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ON STRONG MATRIX SUMMABILITY WITH RESPECT TO A MODULUS AND STATISTICAL CONVERGENCE

BY

JEFF CONNOR

ABSTRACT. The definition of strong Cesaro summability with respect to a modulus is extended to a definition of strong A-summability with respect to a modulus when A is a nonnegative regular matrix summability method. It is shown that if a sequence is strongly A-summable with respect to an arbitrary modulus then it is A-statistically convergent and that A-statistical convergence and strong A-summability with respect to a modulus are equivalent on the bounded sequences.

The original definitions of the summability methods discussed in this note both appeared in the context of Cesaro summability. Recall that if x is a sequence of complex numbers we say that

(a) x is strongly (Cesaro) summable to L if $\lim_n n^{-1}\sum_{k=1}^{n}|x_k - L| = 0$, and

(b) x is statistically convergent to L if $\lim_n n^{-1}|(k \leqq n : |x_k - L| \geqq \epsilon)| = 0$ for all $\epsilon > 0$.

Strong summability and statistical convergence were introduced separately and, until recently, followed independent lines of development (cf. [1]). Strong summability first appeared in the paper by Hardy and Littlewood which improved Fejer's theorem on the Cesaro convergence of a Fourier series [9] and the strong summability of Fourier series continues to be an active area of research (i.e. [10]). Statistical convergence was originally introduced in [4]. Jamison and Flemming [5] have recently characterized the linear isometries which map the strongly summable sequences onto themselves: the ideas underlying the techniques used in this note can be used to characterize the linear isometries which map the bounded strongly summable sequences onto themselves [2]. It should also be noted that strong summability appears in ergodic theory as "weakly mixing" [8] and that the summability methods discussed in this note are also related tp the method of "convergence in density" discussed in [7].

The class of sequences which are strongly Cesaro summable with respect to a modulus was introduced by Maddox [12] as an extension of the definition of strongly Cesaro summable. In this note, following Maddox, we further extend his definition by replacing the Cesaro matrix with an arbitrary nonnegative regular matrix summability method A and establish some elementary connections between strong A-summability,

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strong A-summability with respect to a modulus and A-statistical convergence. In particular, by exploiting an observation regarding closed ideals of bounded sequences, we show that all three of these notions are equivalent for bounded sequences of scalars and that A-statistical convergence includes strong A-summability with respect to modulus for any modulus.

Before continuing with the discussion, we pause to establish some notation. Throughout the following we let e denote the sequence which is identically 1 and let

$$s = \{\text{all complex valued sequences}\},$$

$$l_\infty = \{x \in s : \sup_n |x_n| < \infty\}.$$

If $x, y \in s$, we let xy denote the sequence $(x_k y_k)$, $\|x\| = \sup_n |x_n|$ and, given $\epsilon > 0$, $S(x;\epsilon) = \{k \in \mathbf{N} : |x_k| \geqq \epsilon\}$. If $S \subseteq \mathbf{N}$, we let χ_S denote the characteristic function of S.

If $A = (a_{n,k})$ is a nonnegative regular matrix summability method and Σ denotes the summation from $k = 1$ to ∞ (as it does throughout this note), then we let

$$w_0(A) = \{x \in s : \lim_n \Sigma\, a_{n,k}|x_k| = 0\}$$

$$w(A) = \{x \in s : \text{ there is an } L \in \mathbf{C} \text{ such that } x - Le \in w_0(A)\}.$$

The collection $w(A)$ is commonly referred to as the collection of strongly A-summable sequences. If $x - Le \in w_0(A)$, we say that x is strongly A-summable to L. We extend the notion of strong A-summability by using a modulus in the same fashion as Maddox extends strong summability.

Definition 1. *A function* $f : [0,\infty) \longrightarrow [0,\infty)$ *is called a modulus if*
a) $f(x) = 0$ *if and only if* $x = 0$,
b) $f(x+y) \leqq f(x) + f(y)$,
c) f *is increasing and*
d) f *is continuous from the right at* 0.

Definition 2. *Let* f *be a modulus and* A *be a nonnegative regular matrix summability method. We let*

$$w_0(A,f) = \{x \in s : \lim_n \Sigma\, a_{n,k} f(|x_k|) = 0\} \textit{ and}$$

$$w(A,f) = \{x \in s : \textit{ there is an } L \in \mathbf{C} \textit{ such that } x - Le \in w_0(A,f)\}.$$

If $x - Le \in w_0(A,f)$, *we say that* x *is strongly* A*-summable to* L *with respect to the modulus* f.

Using the same technique as Maddox, it is easy to extend Theorem 4 of [12] to the following result.

PROPOSITION 3. *If f is a modulus and x is strongly A-summable to L, then x is strongly summable to L with respect to the modulus f,*

We now record two useful observations regarding ideals in l_∞ and $w_0(A, f) \cap l_\infty$.

LEMMA 4. *Let M be an ideal in l_∞ and let $x \in l_\infty$ Then x is in the closure of M in l_∞ if and only if $\chi_{S(X;\epsilon)} \in M$ for all $\epsilon > 0$.*

PROOF. Suppose that x is in the closure of M and $\epsilon > 0$ has been given. Select $z \in M$ such that $\|x - z\| < \epsilon/2$ and observe that $S(x; \epsilon) \subseteq S(z; \epsilon/2)$. Define $y \in l_\infty$ by $y_k = 1/z_k$ if $|z_k| \geqq \epsilon/2$ and $y_k = 0$ otherwise. Note that $yz = \chi_{S(z;\epsilon/2)} \in M$ and hence, since $S(x; \epsilon) \subseteq S(z; \epsilon/2)$ and $\chi_{S(x;\epsilon)} \in l_\infty, \chi_{S(x;\epsilon)}\chi_{S(z;\epsilon/2)} = \chi_{S(x;\epsilon)} \in M$.

Conversely, note that if $x \in l_\infty$ then $\|x - x\chi_{S(x;\epsilon)}\| < \epsilon$. It follows that if $\chi_{S(x;\epsilon)} \in M$ for all $\epsilon > 0$, then x is in the closure of M. □

We also use the following result, which appears in [7] and [3].

LEMMA 5. *If A is a nonnegative regular matrix summability method, then $w_0(A) \cap l_\infty$ is a closed ideal of l_∞.*

We also note that $w_0(A, f) \cap l_\infty$ is an ideal in l_∞ for any modulus f. This follows from observing that if $x \in w_0(A, f)$, $y \in l_\infty$ and K is an integer such that $\|y\| \leqq K$, then $\Sigma\, a_{n,k} f(|x_k y_k|) \leqq K \Sigma\, a_{n,k} f(|x_k|)$ for any natural number n. Since $\lim_n \Sigma\, a_{n,k} f(|x_k|) = 0$, it follows that $\lim_n \Sigma\, a_{n,k} f(|x_k y_k|) = 0$, i.e. $xy \in w_0(A, f)$. We are now ready to establish:

THEOREM 6. *Let x be a bounded sequence, f be a modulus and A be a nonnegative regular matrix summability method. Then x is strongly A-summable to L with respect to the modulus f if and only if x is strongly A-summable to L, i.e. $w(A, f) \cap l_\infty = w(A) \cap l_\infty$.*

PROOF. First we establish that $w_0(A) \cap l_\infty = w_0(A, f) \cap l_\infty$. Observe that, once this has been established, the theorem follows immediately from the definition of $w(A, f)$.

Note that proposition 3 yields that $w_0(A) \cap l_\infty \subseteq w_0(A, f) \cap l_\infty$. Now notice that if $S \subseteq \mathbf{N}$, then $\Sigma\, a_{n,k} f(\chi_S(k)) = f(1) \Sigma\, a_{n,k} \chi_S(k)$ for all $n \in \mathbf{N}$. The last observation, in tandem with Lemma 4 and that $w_0(A, f) \cap l_\infty$ is an ideal, yields that $w_0(A, f) \cap l_\infty \subseteq w_0(A) \cap l_\infty$ since $\chi_{S(x:\epsilon)} \in w_0(A) \cap l_\infty$ whenever $x \in w_0(A, f) \cap l_\infty$ and $w_0(A) \cap l_\infty$ is closed.

If A is a regular nonnegative matrix summability method, we can make some connections between strong A-summability with respect to a modulus and A-statistical convergence. The following definition is an extension of the original definition of statistical convergence which appears in [4]. Statistical convergence is discussed in [6], [13] and [3].

DEFINITION 7. *Let A be a nonnegative regular summability method and let x be a sequence. Then x is said to be A-statistically convergent to L if $\chi_{S(x-Le:\epsilon)}$ is contained in $w_0(A)$ for every $\epsilon > 0$.*

It is fairly easy to show that the bounded sequences which are A-statistically convergent to 0 from an ideal which, via Lemma 4, is closed in l_∞.

The proofs of the preceding results can be modified to yield the following result

THEOREM 8. *Let A be a nonnegative regular matrix summability method and f be a modulus.*

(a) *If $x \in s$ is strongly A-summable to L with respect to f, then x is A-statistically convergent to L.*

(b) *If $x \in s$ is bounded and A-statistically convergent to L, then x is strongly A-summable to L with respect to the modulus f.*

PROOF. (a) First recall that if $x \in w_0(A, f)$ and $y \in l_\infty$ then $xy \in w_0(A, f)$. Now suppose that $x \in w_0(A, f)$ and $\epsilon > 0$ has been given. Define $y \in l_\infty$ by $y_k = 1/x_k$ if $|x_k| \geqq \epsilon$ and $y_k = 0$ otherwise. Observe that

$$xy = \chi_{S(x,\epsilon)} \in w_0(A, f) \cap l_\infty = w_0(A) \cap l_\infty$$

and hence x is A-statistically convergent to 0. The remainder of the claim follows immediately.

(b) Now suppose that $x \in l_\infty$ and x is A-statistically convergent to L, then the definition yields that $\chi_{S(x-Le;\epsilon)} \in w_0(A) \cap l_\infty$ for every $\epsilon > 0$. Lemmas 4 and 5 assert that $x - Le$ is strongly A-summable to 0 and hence, via theorem 6, strongly A-summable to L with respect to any modulus f. □

It is easy to check that if a sequence is A-statistically convergent to L, then it must have a subsequence which is convergent to L. The above theorem now yields the following corollary:

COROLLARY 9. *If x is strongly A-summable to L with respect to the modulus f, then x has a subsequence which is convergent to L.*

The special cases of strong summability and statistical convergence can be used to show that a boundedness condition cannot be omitted from the hypothesis of Theorem 8(b). One observation in this direction is to note that if x is strongly summable then $x = 0(n)$ whereas a statistically convergent sequence is not required to satisfy any order growth condition. This observation is sufficient to show that, while the strongly summable sequences can be given a BK topology [11], the statistically convergent sequences cannot be given a BK topology [14, p.58]. (In fact, the statistically convergent sequences cannot even be given a locally convex FK topology [1].)

The above considerations suggest the conjecture that if x is statistically convergent to L and $x = 0(n)$, then x is strongly summable to L. However, it is easy to verify that the sequence defined by $x_k = \sqrt{k}$ if k is a perfect square and 0 otherwise is a counterexample to this conjecture. In fact, $x = o(n)$. The following observation is a partial converse to theorem 8(a). Note that the hypothesis forces the sequence to be statistically convergent to L.

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PROPOSITION 10. *Let $x \in s$ such that $x = 0(\sqrt{n})$ and* $\lim_n(\sqrt{n}^{-1}|\{k \leqq n : |x_k - L| \geqq \epsilon\}| = 0$ *for all* $\epsilon > 0$. *Then x is strongly summable to L.*

PROOF. Note that if $|x_n - L| \leqq M\sqrt{n}$ for all n, then

$$n^{-1}\sum_{k=1}^{n}|x_k - L| \leqq \epsilon + M(\sqrt{n}^{-1}|\{k \leqq n : |x_k - L| \geqq \epsilon\}|$$

for all $\epsilon > 0$. □

This result is unsatisfying since it requires the sequence to be more than statistically convergent. Theorem 8(b) is the best partial converse I have been able to find which only requires that the sequence by statistically convergent.

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Dept. of Mathematics
Ohio University
Athens, Ohio
45701

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