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On some theorems of Tarafdar

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In a recent paper (*Bull. Austral. Math. Soc.* 13 (1975), 241-245), Tarafdar has considered nonexpansive self mappings on a subset X of a locally convex vector space E and proved an extension to E of a theorem of Göhde. The purpose of this paper is to show that the condition $f : X \to X$, in Göhde-Tarafdar's Theorem in the above paper, may be weakened to $f : X \to E$ with $f(\partial X) \subseteq X$. As a consequence, it is further shown that an extension to E of a well-known common fixed point theorem of Belluce and Kirk due to Tarafdar remains true on domains that are not necessarily bounded or quasi-complete.

Introduction

Let X be a subset of a locally convex vector space E. In a recent paper [7], Tarafdar considered nonexpansive mappings $f : X \to X$ and proved extensions of certain results of Göhde [4], Taylor [8], and Belluce and Kirk [2]. The purpose of this paper is to show that the condition $f : X \to X$ in Göhde-Tarafdar's Theorem ([7], Lemma 2.1) may be weakened to $f : X \to E$ with $f(\partial X) \subseteq X$. As a consequence, it is shown that Theorem 2.1, [7], remains true on domains that are not necessarily bounded or quasi-complete.

1.

Throughout this paper, let E be a locally convex, Hausdorff topological vector space, X a nonempty subset of E, and $\mathcal{U}$ a neighborhood base of the origin consisting of absolutely convex subsets of E. For each $U \in \mathcal{U}$, let p_U be the Minkowski's functional of U in

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E . Note that, since for any finite family $\{U_i : i = 1, 2, \ldots, n\} \subseteq \mathcal{U}$, there is a $V \in \mathcal{U}$ with $V \subseteq \cap\{U_i : i = 1, 2, \ldots, n\}$, therefore (see [5]),

(1) $$p_{U_i} \le p_V \text{ for each } i = 1, 2, \ldots, n .$$

For $x, y \in E$, let

$$(x, y) = \{z \in E : z = \alpha x+(1-\alpha)y,\ 0 < \alpha < 1\} ,$$

and $[x, y) = \{x\} \cup (x, y)$. For any subset A of E , let $\mathrm{co}(A)$ denote the convex hull of A , $\mathrm{cl}(A)$ the closure of A , and ∂A the boundary of A in E .

A mapping $f : X \to E$ is a P-contraction (see [3]) iff for each $p \in P$, there exists a $\alpha_p < 1$ such that $p\big(f(x)-f(y)\big) \le \alpha_p p(x-y)$ for all $x, y \in X$. If this inequality holds with $\alpha_p \equiv 1$ for each $p \in P$, then f is called a P-nonexpansive mapping (see [6], [7]). Note that a P-contraction or a P-nonexpansive mapping is continuous.

The following result is proved in [6] and is used in this paper.

LEMMA 1. *Let X be a closed subset of E . If $x \in X$ and $y \notin X$, then there exists a $z \in [x, y) \cap \partial X$; that is $z = (1-\lambda)x + \lambda y \in \partial X$ for some $\lambda \in [0, 1)$.*

Now let X be a closed subset of E and $f : X \to E$ be a mapping. An orbit $O(x, f)$ of any $x \in X$ is a sequence $\{x_n : n \in I,\ x_0 = x\} \subseteq X$ defined inductively as follows: let $x_0 = x$ and for each $n \in I$, if $f(x_n) \in X$, set $x_{n+1} = f(x_n)$ and if $f(x_n) \notin X$, then let x_{n+1} be any element of $[x_n, f(x_n)) \cap \partial X$ (such a x_{n+1} exists by Lemma 1).

It follows from the above definition that for each $n \in I$, there is a $\gamma_n \in [0, 1]$ such that

(2) $$x_{n+1} = \gamma_n x_n + (1-\gamma_n) f(x_n) .$$

Note that if $f(x_n) \notin X$, then $x_{n+1} \in \partial X$. Therefore, if $f(\partial X) \subseteq X$ then $f(x_n) \notin X$ implies $f(x_{n+1}) \in X$. Also, note that a point may have many

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orbits. However, if $f : X \to X$, then for any $x \in X$,

$$O(x, f) = \{f^n(x) : n \in I, f^0(x) = x\} .$$

The following lemma simplifies the proof of the next theorem.

LEMMA 2. *Let* X *be a closed subset of* E *and let* $f : X \to E$ *be a mapping such that* $f(\partial X) \subseteq X$. *If*

(a) f *is* P*-nonexpansive mapping then for any* $p \in P$ *and orbit* $O(x, f) \equiv \{x_n : n \in I, x_0 = x\}$ *of* $x \in X$, *then* $p(x_n - f(x_n)) \le p(x - f(x))$ *for each* $n \in I$, *and if*

(b) f *is a* P*-contraction,*

then for each $p \in P$ *and* $\varepsilon > 0$, *there exists a* $x \in X$ *such that* $p(x - f(x)) < \varepsilon$.

Proof. Let $O(x, f) = \{x_n : n \in I, x_0 = x\}$ be an orbit of $x \in X$ and let $p \in P$. Then by (2),

(3) $x_{n+1} - x_n = (1-\gamma_n)(f(x_n) - x_n)$ and $x_{n+1} - f(x_n) = \gamma_n(x_n - f(x_n))$.

Since for each $n \in I$,

$$p(x_{n+1} - f(x_{n+1})) \le p(x_{n+1} - f(x_n)) + p(f(x_n) - f(x_{n+1})) ,$$

therefore, in either case (a) or (b), it follows by (3) that

(4) $p(x_{n+1} - f(x_{n+1})) \le p(x_n - f(x_n)) \le \ldots \le p(x_0 - f(x_0)) = p(x - f(x))$.

This proves (a). To prove (b), it suffices to show that $p(x_n - f(x_n)) \to 0$. Now, by (4), $\{p(x_n - f(x_n))\}$ is a nonincreasing sequence of nonnegative reals and hence there is $r \ge 0$ such that $p(x_n - f(x_n)) \to r$. We show that $r = 0$. Suppose $r > 0$. Choose an $\varepsilon > 0$ and a $n_0 \in I$ such that $\alpha_p(r+\varepsilon) < r$ and $p(x_n - f(x_n)) < r + \varepsilon$ for all $n \ge n_0$. Now choose a $m \in I$, $m \ge n_0$, such that $x_{m+1} = f(x_m)$. (Let $m = n_0$ if $f(x_{n_0}) \in X$ and if $f(x_{n_0}) \notin X$, let $m = n_0 + 1$. Note $f(x_{n_0+1}) \in X$, see the remark before Lemma 2.) Then for this $m \in I$,

$$r \le p(x_{m+1} - f(x_{m+1})) \le \alpha_p p(x_m - f(x_m)) \le \alpha_p(r+\varepsilon) < r ,$$

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which contradicts that $r > 0$. Thus $p\left(x_n - f(x_n)\right) \to 0$.

A subset X of E is called starshaped if there exists a $q \in X$ such that for each $x \in X$ and $\gamma \in [0, 1]$, $\gamma q + (1-\gamma)x \in X$. The element q is called a star center of X.

The following result improves a result of Göhde [4] and also Lemma 2.1 in [7].

THEOREM 1. *Let X be a closed, starshaped subset of E and let $f : X \to E$ be a P-nonexpansive mapping such that $f(\partial X) \subseteq X$. Suppose the set $f(X)$ bounded and f satisfies the condition:*

(5) *there exists a compact subset $L \subseteq X$ such that for each $x \in X$, there is an orbit $O(x, f)$ with $\mathrm{cl}\left(O(x, f)\right) \cap L \neq \emptyset$.*

Then $\{x \in X : f(x) = x\}$ is a nonempty compact subset of L.

Proof. Let p be a fixed element of P. First, we show that for each $\varepsilon > 0$, there exists an $x \in L$ such that

(6) $$p\left(x - f(x)\right) < \varepsilon .$$

Let q be a star center of X. Since, by hypothesis, $f(X)$ is a bounded subset of E, therefore there exists an $N \in I$ such that

(7) $$\frac{1}{N} \sup\left\{p\left(q - f(x)\right) : x \in X\right\} < \frac{\varepsilon}{4} .$$

Define a mapping $g : X \to E$ by

(8) $$g(x) = \frac{1}{N} q + \left(1 - \frac{1}{N}\right) f(x) .$$

Then g is a P-contraction $\left(\alpha_p \equiv 1 - \frac{1}{N}\right.$ for each $\left. p \in P\right)$ and since $f(\partial X) \subseteq X$, it follows by (8) that $g(\partial X) \subseteq X$. Thus, by Lemma 2 *(b)*, there is a $y \in X$ satisfying $p\left(y - g(y)\right) \le \frac{\varepsilon}{4}$ and hence, by (7) and (8),

(9) $$p\left(y - f(y)\right) \le p\left(y - g(y)\right) + p\left(g(y) - f(y)\right) \le \frac{\varepsilon}{4} + \frac{1}{N} p\left(q - f(y)\right) \le \frac{\varepsilon}{2} .$$

For this $y \in X$, let $O(y, f) = \{y_n : n \in I, y_0 = y\}$ be an orbit of y such that $\mathrm{cl}\left(O(y, f)\right) \cap L \neq \emptyset$. This implies that there is a $y_n \in O(y, f)$ and an $x \in L$ such that

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(10) $$p(x-y_n) \leq \frac{\varepsilon}{4}\ .$$

Since f if P-nonexpansive, it follows, by Lemma 2 *(a)*, (9), and (10), that

$$p(x-f(x)) \leq p(x-y_n) + p(y_n-f(y_n)) + p(f(y_n)-f(x)) \leq \frac{\varepsilon}{2} + p(y-f(y)) < \varepsilon\ .$$

This proves (6). It now follows by (6) that for each $n \in I$, $n \geq 1$, there is an $x_n \in L$ such that

(11) $$p(x_n-f(x_n)) \leq \frac{1}{n}\ .$$

Since L is compact, there is a subnet $\{x'_n\}$ of the net $\{x_n : n \in I, n \geq 1\}$ and an $x_0 \in L$ such that $\{x'_n\} \to x_0$, and it follows, by (11), that

(12) $$p(x_0-f(x_0)) = 0\ .$$

Thus, for each $p \in P$, there exists an $x_p \in L$ such that

(13) $$p(x_p-f(x_p)) = 0\ .$$

Let, for each $p \in P$, $A_p = \{x \in L : p(x-f(x)) = 0\}$ and $F = \{A_p : p \in P\}$. Then, by (12), A_p is a nonempty, closed subset of L for each $p \in P$. Further, since for any finite subset $\{p_i : i = 1, 2, \ldots, n\} \subseteq P$, there exists a $p \in P$ with $p_i \leq p$ for each i (see the first paragraph of Section 1); therefore

$$A_p \subseteq \cap\{A_{p_i} : i = 1, 2, \ldots, n\}\ .$$

Since, by (12), $A_p \neq \emptyset$, it follows that the family F has a finite intersection property. Consequently $F = \cap\{A_p : p \in P\}$ is a nonempty compact subset of L . Clearly $F \subseteq \{x \in X : f(x) = x\}$. Also, if for some $x \in X$, $f(x) = x$, then $\mathrm{cl}(O(x, f)) = \{x\}$ and hence, by (5), $x \in L$, and since $p(x-f(x)) = 0$ for each $p \in P$, it follows that $x \in F$. Thus $F = \{x \in X : f(x) = x\}$.

REMARK. It may be remarked that Theorem 1 extends Lemma 2.1 in [7], where boundedness of X is crucial in the construction of norm $\| \|_B$ for the proof. Our proof does not require such a construction of the norm and appears simpler even in this more general case.

2.

The purpose of this section is to show that the conditions on X being bounded and quasi-complete in Theorem 2.1 in [7] are unnecessary.

The following lemma, whose proof is given in ([7], Lemma 2.2), is stated here for completeness.

LEMMA 3. *Let* L *be a compact subset of* E . *If for some* $p \in P$,

$$d_p = \sup\{p(x-y) : x, y \in L\} > 0 .$$

Then there exists $u \in \mathrm{co}(L)$ *such that*

$$r = \sup\{p(x-u) : x \in L\} < d_p .$$

The proof of the following result is similar to the argument in Bakhtin [1].

THEOREM 2. *Let* X *be a nonempty, closed, and convex subset of* E *and* Γ *a commutative family of* P*-nonexpansive self mappings of* X *satisfying the condition:*

(14) *there exists a* $g \in \Gamma$ *and a compact set* $L \subseteq X$ *such that*

(i) $g(X)$ *is bounded and*

(ii) for each $x \in X$, $\mathrm{cl}\{g^n(x) : n \in I\} \cap L \neq \emptyset$.

Then the family Γ *has a common fixed point in* L .

Proof. Let

$A = \{S \subseteq X : S$ is nonempty, closed, convex, and $f(S) \subseteq S$ for each $f \in \Gamma\}$.

Then $X \in A$. Define a partial order $\leq$ in A by $S_1 \leq S_2$ iff $S_2 \subseteq S_1$. We show that each chain in A has an upper bound in A . Let $\{S_\alpha : \alpha \in \Delta\}$ be a chain in A . Let $A = \cap\{S_\alpha : \alpha \in \Delta\}$. We show that $A \neq \emptyset$. For each $\alpha \in \Delta$, set $L_\alpha = S_\alpha \cap L$. Since S_α is closed and $g(S_\alpha) \subseteq S_\alpha$, it follows from (14) *(ii)* that L_α is a nonempty compact subset of S_α and for any $x \in S_\alpha$, $\mathrm{cl}\{g^n(s) : n \in I\} \cap L_\alpha \neq \emptyset$. Since $g(S_\alpha)$ is bounded, it follows by Theorem 1, that for each $\alpha \in \Delta$,

(15) $$F_\alpha = \{x \in S_\alpha : g(x) = x\}$$

is a nonempty compact subset of S_α. Since $\{F_\alpha : \alpha \in \Delta\}$ is a chain of compact subsets of X, therefore $F = \cap\{F_\alpha : \alpha \in \Delta\} \neq \emptyset$ and $F \subseteq A$. Thus $A \neq \emptyset$. Now it is easy to verify that $A \in \mathcal{A}$ and that A is an upper bound of the chain $\{S_\alpha : \alpha \in \Delta\}$. Therefore, by Zorn's Lemma, there exists a maximal element $S_0 \in \mathcal{A}$. Let

$$F = \{x \in S_0 : g(x) = x\} .$$

Then, by the similar arguments used above, it follows that F is a nonempty compact subset of S_0. Further, since for any $f \in \Gamma$ and $x \in F$,

$$f(x) = f(g(x)) = g(f(x)) ,$$

it follows that $f(F) \subseteq F$ for each $f \in \Gamma$. Let

$$\mathcal{B} = \{C \subseteq S_0 : C \text{ is nonempty, compact, and } f(C) \subseteq C \text{ for each } f \in \Gamma\} .$$

Then $F \in \mathcal{B}$. Define the same partial order in $\mathcal{B}$ as in $\mathcal{A}$. Then it follows by Zorn's Lemma that there is a maximal element $M \in \mathcal{B}$. Clearly

(16) $$M \subseteq S_0 .$$

Also the maximality of M in $\mathcal{B}$ implies that

(17) $$f(M) \equiv M$$

for each $f \in \Gamma$, for if $f_0(M) = M_1 \subseteq M$ for some $f_0 \in \Gamma$, then for each $f \in \Gamma$, $f(M_1) = f(f_0(M_1)) = f_0(f(M_1)) \subseteq f_0(M) \subseteq M_1$, contradicting the maximality of M in $\mathcal{B}$. We show that M consists of a single element. Suppose not. Then, since E is Hausdorff, there exists a $p \in P$ satisfying

(18) $$d_p = \sup\{p(x-y) : x, y \in M\} > 0 ,$$

and hence by Lemma 3, there exists a $u \in \mathrm{co}(M)$ such that

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(19) $$r = \sup\{p(x-u) : x \in M\} < d_p .$$

Since S_0 is convex, therefore, by (15), $u \in S_0$. Let for each $x \in M$,

(20) $$V(x) = \{z \in E : p(x-z) \leq r\} .$$

Then $V(x)$ is a closed and convex subset of E and, by (19), $u \in V(x)$ for each $x \in M$. Set

(21) $$V = \cap\{V(x) : x \in M\} \text{ and } S = S_0 \cap V .$$

Clearly S is a closed and convex subset of X and $u \in S$. We shall show that $f(S) \subseteq S$ for each $f \in \Gamma$. Since $f(S_0) \subseteq S_0$ for each $f \in \Gamma$, it suffices to show that $f(V) \subseteq V$ for each $f \in \Gamma$. Let $z \in V$ and $f \in \Gamma$. Then, by (20),

(22) $$p(x-z) \leq r$$

for each $x \in M$. Now by (17), for each $x \in M$, there is a $y = y(x) \in M$ such that $f(y) = x$ and hence, by (22),

$$p(f(z)-x) = p(f(z)-f(y)) \leq p(z-y) \leq r ,$$

for each $x \in M$. The last inequality implies that $f(z) \in V(x)$ for each $x \in M$; that is $f(V) \subseteq V$ and consequently $f(S) \subseteq S$ for each $f \in \Gamma$. Thus $S \in A$. However, by (21), $S_0 \leq S$ and since S_0 is maximal in A,

(23) $$S_0 = S .$$

Now p being continuous and M compact, there are elements $x, y \in M$ such that $p(x-y) = d_p$. Since $r < d_p$, the last equality implies that $y \notin V(x)$ and hence $y \notin S$. However, since $M \subseteq S_0$, $y \in S_0$. This contradicts (23). Thus $M = \{x\}$ for some $x \in X$. This implies that $f(x) = x$ for each $f \in \Gamma$.

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