A SOCRATIC DIALOGUE ON MATHEMATICS

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Socrates: Are you looking for someone, my dear Hippocrates?

Hippocrates: Nay, Socrates, because I have just found him. I looked for you in many places. Somebody told me at the agora that he saw you walking here along the river Ilissos. Thus it happened that I came here after you.

Socrates: Well then, tell me why you came. After this I want to ask you something about our discussion with Protagoras. Do you remember it still?

Hippocrates: How can you ask? Since that time not a single day passed without my thinking about it. I came today to ask your advice just because this discussion was in my mind.

Socrates: It seems, my dear Hippocrates, that you want to speak with me about the same question; thus the two subjects turned out to be one and the same. It seems that the mathematicians are mistaken in saying two is never equal to one.

Hippocrates: Socrates, you have witchcraft; as a matter of fact I wanted to speak with you about mathematics.

Socrates: My dear Hippocrates, you certainly know that I am not a mathematician; why did you not go with your question to the celebrated Theodoros?

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Hippocrates: I am perplexed, Socrates, that you are able to answer my question even before I tell you what my question is. I came to ask your opinion about my becoming a pupil of Theodoros. The last time I came to you with the intention of becoming a pupil of Protagoras, and we went to him together, you directed the discussion so that it became quite clear that Protagoras does not even know what is the knowledge taught by him. Thus I changed my mind and did not follow him. This discussion helped me to see what I should not do, but did not show me what I should do. I am still wondering about this. I am visiting banquets and the palaestra with youngsters of my age, and I dare say I am having a pleasant time, but this does not satisfy me. It disturbs me that I feel myself ignorant; more exactly I feel that the knowledge I have is rather uncertain. During the discussion with Protagoras I realized that my knowledge about familiar notions like virtue and justice and courage is far from being satisfactory. Nevertheless I feel that I have made great progress in that now I see clearly my own ignorance.

Socrates: I am glad, my dear Hippocrates, that you understand me so well. I always say quite frankly that I do not know anything; the difference between myself and most other people is that I do not imagine I know what in reality I do not know.

Hippocrates: This shows, Socrates, your wisdom. But this sort of wisdom does not satisfy me. I have a strong desire to obtain certain and solid knowledge, and I shall not be happy until I get this. I am constantly pondering on what kind of knowledge I should try to acquire. Recently Theaitetos told me that this sort of knowledge exists only in mathematics and suggested that I should learn mathematics from his master, Theodoros, who is - as he told me - the best expert on numbers and geometry in Athens. Now I do not want to make the same mistake as when I wanted to be a pupil of Protagoras. Therefore tell me, my dear Socrates, shall I find that kind of sound knowledge which I am seeking if I learn mathematics from Theodoros or not?

Socrates: If you want to study mathematics, o son of Apollodoros, then you certainly cannot do better than to go to my highly esteemed friend Theodoros. But you have to decide yourself whether you want to study mathematics or not. Nobody can know better than yourself what you want.

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Hippocrates: Why do you refuse to help me Socrates? Perhaps I offended you without knowing it?

Socrates: You misunderstand me my young friend. I am not angry, only you ask something impossible from me. Everybody has to decide for himself what he wants to do. I cannot do more than assist as a midwife to give birth to your decision.

Hippocrates: Please, my dear Socrates, do not refuse this help to me, and if you are free now, let us start immediately.

Socrates: Well, if you want it, let us lie down in the shadow of that plane-tree and begin. But tell me first: are you ready to conduct the discussion in the way I like it? I shall ask questions and you have to answer them? And do you know that from such a talk you cannot profit more than by seeing clearly what you knew already, and by bringing into blossom the knowledge, the seeds of which were already in your soul? I hope you will not behave like king Darius, who killed the master of his mines because he brought only copper out of a mine while the king thought that it contained gold. I hope you will not forget that no miner can find in a mine anything but what the mine contains.

Hippocrates: I swear that I shall make no reproaches, but by Zeus, let us begin mining at once.

Socrates: All right. Then tell me: do you know what mathematics is? I suppose you know what it is you want to study.

Hippocrates: I think every child knows it. Mathematics is one of the sciences, and even one of the finest.

Socrates: I did not ask you to praise mathematics, but to tell what its nature is. For instance, if I asked you about the art of physicians, you would have answered that this art deals with health and illness, and aims at healing the sick and preserving health. Am I right?

Hippocrates: Certainly.

Socrates: Answer me: does the art of the physician deal with something which exists or with something which does

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not exist? If there were no physicians, would illness still exist?

Hippocrates: Certainly, even more than now.

Socrates: Let us have a look at some other art, say, that of the astronomers. Do you agree with me, that the astronomers are studying the motion of the stars?

Hippocrates: To be sure.

Socrates: And if I ask you whether astronomy deals with something which exists, what is your answer?

Hippocrates: My answer is yes.

Socrates: Would stars exist if there were no astronomers in the world?

Hippocrates: Certainly. And if Zeus in his anger should extinguish all mankind, the stars would still shine in the sky at night. But why do we speak about astronomy instead of about mathematics?

Socrates: Don't be impatient, my good friend. Let us consider a few other arts, in order to compare them with mathematics. What do you call the man who knows all creatures living in the woods or in the depth of the sea?

Hippocrates: I would call him a scientist studying living nature.

Socrates: And do you agree that such a man studies things which exist?

Hippocrates: I agree.

Socrates: And if I say that every art deals with something which exists, do you agree?

Hippocrates: Completely.

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Socrates: Now tell me then, my young friend, what is the object of mathematics, what are the things which a mathematician studies?

Hippocrates: I have asked Theaitetes the same question; he answered that mathematics studies numbers and geometrical forms.

Socrates: Well, the answer is clear, but would you say that these are things which exist?

Hippocrates: Of course. How could we speak about them if they would not exist?

Socrates: Then tell me - if there were no mathematicians would there be prime numbers and if so where would they be?

Hippocrates: Really I do not know what to answer. Clearly if mathematicians are thinking about prime numbers then these exist in their consciousness, but if there were no mathematicians the prime numbers would not be anywhere.

Socrates: Do you mean that we have to say that the mathematicians are studying non-existing things?

Hippocrates: Yes, I think we have to admit this.

Socrates: Let us have a look at the question from another point of view. Look here, I wrote on this wax tablet the number 39. Do you see it?

Hippocrates: Yes, I do.

Socrates: And can you touch it with your hand?

Hippocrates: Certainly.

Socrates: Then perhaps numbers nevertheless exist?

Hippocrates: O Socrates, you want to pull my leg. Look here, I have drawn on the same tablet a dragon with seven heads. Does it follow that such a dragon exists? I have never met anybody who saw a dragon, and I am convinced that dragons do not

exist at all, except in fairy tales. But suppose I am mistaken, and somewhere beyond the pillars of Hercules there really are dragons living, still this has nothing to do with my drawing.

Socrates: You are speaking the truth, Hippocrates, and I agree with you completely. But does this mean that though we can speak about them, and write them down, nevertheless numbers do not exist in reality?

Hippocrates: Certainly.

Socrates: Let us not make hasty conclusions. Let us make another trial. Am I right in saying that we can count the sheep here on the meadow, or the ships in the harbour of Pireus?

Hippocrates: Yes, we can.

Socrates: And the sheep and the ships exist?

Hippocrates: Clearly.

Socrates: But if the sheep exist, their number must be something that exists too?

Hippocrates: You are making fun of me, Socrates. The mathematicians do not count sheep; this is the business of shepherds.

Socrates: Do you mean that what the mathematicians are studying is not the number of sheep or ships, or other existing things, but they are studying the number itself, and thus are concerned with something that exists only in their minds?

Hippocrates: Yes, this is what I mean.

Socrates: You told me that according to Theaitetos the mathematicians study numbers and geometrical forms. How about forms? If I ask you whether forms exist, what is your answer?

Hippocrates: They certainly exist. We can see the form of a beautiful vessel, say, and feel it with our hands too.

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Socrates: Yet I have still one difficulty. If you look at a vessel what do you see: the vessel itself or its form?

Hippocrates: I see both.

Socrates: Is this the same thing as looking at a lamb: you see the lamb and also its hair?

Hippocrates: I find the simile very well chosen.

Socrates: Well, I think that this simile is as limping as Hephaestus. You can cut the hair of the lamb and then you see separately the lamb without its hair, and the lamb's hair without the lamb. Can you separate in a similar way the form of a vessel from the vessel itself?

Hippocrates: Certainly not, and I dare say nobody can do this.

Socrates: And nevertheless you still believe that you can see a geometric form?

Hippocrates: I am beginning to doubt it.

Socrates: Besides, if the mathematicians would study the forms of vessels, shouldn't we call them potters?

Hippocrates: Certainly.

Socrates: If the mathematicians would study the form of vessels, would not Theodoros be the best potter? I have heard many people praising him, but nobody told me that he understands anything about pottery. I doubt whether he could make even the simplest pot. Or do mathematicians - deal with the form of statues or buildings?

Hippocrates: If they did this they would be sculptors and architects.

Socrates: Well, my friend, we came to the conclusion that the mathematicians when studying geometry, are not concerned with the forms of existing objects such as vessels, but with forms which exist only in their thoughts. Do you agree?

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Hippocrates: I have to agree.

Socrates: After having established that the mathematicians are concerned with things which do not exist in reality, but only in their thoughts, let us examine the statement of Theaitetos which you mentioned, that mathematics gives us more reliable and more trustworthy knowledge than any other branch of science. Tell me, did Theaitetos give you some examples?

Hippocrates: Yes, he said for instance that one cannot know exactly how far Athens is from Sparta. Of course people who have travelled that way agree on how many days one has to walk, but it is impossible to know exactly how many feet the distance is. On the other hand, one can tell, by means of the theorem of Pythagoras what the length of the diameter of a square is. He said also that it is impossible to give the exact number of people living in Hellas. If somebody tried to count all the people, he would never get the exact figure, because during the counting some old people would die, children would be born, and thus the total number would be only approximately correct. But if you ask the mathematician how many edges a regular dodecahedron has, he will tell you that the dodecahedron is bounded by 12 faces, each having 5 edges, a total of 60. But as each edge belongs to two faces and thus has been counted twice, the number of edges of the dodecahedron is equal to 30, and this figure is beyond any doubt.

Socrates: Did he mention other examples?

Hippocrates: Quite a lot, but I do not remember all of them. He said that in reality you never find two things which are exactly the same. No two eggs are exactly the same; even the pillars of Poseidon's temple are slightly different from each other, but you may be sure that the two diagonals of a rectangle are exactly equal. He quoted Herakleitos who said that everything which exists is constantly changing, and that sure knowledge is only possible about things which never change, as for instance the odd and the even, the straight line and the circle.

Socrates: This will do. These examples convince me that in mathematics we can get knowledge which is beyond any doubt, while in other sciences or in everyday life this is impossible. Let us try to summarize the results of our inquiry into

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the nature of mathematics. Am I right if I say that we came to the conclusion that mathematics studies non-existing things and is able to find out the full truth about them?

Hippocrates: Yes, this is what we established.

Socrates: But tell me, for Zeus' s sake, my dear Hippocrates, is it not a mysterious thing that one can know more about things which do not exist than about things which exist?

Hippocrates: If you put it like this, it is certainly a mystery. I am sure there is some mistake in our arguments.

Socrates: Nay, we proceeded with the utmost care and we controlled every step of the argument; there cannot be any mistake in our reasoning. But listen, I remember something which may help us to solve the riddle.

Hippocrates: Tell it quickly, because I am quite bewildered.

Socrates: This morning I was in the hall of the second archon, where the wife of a carpenter from the village Pitthos was accused that she had betrayed and, with the aid of her lover, had murdered her husband. The woman protested and swore to Artemis and Aphrodite that she was innocent, that she never loved anybody else than her husband, and that her husband was killed by pirates. Many people were called to witness; some said that the woman was guilty, others said that she was innocent. It was impossible to find out what really happened.

Hippocrates: Are you again mocking me? First you confused me completely, and now instead of helping me to find the truth you tell me such stories.

Socrates: Don't be angry, my friend. I have serious reasons for speaking about this woman, about whom it was impossible to find out whether she was guilty or not. But one thing is sure: this woman exists. I saw her with my own eyes, and you can ask anybody who was there, amongst whom there were quite a few trustworthy men, who never lied in their life.

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Hippocrates: Your testimony is sufficient for me, my dear Socrates. Let it be granted that the woman exists. But what has this to do with mathematics?

Socrates: More than you imagine. But tell me first, - do you know the story about Agamemnon and Klytaimnestra?

Hippocrates: Everybody knows the story. I saw the trilogy of Aeschylos in the theatre last year.

Socrates: Then tell me the story in a few words.

Hippocrates: While Agamemnon, the king of Argos, fought at Troy, his wife Klytaimnestra committed adultery with Aigisthos, the cousin of her husband. When after the fall of Troy, Agamemnon returned home, his wife and her lover murdered him.

Socrates: Tell me Hippocrates, is it quite sure that Klytaimnestra was guilty?

Hippocrates: I do not understand why you ask me such questions. There can be no doubt about the story. It is related by Homer that when Odysseus visited the netherworld he met Agamemnon, who himself told him his sad fate.

Socrates: But are you sure that Klytaimnestra and Agamemnon and all the other characters of the story really existed?

Hippocrates: Perhaps I would be ostracized if I said this in public, but my opinion is that it is impossible either to prove or to disprove today, after so many centuries, that the stories which Homer tells us are true or not. But this is quite irrelevant. When I told you that Klytaimnestra was guilty I did not speak about the real Klytaimnestra, whether such a person lived or not, but about the Klytaimnestra of our Homeric tradition, about the Klytaimnestra in the trilogy of Aeschylos.

Socrates: May I say than that about the real Klytaimnestra we do not know anything - even her existence is uncertain - but as regards the Klytaimnestra who is a character in the trilogy of Aeschylos we are sure that she was guilty and murdered Agamemnon, because this is what Aeschylos tells us?

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Hippocrates: Yes, of course, but why do you insist on all this?

Socrates: You will see in a moment. Let me summarize what we have found out: as for the flesh and blood woman who was tried in Athens today, it is almost impossible to find out whether she was guilty or not, while as regards Klytaimnestra, who is a character in a play and who probably never existed, there can be no doubt that she was guilty. Do you agree?

Hippocrates: Now I am beginning to understand what you want to say. But it would be better if you would draw the conclusions yourself.

Socrates: The conclusion is this: we have much more certain knowledge about persons who exist only in our imagination, as for example about characters in a play, than about living persons. If we say that Klytaimnestra was guilty, this does not mean anything else than this is how Aischylos imagined her and presented her in his play. The situation is exactly the same as regarding mathematics; we may be sure that the diagonals of a rectangle are equal because this follows from the definition of a rectangle given by the mathematicians.

Hippocrates: Do you mean, Socrates, that our paradoxical result is really true and one can have much more reliable knowledge about non-existent things - as for instance the things which are the objects of mathematics - than about the real objects of nature? I think that I see now the reason for this too. The notions which we ourselves have created are by their very nature completely known to us, and we can find out the full truth about them because they are exactly as we imagine them, as they have no other reality outside our imagination. However, objects which exist in the real word are not identical with our picture of them, which is always incomplete and approximate, and therefore our knowledge of these real things can never be complete and certain.

Socrates: This is the truth, my young friend, and you have stated it better than I could.

Hippocrates: This is your merit, Socrates, because you led me to understand these things. I see now not only that

Theaitetos was quite right in telling me that if I want to obtain unfailing knowledge, I have to study mathematics; it becomes clear to me why this is true. But if you guided me with patience up to now, please do not abandon me yet, because one of my questions, in fact the most important one, is still unanswered.

Socrates: What is this question?

Hippocrates: Please remember, o Socrates, that I came to you to ask your advice whether I should study mathematics or not. You helped me to realize that mathematics and only mathematics can give me that sort of sound knowledge which I want. But what is the use of this knowledge? It is clear that if one obtains some knowledge about the existing world, even if this knowledge is not quite certain and is incomplete, this nevertheless is of value for the individual as well as for the state. Even if one gets some knowledge about things as far away as the stars this may be useful, for instance in navigation at night. But what is the use of the knowledge of non-existent things like that which mathematics offers? Even if this knowledge is complete and beyond any doubt, what is the use of knowledge concerning things which do not exist in reality?

Socrates: My dear friend, I am quite sure you know the answer, only you want to examine me.

Hippocrates: By Hercules, I cannot answer the question Please help me.

Socrates: Well, let us try it. We have established that the notions of mathematics are created by the mathematician himself. Tell me: does this mean that the mathematician chooses his notions quite arbitarily as it pleases him?

Hippocrates: As I told you, I do not know much yet about mathematics. It seems to me that the mathematician is as free to choose the mental objects of his study as the poet is free to choose the characters of his play; and as the poet invests his characters with whatever traits pleases him, the mathematician can endow his notions with such properties as he likes.

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Socrates: If this would be true, there would be as many mathematics as mathematicians. How do you explain then that all mathematicians study the same notions and problems? How do you explain what often happens, that mathematicians living far from each other and having no contact discover independently the same truths? I never heard that two poets wrote the same poem.

Hippocrates: Nor have I heard such a thing, but I remember Theaitetos telling me about some very interesting theorem on incommensurable distances which he discovered. He showed his result to his master Theodoros, who produced a letter by Archytas in which the same theorem was contained, almost word for word.

Socrates: In poetry this would be impossible. Now you see that there is a problem. But let us continue: how do you explain that the mathematicians of different countries can usually agree about the truth, while about questions concerning the state, for instance, not only the Persians but also the Spartans have quite opposite views to ours in Athens? Moreover, we in Athens often do not agree with each other.

Hippocrates: I can answer this last question. In matters of the state everybody is personally interested, and these personal interests are often in contradiction; this is why it is difficult to come to an agreement. However the mathematician is led purely by his desire to find the truth.

Socrates: Do you mean to say that the mathematicians are trying to find a truth which is completely independent from their own person?

Hippocrates: Yes, I do.

Socrates: Well, but then we were mistaken when we thought that mathematicians choose the object of their study at their own will. It seems that such an object has some sort of existence which is independent of their own person. We have to solve this new riddle.

Hippocrates: I do not see how to start.

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Socrates: If you still have patience, let us try it together. Tell me, what is the difference between the sailor who finds an uninhabited island and the painter who finds a new colour, which no other painter used before him?

Hippocrates: I think that the sailor may be called a discoverer, while the painter an inventor. The sailor discovers an island which existed before him only it was unknown, while the painter invents a new colour which before this did not exist at all.

Socrates: Nobody could answer this question better. But tell me, the mathematician who finds a new truth, does he discover it or invent it? Is he a discoverer like the sailor or an inventor like the painter?

Hippocrates: It seems to me that the mathematician is more like a discoverer: he is a bold sailor who sails on the unknown sea of thought and explores its coasts, islands and whirlpools.

Socrates: Well said, and I agree with you completely. I would add only that to a lesser extent the mathematician is an inventor too, especially when he invents new concepts. But every discoverer has to be to a certain extent an inventor also. For instance if a sailor wants to get to places which other sailors before him were unable to reach, he has to build a ship which is better than the ships which other sailors used. The new concepts invented by the mathematicians are like new ships which carry the discoverer farther on the great sea of thought.

Hippocrates: My dear Socrates, you have helped me to find the answer to the question which seemed so difficult to me. The main aim of the mathematician is to explore the secrets and riddles of the sea of human thoughts. These exist independently of the person of the mathematician, though not from humanity as a whole. The mathematician has a certain independence in inventing new concepts, as tools, and it seems that he could do this at his discretion; but he is not quite free in doing this because the new concepts have to be useful for his work. The sailor can build any sort of ship at his discretion, but of course he will not be so mad as to build a ship which would be crushed to pieces by the first storm. Now I think that everything is clear.

Socrates: If you see everything clearly, try to answer again the question: what is the object of mathematics?

Hippocrates: We came to the conclusion that besides the world in which we are living there exists another world, the world of human thought, and the mathematician is the fearless sailor who explores this world, not shrinking back from the troubles, dangers and adventures which await him.

Socrates: My friend, your youthful vigour almost sweeps me off my feet; but I am afraid that in the ardour of your enthusiasm you overlook certain questions.

Hippocrates: What are these questions?

Socrates: I do not want to disappoint you, but I feel that your main question has not yet been answered. We have not yet answered the question: what is the use of exploring the wonderful sea of human thought?

Hippocrates: You are right, my dear Socrates, as always. But wouldn't you this time put aside your method and tell me the answer immediately?

Socrates: Nay, my friend, even if I could, I would not do this, and that for your sake. The knowledge which somebody gets without having to work for it is almost worthless to him; we understand thoroughly only that which - perhaps with some outside help - we find out ourselves, just as a plant can use only the water which it sucks up through its own roots from the soil.

Hippocrates: All right, let us continue our search by the same method, but at least help me by a question.

Socrates: Let us go back to the point where we established that the mathematician is not dealing with the number of sheep or ships or other existing things but with the numbers themselves. Don't you think, however, that nevertheless what the mathematicians discover to be true for pure numbers, are valid for the number of existing things too? For instance the mathematician finds that 17 is a prime number. Does it not follow that you cannot distribute 17 living sheep among some persons so that

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each should get the same number of sheep in another way than giving to 17 persons one sheep each?

Hippocrates: Of course this is true.

Socrates: Well, how about geometry? Can it not be applied in building houses, in making pots or computing the amount of grain which a ship can hold?

Hippocrates: It can; moreover it is applied, though it seems to me that for these practical purposes of the craftsman not too much mathematics is needed. The simple rules known already by the clerks of the pharaohs in Egypt are sufficient for most such purposes, and the new discoveries about which Theaitetos spoke to me with such overflowing fervour are neither used nor needed in practice.

Socrates: Perhaps not for the moment, but they may be used in the future.

Hippocrates: I am interested in the present time.

Socrates: If you want to be a mathematician, you have to realize that you will be working mostly for the future. But let us return to the main question. We saw that the knowledge about another world, the world of thought, about things which do not exist in the usual sense of the word, can be used in everyday life to answer questions of the real world. Is this not surprising?

Hippocrates: More than that, it is even incomprehensible, it is really a miracle.

Socrates: Perhaps it is not so mysterious after all, and if we open the shell of this question we find in it a real pearl.

Hippocrates: Please, my dear Socrates, do not speak in riddles, like the Pythia.

Socrates: Tell me then, if somebody who has travelled abroad in far countries has seen and experienced much, and then returns to his city, and using his experience gives wise advice to his fellow-citizens, do you find this surprising?

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Hippocrates: Not at all.

Socrates: Even if the countries which the traveller has visited are very far, are inhabited by a quite different sort of people speaking another language, worshipping other gods?

Hippocrates: Not even in this case, because there is much that is common between different people.

Socrates: Now tell me: if it turned out that the world of mathematics is, in spite of its peculiarities, in some sense similar to our real world, would you still find it miraculous that mathematics can be applied to the study of the real world?

Hippocrates: In that case not. But I do not see any similarity between the real world and the imaginary world of mathematics.

Socrates: Do you see that rock on the other side of the river, there where the river broadens out and forms a lake?

Hippocrates: I see it.

Socrates: And do you see the reflected image of the rock in the water?

Hippocrates: Certainly I do.

Socrates: Then tell me: what is the difference between the rock and its reflection?

Hippocrates: The rock is a solid piece of hard matter. It is made warm by the sun. If you would touch it you would feel that it is rough. The reflected image cannot be touched; if I would put my hand on it, I would touch only the cool water. As a matter of fact the reflected image does not exist really; it is illusion, nothing else.

Socrates: Is there nothing in common between the rock and its reflected image?

Hippocrates: Well, in a certain sense the reflected image is a faithful picture of the rock. The contour of the rock, even

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its small abutments, are clearly visible in the reflected image. But what about all this? Do you want to say that the world of mathematics is a reflected image of the real world in the mirror of our thinking?

Socrates: You said it, and it is very well said.

Hippocrates: But how is this possible?

Socrates: Let us recall how the abstract concepts of mathematics developed. We have said that the mathematician deals with pure numbers and not with numbers of real objects. But do you think that somebody who has never counted real objects can understand the abstract notion of number? When a child learns counting, he counts first pebbles and small sticks; only if he knows that two pebbles and three pebbles make five pebbles, and the same is true about sticks or coins, is he able to understand that two and three make five. The situation is essentially the same with geometry. The child arrives at the notion of a sphere through experiences with round objects like balls. Mankind developed all fundamental notions of mathematics in a similar way. These notions crystallized from the knowledge of the real world, and thus it is not surprising but quite natural that they bear marks of their origin, as children are similar to their parents. And exactly as the children when they grow up become the supporters of their parents, so any branch of mathematics, if it is sufficiently developed, becomes a useful tool in exploring the real world.

Hippocrates: Now it is quite clear to me how the knowledge of the non-existent things of the world of mathematics can be used in everyday life. You rendered me a great service in helping me to understand this.

Socrates: I envy you, my dear Hippocrates, because I still marvel at one thing about which I should like to have my mind set at rest, but perhaps you can help me.

Hippocrates: I would do this with pleasure, but I am afraid you are again mocking me. Do not make me ashamed by asking my help but tell me frankly what is the question which I overlooked?

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Socrates: You will see it yourself if you try to summarize the results of our discussion.

Hippocrates: Well, when it became clear why mathematics is able to give sure knowledge about another world, different from the world in which we are living, about the world of human thought, there remained the question: what is the use of this knowledge? Now we have found that the world of mathematics is nothing else than a reflection of the real world in our mind, and this makes clear that every discovery about the world of mathematics gives us some information about the real world. I am completely satisfied with this answer.

Socrates: If I tell you that the answer is not yet complete, I say this not because I want to confuse you, but because I am sure that sooner or later you would raise the question yourself and would reproach me for not having called your attention to it. You would ask me: "Tell me, Socrates, what is the sense of studying the reflected image if we can study the object itself?"

Hippocrates: Well, you are perfectly right that this is an obvious question. You are a wizard Socrates. You can confuse one totally by a few words, and you can knock down, by an innocent-looking question, the whole edifice which we have built up with so much trouble. I should of course answer that if we are able to look at the original thing there is no sense in looking at the reflected image. But I am sure that this shows only that our analogy fails at this point. Certainly there is an answer, only I do not know how to find it.

Socrates: Your guess is correct; the paradox arose because we stuck too closely to the analogy of the reflected image. An analogy is like a bow; if you stretch it too far it breaks. Let us drop it and choose another one. You certainly know that travellers and sailors make good use of maps.

Hippocrates: I have experienced this myself. Do you mean that mathematics furnishes a map of the real world?

Socrates: You said it. Can you now answer the question: what advantage is there in looking at the map instead of looking at the landscape?

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Hippocrates: This is clear: using the map we can look over large distances which can be covered only by travelling many weeks or months. The map shows us not every detail, only the most important things and therefore it is useful if we want to plan a long voyage.

Socrates: Very well. But there is something else which occurs to me.

Hippocrates: What is it?

Socrates: There is another reason why the study of the mathematical image of the world may be of use. If the mathematicians discover some property of the circle, this gives us at once some information about any object of circular shape. Thus the method of mathematics enables us to deal with different things at the same time.

Hippocrates: What about the following analogies? If somebody looks at a city from the top of a near mountain he gets a more comprehensive view than walking through the crooked streets; the general watching the movements of the army of the enemy from a hill gets a clearer picture of the situation than the soldier in the front line who sees only those directly opposite him.

Socrates: Well, you have surpassed me in inventing new similes; but since I do not want to fall behind, let me also add one parable. Recently I looked at a picture made by Aristophon, the son of Aglaophon, and the painter warned me, saying "if you go so near to the picture, Socrates, you will see only coloured spots but you will not see the whole picture".

Hippocrates: Of course he was right, and you also when you did not let us finish our discussion before we got to the heart of the question. However, I think it is time to return to the city because the shades of night are falling and I am hungry and thirsty. If you have still some patience, I would like to ask you something while we are walking towards the city.

Socrates: All right, let us start and you may ask your question.

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Hippocrates: Our discourse has convinced me fully that I should start studying mathematics, and I am very grateful to you for this. Now tell me: why are you yourself not doing mathematics? Judging from your deep understanding of its real nature and importance, I guess that you would surpass all other mathematicians of Hellas if you would concentrate on it. I would be glad to follow you as your pupil if you accepted me.

Socrates: No, my dear Hippocrates, this is not my business. Theodoros knows much more about mathematics than I do, and you cannot find a better master than he. As to your question why I myself am not a mathematician, I shall give you the reasons. I did not conceal my high opinion of mathematics. I think that we Hellenes have in no other art made such important progress as in mathematics, and this is only the beginning; if we do not extinguish each other in mad wars, we shall obtain wonderful results as discoverers and also as inventors. You asked me why I do not join the ranks of those who develop this wonderful science. As a matter of fact, I too am some sort of a mathematician, only of a different kind. An inner voice, you may call it an oracle to which I always listen carefully, asked me many years ago: "What is the source of the great advances which the mathematicians have made in their noble science?" I answered: "I think the source of the successes of the mathematicians lies in their method: the high standards of their logic, their striving without the least compromise towards the full truth, their habit of starting always from first principles and defining every notion used exactly and avoiding self-contradictions." My inner voice answered: "Very well, but why do you think, Socrates, that this method of thinking and arguing can be used only for the study of numbers and geometric forms? Why do you not try to convince your fellow-citizens to apply the same high logical standards in every other field, for instance in philosophy and politics; in discussing the problems of every-day private and public life?" From that time, this is what I have always tried to do. I have demonstrated - you remember for instance our discussion with Protagoras - that those who are thought to be wise men are mostly ignorant fools, and all their arguing lacks any solid foundation, since they use - in contrast to mathematicians - undefined and only half-understood notions. By this activity I have succeeded in making almost

everybody my enemy. This is not surprising, because for all people who are sluggish in thinking and idly contented to use obscure terms, I am a living reproach. People do not like those who constantly remind them of their faults which they are unable or unwilling to correct. The day will come when these people will fall upon me and exterminate me. But until that day comes I shall continue to follow my calling. You, however, go to Theodoros!

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