span of $\{\Phi_{\alpha,\beta} : \alpha, \beta \in \mathbf{C}\}$. Let $M(\mathbf{C})$ denote the space of complex Borel measures on $\mathbf{C}$. If $\Psi \in C(\mathbf{C})$, $\mu \in M(\mathbf{C})$, and

$$(1) \qquad \int_{\mathbf{C}} \Psi(z) d\mu(z) = 0,$$

we say μ *annihilates* Ψ, written $\mu \perp \Psi$. Note that implicit in (1) is the integrability condition

$$\int_{\mathbf{C}} |\Psi(z)| d|\mu|(z) < \infty.$$

If $\Phi \in C(\mathbf{C})$ and $\mu \perp \Psi$ for all $\Psi \in [\Phi]$, we write $\mu \perp [\Phi]$.

2.2. *Induced measures.* Given our functions f and g as before, let $\mu \in M(\mathbf{C})$ be the difference of the measures induced on $\mathbf{C}$ by f and g. That is, for each Borel set $E \subseteq \mathbf{C}$,

$$\mu(E) = \eta(f^{-1}(E)) - \nu(g^{-1}(E)).$$

Clearly, $\mu \equiv 0$ if and only if $f \sim g$. Also, (H1) implies $\mu \perp [\Phi]$, so our original problem is solved if we answer the question: *For which $\Phi \in C(\mathbf{C})$ does $\mu \perp [\Phi]$ imply $\mu \equiv 0$?*

In this form, the only restriction on μ, a priori, is that $\mu \in M(\mathbf{C})$. However, much of our later work will require that we restrict μ to be in some subspace of $M(\mathbf{C})$, so we make one further definition:

2.3. *Definition.* Let $\tilde{M}$ be a subspace of $M(\mathbf{C})$. We say $\Phi \in C(\mathbf{C})$ *separates* $\tilde{M}$ if $\mu \in \tilde{M}$ with $\mu \perp [\Phi]$ implies $\mu \equiv 0$.

Thus, in its most general form, the question we want to answer is this: *Given a subspace $\tilde{M} \subseteq M(\mathbf{C})$, which functions $\Phi \in C(\mathbf{C})$ separate $\tilde{M}$?* We will be considering only two types of subspaces in addition to $M(\mathbf{C})$:

a) $MB(\mathbf{C}) = \{\mu \in M(\mathbf{C}): \mu \text{ has bounded support}\}$. This type of measure arises when f and g are bounded functions.

b) $M^p(\mathbf{C}) = \{\mu \in M(\mathbf{C}): \int_{\mathbf{C}} |z|^p \, d|\mu|(z) < \infty\}$, $0 < p < \infty$. This type of measure arises when $f \in L^p(X, \eta)$ and $g \in L^p(Y, \nu)$.

3. Measures of bounded support.

3.1. THEOREM. *$\Phi \in C(\mathbf{C})$ separates $MB(\mathbf{C})$ if and only if Φ is not polyharmonic.*

Proof. If we endow $C(\mathbf{C})$ with the compact-open topology (the topology of uniform convergence on compact sets), then $MB(\mathbf{C})$ is its dual space. For Φ to separate $MB(\mathbf{C})$, it is necessary and sufficient that $[\Phi]$ be dense in $C(\mathbf{C})$. This theorem is therefore an immediate consequence of the following theorem of Schwartz [**7**]:

3.2. THEOREM (Schwartz). *Suppose $\Phi \in C(\mathbf{C})$. Then $[\Phi]$ is dense in $C(\mathbf{C})$ in the compact-open topology if and only if Φ is not polyharmonic.*

This is actually the 2-dimensional case of a more general result which we quote in Section 7. From Schwartz's result we can give a characterization of polyharmonic functions which will be of use in the sequel.

3.3. *Definitions.* Recall that $\Phi \in C(\mathbf{C})$ is polyharmonic if it is annihilated by some power of the Laplacian. More precisely, Φ is *polyharmonic of order m* if $\Delta^m \Phi \equiv 0$.

Let $\tau \in MB(\mathbf{C})$ be defined as $(\sigma - \delta)$, where σ is normalized Lebesque measure on T and δ is the Dirac measure. For $\lambda > 0$, let τ_λ be its dilation by λ, that is,

$$\tau_\lambda(E) = \tau\{z/\lambda : z \in E\}$$

for all Borel sets $E \subseteq \mathbf{C}$. If $\Psi \in C(\mathbf{C})$, then the convolution $\Psi * \tau_\lambda$ is in $C(\mathbf{C})$ and may be represented as

$$(\Psi * \tau_\lambda)(z) = \int_T [\Psi(z + \lambda w) - \Psi(z)]d\sigma(w), \quad \text{for all } z \in \mathbf{C}.$$

For $z_0 \in \mathbf{C}$, define the radial function $R_{z0}\Psi \in C(\mathbf{C})$ by

$$(R_{z_0}\Psi)(z) = \int_T \Psi(z_0 + |z|w)d\sigma(w), \quad \text{for all } z \in \mathbf{C}.$$

If we fix $\Psi \in C(\mathbf{C})$, $z_0 \in \mathbf{C}$, and $\lambda > 0$, then easy calculations show:

1) $R_{z_0}(\Psi * \tau_\lambda) = ((R_{z_0}\Psi) * \tau_\lambda)$.

2) Assume Ψ is radial and an even polynomial. Then Ψ has degree $2m > 0$ if and only if $\Psi * \tau_\lambda$ is an even polynomial of degree $2m - 2$. Ψ is constant if and only if $\Psi * \tau_\lambda \equiv 0$.

The first property just involves a change in order of integration. For the second, it is enough to consider even mononomials $\Psi(z) = |z|^{2k}$. Writing

$$(\Psi * \tau_\lambda)(z) = \int_T (|z + \lambda w|^{2k} - |z|^{2k})d\sigma(w),$$

the result follows by expanding the integrand.

3.4. THEOREM. *$\Phi \in C(\mathbf{C})$ is polyharmonic of order m if and only if*

$$(2) \qquad \Phi * \underbrace{\tau_\lambda * \tau_\lambda * \ldots * \tau_\lambda}_{m \text{ times}} \equiv 0,$$

for all $\lambda > 0$.

Proof. For $\lambda > 0$, define μ_λ by

$$\mu_\lambda = \underbrace{\tau_\lambda * \tau_\lambda * \ldots * \tau_\lambda}_{m \text{ times}}.$$

Assume Φ is polyharmonic of order m, that is, $\Phi \in C^\infty(\mathbf{C})$ and $\Delta^m\Phi \equiv 0$.

Fix $z_0 \in C(\mathbf{C})$. Then $R_{z_0}\Phi \in C^\infty(\mathbf{C})$, and since the Laplacian is invariant under translation and rotation (i.e. multiplication by an element of T),

$$\Delta^m(R_{z_0}\Phi) \equiv R_{z_0}(\Delta^m\Phi) \equiv 0.$$

Since $R_{z_0}\Phi$ is radial, let $h(t)$ be the associated (smooth) function on $[0, \infty)$. The partial differential equation $\Delta^m(R_{z_0}\Phi) \equiv 0$ leads as follows to an ordinary differential equation for $h(t)$: Letting x and y be the real and imaginary parts of z, we have

$$t^2 = x^2 + y^2.$$

If $\Psi(z)$ is any radial function in $C^\infty(\mathbf{C})$ with $\psi(t)$ the associated function on $[0, \infty)$, then $\Delta\Psi$ is also radial and for $z \neq 0$ is given by

$$(\Delta\Psi)(z) = \frac{d^2\psi}{dt^2}(t) + \frac{1}{t}\frac{d\psi}{dt}(t).$$

Applying this operation m times to $R_{z_0}\Phi$, we arrive at an ordinary differential equation of degree $2m$ of the form

$$(3) \qquad \sum_{j=0}^{2m-1} c_j t^{-j} h^{(2m-j)}(t) \equiv 0, \quad t \in (0, \infty),$$

where the c_j are constants, $c_0 = 1$.

In particular, (3) implies that there are $2m$ linearly independent radial solutions to $\Delta^m\Psi \equiv 0$ in $\mathbf{C}\backslash\{0\}$; it is an easy exercise to show that they are

$$\{1, |z|^2, |z|^4, \ldots, |z|^{2m-2}, \log|z|, |z|^2\log|z|, \ldots, |z|^{2m-2}\log|z|\}.$$

$h(|z|)$ must be a linear combination of these, but the fact that $(R_{z_0}\Phi)(z) = h(|z|)$ is smooth at $z = 0$ means that none of the terms involving $\log|z|$ can appear. That is, $R_{z_0}\Phi$ must be an even polynomial of degree $< 2m$. For $\lambda > 0$, Property 2) above implies

$$(R_{z_0}\Phi) * \mu_\lambda \equiv 0,$$

and Property 1) implies

$$R_{z_0}(\Phi * \mu_\lambda) \equiv 0.$$

This holds for all $z_0 \in \mathbf{C}$ and $\lambda > 0$, which clearly implies (2).

Conversely, assume (2) holds for $\Phi \in C(\mathbf{C})$. Consider μ_1 $(= \tau * \ldots * \tau)$. Since $\mu_1 \not\equiv 0$, it suffices by Schwartz's result to prove that

$$(4) \qquad \mu_1 \perp \Phi_{\alpha,\beta}$$

for all $\alpha, \beta \in \mathbf{C}$, hence $\mu \perp [\Phi]$. (4) is clear if $\beta = 0$, so fix $\alpha, \beta \in \mathbf{C}$, $\beta \neq 0$ and let $\lambda = |\beta|^{-1}$. Then

$$\int_{\mathbf{C}} \Phi_{\alpha,\beta}(z) d\mu_1(z)$$

$$= \int_{\mathbf{C}} \cdots \int_{\mathbf{C}} \Phi_{\alpha,\beta}(z_1 + z_2 + \ldots + z_m) d\tau(z_1) \ldots d\tau(z_m)$$

$$= \int_{\mathbf{C}} \cdots \int_{\mathbf{C}} \Phi(\alpha + \beta z_1 + \beta z_2 + \ldots + \beta z_m) d\tau(z_1) \ldots d\tau(z_m)$$

$$= \int_{\mathbf{C}} \cdots \int_{\mathbf{C}} \Phi(\alpha - w_1 - w_2 - \ldots - w_m) d\tau\left(\frac{w_1}{-\beta}\right) \ldots d\tau\left(\frac{w_m}{-\beta}\right)$$

$$= \int_{\mathbf{C}} \cdots \int_{\mathbf{C}} \Phi(\alpha - w_1 - w_2 - \ldots - w_m) d\tau_\lambda(w_1) \ldots d\tau_\lambda(w_m)$$

$$= \int_{\mathbf{C}} \Phi(\alpha - z) d\mu_\lambda(z) = (\Phi * \mu_\lambda)(\alpha) = 0.$$

This gives (4) and completes the proof.

3.5. *Remarks.* Polyharmonic functions of order 1 are just harmonic functions, so this theorem reduces to the usual mean value property when $m = 1$. Considerable work has been done on generalizations of the mean value property; interested readers will find references in Zalcman [**9**].

4. A counterexample.

4.1. PROPOSITION. *There are functions $\Phi \in C(\mathbf{C})$ which do not separate $M(\mathbf{C})$, yet are not polyharmonic.*

Proof. Let μ be the measure defined on the positive real axis (x-axis) in $\mathbf{C}$ by

$$\frac{d\mu}{dx} = e^{-x^{1/4}} (\sin x^{1/4}).$$

Then for $p = 0, 1, 2, \ldots,$

$$\int_0^\infty x^p e^{-x^{1/4}} |\sin x^{1/4}| dx < 4\Gamma(4p + 4) < \infty,$$

and using Cauchy's theorem,

$$\int_0^\infty x^p e^{-x^{1/4}} (\sin x^{1/4}) dx = \Gamma(4p + 4) 2^{-2p} \sin (p + 1)\pi.$$

Therefore, μ annihilates all polynomials on $\mathbf{C}$ in the real variable x. Since μ is concentrated on the real axis, we see

$$\mu \perp P$$

whenever P is a polynomial in the two real variables x and y. Clearly $\mu \not\equiv 0$

and $\mu \in M(\mathbf{C})$. For $k = 0, 1, 2, \ldots$, let

$$\gamma_k = \int_{\mathbf{C}} |z|^k d|\mu|(z) < \infty.$$

Define the function Φ on $\mathbf{C}$ by

$$\Phi(z) = \sum_{j=0}^{\infty} a_j |z|^{2j}, \tag{5}$$

where the $a_j > 0$ satisfy the conditions

$$\lim_{j\to\infty} \frac{a_{j+1}}{a_j} = 0, \tag{6}$$

$$\lim_{j\to\infty} \frac{a_{j+1}(\gamma_{2j+2})}{a_j(\gamma_{2j})} = 0. \tag{7}$$

For example, let $a_0 = 1$, and for each $j \geqq 0$ inductively define

$$a_{j+1} = \left(\frac{1}{2}\right)^j \min \left\{a_j, \frac{a_j(\gamma_{2j})}{(\gamma_{2j+2})}\right\}.$$

Condition (6) ensures in the first instance that (5) converges for all $z \in \mathbf{C}$, so $\Phi \in C(\mathbf{C})$. Φ is not polyharmonic since it is radial, and by the proof of Theorem 3.4, the only radial polyharmonic functions are even polynomials. It remains only to show that

$$\mu \perp \Phi_{\alpha,\beta} \tag{8}$$

for all $\alpha, \beta \in \mathbf{C}$. This is clear if $\beta = 0$; and if $\beta \neq 0$ then the fact that Φ is radial implies

$$\Phi_{\alpha,\beta} \equiv \Phi_{\gamma,|\beta|}, \quad \gamma = \frac{|\beta|\alpha}{\beta}.$$

Therefore, it will suffice to prove (8) for $\alpha \in \mathbf{C}$, $\beta = \lambda > 0$.

First we prove that $\Phi_{\alpha,\lambda}$ is μ-integrable: Because of the positivity of (5), for $\alpha = 0$, $\lambda > 0$ we can integrate termwise to get

$$\int_{\mathbf{C}} \Phi(\lambda z) d|\mu|(z) = \sum_{j=0}^{\infty} a_j \lambda^{2j} \int_{\mathbf{C}} |z|^{2j} d|\mu|(z) = \sum_{j=0}^{\infty} a_j \lambda^{2j} \gamma_{2j}.$$

By condition (7), the latter is finite for all $\lambda > 0$. When $\alpha \neq 0$, note that for $|z| \geqq |\alpha|/\lambda$, $|\alpha + \lambda z| \leqq 2\lambda|z|$, hence

$$\Phi(\alpha + \lambda z) \leqq \Phi(2\lambda z).$$

Thus

$$\int_{\mathbf{C}} \Phi(\alpha + \lambda z) d|\mu|(z) \leqq \max_{|z|<|\alpha|/\lambda} \{\Phi(\alpha + \lambda z)|\mu|(\mathbf{C})\} + \int_{\mathbf{C}} \Phi(2\lambda z) d|\mu|(z) < \infty.$$

Now, for $j = 0, 1, 2, \ldots$, the term $|\alpha + \lambda z|^{2j}$ is a polynomial (in two real variables); so our construction of μ implies

$$\int_{\mathbf{C}} |\alpha + \lambda z|^{2j} d\mu(z) = 0.$$

By the dominated convergence theorem,

$$\int_{\mathbf{C}} \Phi(\alpha + \lambda z) d\mu(z) = \int_{\mathbf{C}} \lim_{N\to\infty} \sum_{j=0}^{N} a_j |\alpha + \lambda z|^{2j} d\mu(z)$$

$$= \lim_{N\to\infty} \sum_{j=0}^{N} a_j \int_{\mathbf{C}} |\alpha + \lambda z|^{2j} d\mu(z) = 0.$$

This proves (8) and completes the counterexample.

4.2. *Remark.* It is worth noting that the Φ we constructed, though not polyharmonic, is extremely well behaved. It is positive, radial, C^∞, and as a function of radius is increasing and convex. In fact, Φ is real analytic, hence is the uniform limit on compact sets of polyharmonic functions.

5. Measures with unbounded supports. 5.1. Our first result, the reformulation of Theorem II, forms the basis for our later work. It is any easy consequence of the case $n = 2$ of the following lemma due to Rudin [**5**].

5.2. Lemma (Rudin). *Assume $v \in C_0(\mathbf{R}^n)$ is radial, $v \not\equiv 0$. Let V be the smallest (supremum norm) closed translation-invariant subspace of $C_0(\mathbf{R}^n)$ which contains the dilations*

$$v_r(x) = v(rx), \quad r > 0.$$

Then $V = C_0(\mathbf{R}^n)$.

5.3. Theorem. *If $\Phi \in C_0(\mathbf{C})$, $\Phi \not\equiv 0$, then Φ separates $M(\mathbf{C})$.*

Proof. Suppose $\mu \in M(\mathbf{C})$, $\mu \perp [\Phi]$. Consider $z_0 \in \mathbf{C}$ and $R_{z_0}\Phi$ as defined in Section 3.3,

$$(R_{z_0}\Phi)(z) = \int_T \Phi(z_0 + |z|w) d\sigma(w), \quad \text{for all } z \in \mathbf{C}.$$

For $\alpha, \beta \in \mathbf{C}$,

$$\int_{\mathbf{C}} (R_{z_0}\Phi)(\alpha + \beta z) d\mu(z) = \int_{\mathbf{C}} d\mu(z) \int_T \Phi(z_0 + |\alpha + \beta z|w) d\sigma(w).$$

Using the invariance of σ under multiplication by an element of T, this is

$$= \int_{\mathbf{C}} d\mu(z) \int_T \Phi(z_0 + (\alpha + \beta z)w) d\sigma(w)$$

$$= \int_T d\sigma(w) \int_{\mathbf{C}} \Phi(z_0 + \alpha w + (\beta w)z) d\mu(z).$$

Now $\mu \perp [\Phi]$ implies each inner integral vanishes, so the whole integral vanishes. This is true for each $\alpha, \beta \in \mathbf{C}$, so $\mu \perp [R_{z_0}\Phi]$ for any $z_0 \in \mathbf{C}$.

Since $\Phi \not\equiv 0$, we can fix some z_0 so that $R_{z_0} \not\equiv 0$. Clearly, $R_{z_0}\Phi \in C_0(\mathbf{C}) = C_0(\mathbf{R}^2)$. By Rudin's lemma, with $n = 2$ and $v = R_{z_0}\Phi$, we see that μ annihilates all of $C_0(\mathbf{R}^2)$. As is well known, this implies $\mu \equiv 0$,,so Φ separates $M(\mathbf{C})$.

Throughout the remainder of this section, Φ will denote a *radial* and *real-valued* function in $C(\mathbf{C})$ and ϕ will be the associated function on $[0, \infty)$. Recall from the proof of Theorem 3.4, a radial ϕ is polyharmonic if and only if it is an even polynomial.

5.4. THEOREM. *Let $\phi(t)$ be increasing and k-times differentiable for large t, ϕ not an even polynomial. Assume $\phi^{(k)}(t)$ tends to a finite limit as $t \to \infty$. Then ϕ separates $M(\mathbf{C})$.*

Proof. Suppose $\mu \in M(\mathbf{C})$, $\mu \perp [\Phi]$. For $\lambda > 0$, define

$$\Psi_\lambda = \Phi * \underbrace{\tau_\lambda * \ldots * \tau_\lambda}_{k+1 \text{ times}}$$

where τ_λ is as defined in Section 3.3. It suffices to prove that for some $\lambda > 0$ we have 1) $\Psi_\lambda \not\equiv 0$, 2) $\mu \perp [\Psi_\lambda]$, and 3) $\Psi_\lambda \in C_0(\mathbf{C})$. For then, by Theorem 5.3. $\mu \equiv 0$, so Φ separates $M(\mathbf{C})$.

1) Since Φ is not polyharmonic, Theorem 3.4 implies that there is a $\lambda > 0$ with $\Psi_\lambda \not\equiv 0$. For convenience and without loss of generality, assume $\Psi = \Psi_1 \not\equiv 0$.

2) To prove $\mu \perp [\Psi]$, we use the fact that $\phi(t)$ is increasing for large t to arrive at a more general result.

CLAIM. *Let $\nu \in MB(\mathbf{C})$. If $\mu \perp [\Phi]$, then $\mu \perp [\Phi * \nu]$.*

We must show for each $\alpha, \beta \in \mathbf{C}$ that

(9) $\quad \mu \perp (\Phi * \nu)_{\alpha,\beta}.$

This is clear if $\beta = 0$, so fix $\alpha, \beta \in \mathbf{C}$, $\beta \neq 0$. Suppose ν is supported on

$$B_K = \{z \in \mathbf{C} : |z| \leqq K\}.$$

Then

$$(\Phi * \nu)_{\alpha,\beta}(z) = \int_{\mathbf{C}} \Phi(\alpha + \beta z - \xi)d\nu(\xi)$$
$$= \int_{B_K} \Phi(\alpha + \beta z - \xi)d\nu(\xi) = \int_{B_K} \phi(|\alpha + \beta z - \xi|)d\nu(\xi).$$

For $|z|$ large, $|\alpha + \beta z - \xi| \leqq |2\alpha + 2\beta z|$ for all $\xi \in B_K$. Also, since $\phi(t)$ is increasing for large t, we can choose $|z|$ even larger, if necessary, say $|z| \geqq M$, so that

$$\Phi(\alpha + \beta z - \xi) = \phi(|\alpha + \beta z - \xi|) \leqq \phi(|2\alpha + 2\beta z|) = \Phi(2\alpha + 2\beta z), \quad \text{for all } \xi \in B_K.$$

Now $\mu \perp [\Phi]$ implies in particular that

$$\int_{\mathbf{C}} |\Phi(2\alpha + 2\beta z)| d|\mu|(z) = C < \infty.$$

Therefore,

$$\begin{aligned}
&\int_{B_K} d|\nu|(\xi) \int_{\mathbf{C}} |\Phi(\alpha + \beta z - \xi)| d|\mu|(z) \\
&\leqq \int_{B_K} d|\nu|(\xi) \left\{ \max_{\substack{|\xi| \leqq K \\ |z| \leqq M}} |\Phi(\alpha + \beta z - \xi)| |\mu|(\mathbf{C}) \right. \\
&\qquad\qquad \left. + \int_{|z| \geqq M} |\Phi(\alpha + \beta z - \xi)| d|\mu|(z) \right\} \\
&\leqq \int_{B_K} \left\{ \max_{\substack{|\xi| \leqq K \\ |z| \leqq M}} |\Phi(\alpha + \beta z - \xi)| |\mu|(\mathbf{C}) + C \right\} d|\nu|(\xi) < \infty.
\end{aligned}$$

This justifies the change in order of integration which gives

$$\begin{aligned}
\int_{\mathbf{C}} (\Phi * \nu)_{\alpha,\beta}(z) d\mu(z) &= \int_{\mathbf{C}} d\mu(z) \int_{\mathbf{C}} \Phi(\alpha + \beta z - \xi) d\nu(\xi) \\
&= \int_{\mathbf{C}} d\nu(\xi) \int_{\mathbf{C}} \Phi(\alpha + \beta z - \xi) d\mu(z).
\end{aligned} \tag{10}$$

This proves (9) since μ annihilates $\Phi(\alpha + \beta z - \xi)$ for each fixed ξ, and our claim is proven.

In our particular instance, take

$$\nu = \underbrace{\tau * \tau * \ldots * \tau}_{k + 1 \text{ times}}$$

so that $\Psi = \Phi * \nu$. Then $\mu \perp [\Psi]$.

3) Suppose γ is the function on $[0, \infty)$ associated with a radial function $\Gamma \in C(\mathbf{C})$. Let $D^j\gamma$ be the function on $[0, \infty)$ associated with the radial function

$$\underbrace{\Gamma * \tau * \ldots * \tau}_{j \text{ times}}.$$

Then $D(D^j\gamma) = D^{j+1}\gamma$.

Given ϕ hypothesized, our aim is to prove $(D^{k+1}\phi)(t) \to 0$ as $t \to \infty$. We use induction on k:

For $k = 0$, the hypothesis is that $\phi(t) \to c$ as $t \to \infty$, $|c| < \infty$. By the dominated convergence theorem,

$$(D\phi)(t) = \int_T [\Phi(t+w) - \Phi(t)]d\sigma(w) = \int_T [\phi(|t+w|) - \phi(t)]d\sigma(w)$$

converges to 0 as $t \to \infty$.

For $k = 1$, we assume $\phi^{(1)}(t) \to c$ as $t \to \infty$, $|c| < \infty$. Then for t large,

$$[\phi(|t+w|) - \phi(t)]$$

converges boundedly to $c \cdot [\cos(\arg w)]$. Again by dominated convergence, $(D\phi)(t)$ converges as $t \to \infty$ to

$$\int_T c[\cos(\arg w)]d\sigma(w) = c\int_T \cos\theta \frac{d\theta}{2\pi} = 0.$$

By the previous case, replacing $\phi(t)$ by $(D\phi)(t)$, we see

$$(D(D\phi))(t) = (D^2\phi)(t) \to 0 \quad \text{as } t \to \infty.$$

Assume that $k \geqq 2$ and that our result has been proven for $k - 1$. If $\phi^{(k)}(t) \to c$ as $t \to \infty$, $|c| < \infty$, then by induction it is enough to prove that

$$(D\phi)^{(k-1)}(t) \to 0 \quad \text{as } t \to \infty.$$

First we need some preliminary estimates. For $w \in T$, $t \in [0, \infty)$, define

$$h_w(t) = |t + w|.$$

For large t, easy computations show the following estimates, all of which are independent of $w \in T$:

(i) $h_w^{(1)}(t) = O(1)$ as $t \to \infty$,

(ii) $[h_w(t)]^{n+1}h_w^{(n)}(t) = O(1)$ as $t \to \infty$, $n \geqq 2$, and

(iii) $h_w(t)[(h_w^{(1)})^{k-1} - 1] = o(1)$ as $t \to \infty$.

Now, write

$$(D\phi)(t) = \int_T [\phi(|t+w|) - \phi(t)]d\sigma(w) = \int_T [\phi \circ h_w(t) - \phi(t)]d\sigma(w).$$

For large t we can differentiate $(k-1)$ times under the integral sign to get (suppressing the variable t)

$$\begin{aligned}(D\phi)^{(k-1)} &= \int_T \left[\frac{d^{k-1}}{dt^{k-1}}(\phi \circ h_w) - \phi^{(k-1)}\right]d\sigma(w)\\ &= \int_T [(\phi^{(k-1)} \circ h_w)(h_w^{(1)})^{k-1} - \phi^{(k-1)}]d\sigma(w) \qquad (11)\\ &\quad + \sum_{j=1}^{k-2} \sum_{\substack{l_1+\ldots+l_j=k-1\\ l_i \geqq 1}} b_{l_1,\ldots,l_j} \int_T (\phi^{(j)} \circ h_w)\\ &\qquad\qquad \times (h_w^{(l_1)})(h_w^{(l_2)}) \ldots (h_w^{(l_j)})d\sigma(w).\end{aligned}$$

We will prove that each of the integrals on the right vanishes as $t \to \infty$.

Rewrite the first integral in (11) as

$$\int_T \left(\frac{\phi^{(k-1)} \circ h_w}{h_w}\right)(h_w)[(h_w{}^{(1)})^{k-1} - 1]d\sigma(w)$$
$$+ \int_T [(\phi^{(k-1)} \circ h_w) - \phi^{(k-1)}]d\sigma(w).$$

The first of these vanishes by dominated convergence using (iii) above and the fact that

$$(\phi^{(k-1)} \circ h_w)(t) = O(h_w(t)) \quad \text{as } t \to \infty,$$

independent of w. The second integral vanishes by the argument used earlier in the $k = 1$ case.

Consider one of the latter integrals in (11), say

$$(12) \quad \int_T (\phi^{(j)} \circ h_w)(h_w{}^{(l_1)})(h_w{}^{(l_2)}) \ldots (h_w{}^{(l_j)})d\sigma(w).$$

Here $1 \leqq j \leqq k - 2$, $l_1 + l_2 + \ldots l_j = k - 1$, and $l_1, l_2, \ldots, l_j \geqq 1$. Not all of the l_i can be 1, so assume

$$l_{j_0}, l_{j_0+1}, \ldots, l_j \geqq 2, \quad 1 \leqq j_0 \leqq j,$$

and let

$$m = (l_{j_0} + 1) + (l_{j_0+1} + 1) + \ldots + (l_j + 1).$$

We may rewrite (12) as

$$(13) \quad \int_T \frac{(\phi^{(j)} \circ h_w)}{(h_w)^m} [(h_w{}^{(1)})^{j_0-1}][(h_w)^{l_{j_0}+1} h_w{}^{(l_{j_0})}] \ldots [(h_w)^{l_j+1} h_w{}^{(l_j)}]d\sigma(w).$$

By (i) and (ii) above, each term in square brackets is bounded as $t \to \infty$, independent of w. Now $m \geqq k - j + 1$ and $\phi^{(k)}(t) \to c$ as $t \to \infty$, so

$$\frac{\phi^{(j)}(t)}{t^m} \to 0 \quad \text{as } t \to \infty.$$

Since $h_w(t) \geqq (t - 1)$ for large t and all w, the first term in (13) goes boundedly to 0 as $t \to \infty$. By the dominated convergence theorem, (13) vanishes as $t \to \infty$.

This completes the proof of the theorem.

5.5. *Further results.* The previous theorem applies to many of the functions which are of importance in function theory, e.g., $\phi(t) = t^p$, $\phi(t) = \log^+(t)$, $\phi(t) = \log(1 + t)$. There are many other classes of continuous functions ϕ for which the same proof would apply. The two key elements of the proof are that $\Phi * \tau * \ldots * \tau \in C_0(\mathbf{C})$ and that we can justify the change in order of integration used in formula (10). We list without proof some other classes of functions for which these hold and which therefore separate $M(\mathbf{C})$.

a) $\phi(t)$ *is positive and concave for large* t.

b) $\phi(t)$ *is increasing and* $\phi(\sqrt{t})$ *is strictly concave for large* t.

c) $\phi(t)$ *is convex for large* t *and the left-hand derivative of* ϕ *is bounded as* $t \to \infty$.

Even when Φ does not satisfy any of the hypotheses we have listed, it may be possible to modify Φ by convolution to show that it separates $M(\mathbf{C})$. For example, suppose $\phi(t)$ is increasing but not differentiable. Let Ψ be a radial $C^\infty(\mathbf{C})$ function supported in $\{z \in \mathbf{C} : |z| \leqq 1\}$. By the claim in the proof of Theorem 5.4, $\mu \perp [\Phi]$ implies $\mu \perp [\Phi * \Psi]$. $\Phi * \Psi$ is now $C^\infty(\mathbf{C})$. If it satisfies one of our hypotheses, then $\mu \equiv 0$, hence Φ separates $M(\mathbf{C})$.

5.6. *Remark*. Another way to justify the change in order of integration needed in the proof of Theorem 5.4 would be to restrict the class of measures μ under consideration. For example, we can eliminate the hypothesis that $\phi(t)$ be increasing if we weaken the conclusion to read: ϕ *separates* $M^k(\mathbf{C})$. Another example where we must restrict the class of measures is the following:

5.7. THEOREM. *Suppose* $0 < p < \infty$, $p \neq 2, 4, 6, \ldots$, *and assume* $|\phi(t)|/t^p$ *is bounded for all* t. *If* $\phi(t)/t^p$ *tends to a finite limit as* $t \downarrow 0$ *or if* $\phi(t)/t^p$ *tends to a finite limit as* $t \to \infty$, *then* ϕ *separates* $M^p(\mathbf{C})$.

Proof. Assume $\mu \in M^p(\mathbf{C})$, $\mu \perp [\Phi]$. Fix $\alpha, \beta \in \mathbf{C}$. For $\lambda > 0$,

$$\int_{\mathbf{C}} \frac{\phi(|\lambda\alpha + \lambda\beta z|)}{|\lambda\alpha + \lambda\beta z|^p} |\alpha + \beta z|^p d\mu(z) = 0.$$

Letting $\lambda \downarrow 0$ (or $\lambda \to \infty$ as appropriate) and applying the dominated convergence theorem, we have

$$\int_{\mathbf{C}} |\alpha + \beta z|^p d\mu(z) = 0.$$

Since this holds for all $\alpha, \beta \in \mathbf{C}$ and since the function

$$z \to |z|^p$$

separates $M(\mathbf{C})$ by Theorem 5.4, we conclude that $\mu \equiv 0$. Thus ϕ separates $M^p(\mathbf{C})$.

5.8. COROLLARY. *If* $|\phi(t)|/t$ *is bounded for all* t *and* $\phi(t)$ *has a non-vanishing right-hand derivative at* 0, *then* ϕ *separates* $M^1(\mathbf{C})$.

In all of the above results, $\phi(t)$ has slow, i.e., polynomial, growth while in in the counterexample of Section 4, our ϕ grows more quickly than any polynomial. That difference in growth is not alone the determining factor is shown by the following result. The proof is motivated by the methods of Andersen in [**1**].

5.9. THEOREM. *The function* $\phi(t) = e^t$ *separates* $M(\mathbf{C})$.

Proof. Suppose $\mu \in M(\mathbf{C})$, $\mu \perp [\Phi]$. Then

$$\int_{\mathbf{C}} |\alpha + \beta z| d|\mu|(z) \leqq \int_{\mathbf{C}} \exp \{|\alpha + \beta z|\} d|\mu|(z) < \infty$$

for all $\alpha, \beta \in \mathbf{C}$. By Theorem 5.4, it suffices to prove

$$(14) \quad \int_{\mathbf{C}} |\alpha + \beta z| d\mu(z) = 0, \quad \text{for all } \alpha, \beta \in \mathbf{C}.$$

Fix α and β and let $\nu = \nu(\alpha, \beta)$ be the measure defined on Borel sets $E \subseteq [0, \infty)$ by

$$\nu(E) = \mu\{z \in \mathbf{C} : |\alpha + \beta z| \in E\}.$$

ν is a finite real Borel measure whose total variation measure satisfies

$$|\nu|(E) \leqq |\mu| \{z \in \mathbf{C} : |\alpha + \beta z| \in E\}.$$

For $s = x + iy \in \mathbf{C}$, define the Laplace-Stieltjes transform

$$(15) \quad F(s) = \int_0^\infty e^{-st} d\nu(t).$$

Since ν is finite, this clearly converges for $x \geqq 0$. For $x < 0$ note that by the definition of ν and the fact $\mu \perp [\Phi]$, we have

$$\int_0^\infty e^{-xt} d|\nu|(t) \leqq \int_{\mathbf{C}} e^{|-x\alpha - x\beta z|} d|\mu|(z) < \infty.$$

Therefore, the integral in (15) converges for all $s \in \mathbf{C}$ and $F(s)$ is an entire function. Let $0 < p < \infty$ and let $a = p\alpha$, $b = p\beta$. Then $\mu \perp \Phi_{a,b}$ implies

$$F(-p) = \int_0^\infty e^{pt} d\nu(t) = \int_{\mathbf{C}} e^{|p\alpha + p\beta z|} d\mu(z) = 0.$$

Thus $F \equiv 0$ and by the uniqueness of the Laplace-Stieltjes transform, $\nu \equiv 0$. (See, for example, Widder [**8**, Chapter 5].) In particular,

$$0 = \int_0^\infty t d\nu(t) = \int_{\mathbf{C}} |\alpha + \beta z| d\mu(z).$$

This proves (14) and hence the theorem.

6. Equimeasurability and homomorphisms. 6.1 Interpreting our results in the original setting of functions and equimeasurability, we can extend some work of Forelli [**2**; **3**], Schneider [**6**], and Rudin [**5**] concerning homomorphisms.

Assume (X, η) and (Y, ν) are complex measure spaces. Let B denote the space of complex measurable functions on Y, and let A be an algebra (under pointwise multiplication) of complex measurable functions on X containing the constant function **1**.

6.2. THEOREM. *Suppose* $A \subseteq L^1(X, \eta)$ *and* $\Gamma : A \to B$ *is a linear mapping with* $\Gamma 1 = 1$. *If we have*

$$(16)\quad \int_X \Phi(f(x))d\eta(x) = \int_Y \Phi((\Gamma f)(y))d\nu(y), \quad \text{for all } f \in A,$$

for some function $\Phi \in C(\mathbf{C})$ *which separates* $M(\mathbf{C})$, *then* Γ *is a homomorphism on* A, *that is,*

$$\Gamma(fg) = (\Gamma f)(\Gamma g), \quad \text{for all } f, g \in A.$$

Proof. Fix $f \in A$. The hypotheses imply that

$$(17)\quad \int_X \Phi(\alpha + \beta f(x))d\eta(x) = \int_Y \Phi(\alpha + \beta(\Gamma f)(y))d\nu(y)$$

for all $\alpha, \beta \in \mathbf{C}$. If $\mu = \eta_f - \nu_{\Gamma f}$, where η_f and $\nu_{\Gamma f}$ are the measures induced on $\mathbf{C}$ by f and Γf, then $\mu \in M(\mathbf{C})$ and (17) implies $\mu \perp [\Phi]$. Since Φ separates $M(\mathbf{C})$, $\mu \equiv 0$.

In other words, for every $f \in A, f \sim \Gamma f$ and $(1 + af + bf^2) \sim (1 + a\Gamma f + b\Gamma(f^2))$ for every $a, b \in \mathbf{C}$. Since $A \subseteq L^1(X, \eta)$, we can apply the argument in Section 3.3 of Rudin [0] to complete the proof.

The class of functions Φ satisfying (16) for which the conclusion holds depends on the algebras A and $\Gamma(A) \subseteq B$. For example, if we add the hypothesis that $\Gamma(A) \subseteq L^1(Y, \nu)$, then Φ need only separate $M^1(\mathbf{C})$ for the conclusion to hold. In view of Theorem 3.1, another example is this:

6.3. COROLLARY. *Let* $A \subseteq L^\infty(X, \eta)$ *and let* $\Gamma : A \to L^\infty(Y, \nu)$ *be linear with* $\Gamma 1 = 1$. *Suppose that for some* $\Phi \in C(\mathbf{C})$ *which is not polyharmonic, we have*

$$\int_X \Phi(f(x))d\eta(x) = \int_Y \Phi((\Gamma f)(y))d\nu(y), \quad \text{for all } f \in A.$$

Then Γ *is a homomorphism.*

7. Generalizations. 7.1. Until now we have looked at functions f and g taking values in $\mathbf{C}$ (or equivalently, at measures in $\mathbf{C}$). However, we can prove analogous results for functions with values in any of the $\mathbf{R}^n$. We discuss two approaches to generalization; the first, which occurred in the original work on equimeasurability by Rudin [**5**], applies to $\mathbf{C}^n$-valued functions. The second, and perhaps more natural, approach applies to functions with values in $\mathbf{R}^n$ for any $n \geqq 1$.

Let $F = (f_1, f_2, \ldots, f_n)$ and $G = (g_1, g_2, \ldots, g_n)$ be two measurable $\mathbf{C}^n$-valued functions on (X, η) and (Y, ν). F and G are equimeasurable, $F \sim G$, if

$$\eta(F^{-1}(E)) = \nu(G^{-1}(E))$$

for all Borel sets $E \subseteq \mathbf{C}^n$. Let $\langle \cdot, \cdot \rangle$ denote the usual complex inner product on $\mathbf{C}^n$. For $\Phi \in C(\mathbf{C})$ we make the following standing hypothesis (H2) relating Φ, F, and G:

(H2) *For all* $\alpha \in \mathbf{C}$ *and* $\beta \in \mathbf{C}^n$, *we have*

a) $\Phi(\alpha + \langle\beta, F\rangle) \in L^1(X, \eta)$,

b) $\Phi(\alpha + \langle\beta, G\rangle) \in L^1(Y, \nu)$, *and*

c) $\int_X \Phi(\alpha + \langle\beta, F(x)\rangle)d\eta(x) = \int_Y \Phi(\alpha + \langle\beta, G(y)\rangle)d\nu(y)$.

7.2 THEOREM. *If* Φ *separates* $M(\mathbf{C})$, *then* (H2) *implies* $F \sim G$.

Proof. Let μ be the difference of the measures induced by F and G on $\mathbf{C}^n$. For $w = (w_1, w_2, \ldots, w_n) \in \mathbf{C}^n$ with $|w|^2 = \sum_{i=1}^n |w_i|^2 = 1$, let $\mu_w \in M(\mathbf{C})$ be defined on Borel sets $E \subseteq \mathbf{C}$ by

$$\mu_w(E) = \mu\{z \in \mathbf{C}^n : \langle w, z\rangle \in E\}.$$

(H2) implies that for all $\alpha, \gamma \in \mathbf{C}$, $w \in \mathbf{C}^n$, $|w| = 1$,

$$0 = \int_{\mathbf{C}^n} \Phi(\alpha + \gamma\langle w, z\rangle)d\mu(z) = \int_{\mathbf{C}} \Phi(\alpha + \gamma\xi)d\mu_w(\xi).$$

Since Φ separates $M(\mathbf{C})$, $\mu_w \equiv 0$ whenever $|w| = 1$. It remains only to show that this implies $\mu \equiv 0$ on $\mathbf{C}^n$.

If we identify $\mathbf{C}^n$ with $\mathbf{R}^{2n}$ in the usual way, the real Euclidean inner product is Re $\langle\cdot,\cdot\rangle$. Fix $z \in \mathbf{C}^n$ with $|z| \neq 0$ and let $w = z/|z|$. The Fourier transform of μ at z is

$$\hat{\mu}(z) = \int_{\mathbf{R}^{2n}} e^{-i\mathrm{Re}\langle z,x\rangle}d\mu(x)$$

$$= \int_{\mathbf{R}^{2n}} e^{-i|z|\mathrm{Re}\langle w,x\rangle}d\mu(x) = \int_{\mathbf{C}} e^{-i|z|\mathrm{Re}\xi}d\mu_w(\xi) = 0.$$

Therefore, $\hat{\mu} \equiv 0$, which implies $\mu \equiv 0$, and the proof is complete.

7.3. COROLLARY. *If* $\Phi \in C(\mathbf{C})$ *is not polyharmonic and if* F *and* G *are bounded, then* (H2) *implies* $F \sim G$.

7.4. *Second approach.* We now consider $\mathbf{R}^n$-valued functions f and g, where $n \geqq 1$. Our first need is for an appropriate analogue of the rotated, dilated, and translated functions $\Phi_{\alpha,\beta}$.

Let $SO(n)$ denote the *special orthogonal group* of transformations of $\mathbf{R}^n$, i.e., those orthogonal transformations with determinant $+1$. Let $\Phi \in C(\mathbf{R}^n)$. For each $\alpha \in \mathbf{R}^n$, $b \in [0, \infty)$, and $U \in SO(n)$, define $\Phi_{\alpha,b,U} \in C(\mathbf{R}^n)$ by

$$\Phi_{\alpha,b,U}(x) = \Phi(\alpha + bUx), \quad \text{for all } x \in \mathbf{R}^n,$$

and let $[\Phi]$ denote the linear span of these functions. We have the standing hypothesis (H3) relating Φ, f, and g:

(H3) *For every* $\alpha \in \mathbf{R}^n$, $b \in [0, \infty)$, *and* $U \in SO(n)$, *we have*

a) $(\Phi_{\alpha,b,U} \circ f) \in L^1(X, \eta)$,

b) $(\Phi_{\alpha,b,U} \circ g) \in L^1 Y, \nu)$, *and*

c) $\int_X (\Phi_{\alpha,b,U} \circ f)d\eta = \int_Y (\Phi_{\alpha,b,U} \circ g)d\nu$.

Note that, after identifying $\mathbf{R}^2$ with $\mathbf{C}$, $SO(2)$ is just the group of multiplications by complex numbers of modulus 1, and (H3) is precisely the same as (H1).

A function $\Phi \in C(\mathbf{R}^n)$ is radial if

$$\Phi(x) = \Phi(Ux), \quad \text{for all } x \in \mathbf{R}^n,\ U \in SO(n),$$

in which case it is associated with a function ϕ on $[0, \infty)$.

The Laplacian in $\mathbf{R}^n$ is $\Delta = \sum_{i=1}^n (\partial/\partial x_i)^2$, and $\Phi \in C^\infty(\mathbf{R}^n)$ is polyharmonic of order m if

$$\Delta^m \Phi \equiv 0.$$

7.5. *Results in* $\mathbf{R}^n$. *The analogues of all our previous results are true for* $\mathbf{R}^n$-*valued functions* f *and* g, $n \geqq 1$. The reformulation in Section 2 and the subsequent proofs are essentially unchanged.

The proof of the result when f and g are assumed bounded follows from Schwartz's more general result (see [**7**]):

7.6. Theorem (Schwartz). *Suppose* $\Phi \in C(\mathbf{R}^n)$. *Then* $\Gamma[\Phi]$ *is dense in* $C(\mathbf{R}^n)$ *in the compact-open topology if and only if* Φ *is not polyharmonic.*

The measure τ used in Theorems 3.4 and 5.4 must now be defined as normalized Lebesque measure (denoted σ) on the unit sphere S in $\mathbf{R}^n$ minus the Dirac δ. Convolution with τ can be represented as

$$(\Psi * \tau)(x) = \int_S [\Psi(x + w) - \Psi(x)]d\sigma(w)$$

$$= \int_{SO(n)} [\Psi(x + Uw_0) - \Psi(x)]d\sigma_n(U),$$

where $w_0 \in S$ and σ_n is Haar measure on the group $SO(n)$.

7.7. *Remaining questions.* Is there an analogue of Theorem I when f and g are unbounded? Perhaps the exceptional functions are some appropriate generalization of polyharmonic functions. Are there functions which separate $M^p(\mathbf{C})$ but not $M(\mathbf{C})$? Does $\Phi(z) = e^{|z|^2}$ separate $M(\mathbf{C})$?

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