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A NOTE ON REGULAR METABELIAN GROUPS OF PRIME-POWER ORDER

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Let p be a prime and d, e positive integers. We prove that a regular d-generator metabelian p-group whose commutator subgroup has exponent p^e has nilpotency class at most $e(p-2)+1$ unless $e = 1, d > 2, p > 2$ when the class can be p and these bounds are best possible.

It is known [3] that d-generator metabelian groups of exponent p^e have nilpotency class at most $d(p^{e-1}-1)+(p-2)p^{e-1}+1$ when $e \geqslant 2$ and $d \geqslant (p+2)(e-1)$ and this bound is best possible [6]. Here we report on the corresponding result under the additional condition that the groups are regular.

THEOREM. *Let p be a prime and d, e positive integers. A regular d-generator metabelian p-group G whose commutator subgroup has exponent p^e has nilpotency class at most $e(p-2)+1$ unless $e = 1, d > 2, p > 2$ when the class can be p. These bounds are best possible.*

We acknowledge that finding the result was considerably eased by using a program for computing with metabelian p-groups (see [6]). However the proof given below is self-contained.

The case of 2-groups is covered by the well-known result that a 2-group is regular if and only if it is abelian. For the rest of this note p is taken to be odd.

Our terminology and notation follow [4] except that we use G_n to denote the nth term of the lower central series of G, and G^m the subgroup generated by all mth powers of elements in G. We use the left-norming convention for commutators. For $a_1, a_2, \ldots, a_s \in G$ and positive integers $n_1, n_2, \ldots, n_s$, we make the convention

$$[n_1a_1, n_2a_2, \ldots, n_sa_s] = [a_1, a_2, \underbrace{a_1, \ldots, a_1}_{n_1-1}, \underbrace{a_2, \ldots, a_2}_{n_2-1}, \underbrace{a_3, \ldots, a_3}_{n_3}, \ldots, \underbrace{a_s, \ldots, a_s}_{n_s}].$$

Recall that in a metabelian group G: $[a,b,c][b,c,a][c,a,b] = 1$ for all a, b, c in G (Jacobi identity) and $[u,a,b] = [u,b,a]$ for all a, b in G and all u in G'.

To prove our theorem, we use the following two lemmas.

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LEMMA 1. [**1**, Theorem 3.1], [**7**, Theorem 2.2] *A two-generator metabelian p-group G is regular if and only if $G_p \leqslant G_2^p$.*

LEMMA 2. *Let G be a metabelian p-group and k, r integers such that $r \geqslant 2$ and $k \leqslant r(p-1)-1$ and $k \neq p-1$ when $r = 2$. If every r-generator subgroup of G has nilpotency class at most k, then the nilpotency class of G is at most k.*

PROOF: Since $k \leqslant r(p-1)-1$, there are r positive integers $n_1, n_2, \dots, n_r$ with $n_i < p$ for all i such that $n_1 + n_2 + \dots + n_r = k+1$. Since every r-generator subgroup of G has nilpotency class at most k, it follows that $[n_1 a_1, n_2 a_2, \dots, n_r a_r] = 1$ for all $a_1, a_2, \dots, a_r$ in G. The theorem in [**2**] gives that G_{k+1}/G_{k+2} has exponent prime to p and it follows that G_{k+1} is trivial. $\square$

PROOF OF THEOREM:

(1) $d = 2$: by induction on e. When $e = 1$, the conclusion is given by Lemma 1. When $e > 1$, the induction hypothesis applied to $G/G_2^{p^{e-1}}$ yields

$$G_{2+(e-1)(p-2)} \leqslant G_2^{p^{e-1}},$$

and it follows that

$$\begin{aligned} G_{2+e(p-2)} &= [G_{2+(e-1)(p-2)}, \underbrace{G, \dots, G}_{p-2}] \\ &\leqslant [G_2^{p^{e-1}}, \underbrace{G, \dots, G}_{p-2}] \\ &= G_p^{p^{e-1}} \\ &\leqslant G_2^{p^e} \\ &= 1, \end{aligned}$$

the desired result.

(2) $d > 2$: by induction on e.

(a) $e = 1$: in this case every two-generator subgroup has nilpotency class at most $p-1$. It follows from Lemma 2 (with $k = p$) that G has nilpotency class at most p.

(b) $e = 2$: every two-generator subgroup of G has nilpotency class at most $2p-3$ and the conclusion follows from Lemma 2.

(c) $e = 3$: it suffices by Lemma 2 to prove that every three-generator subgroup of G has nilpotency class at most $3p-5$. In a commutator of weight $3p-4$ with entries a, b or c at least one element, say b, occurs (at least) $p-1$ times. Hence, without loss of generality, the commutator has the form $[a,(p-1)b,\dots]$ or $[a,c,(p-1)b,\dots]$ where "…" represents $2p-4$ or $2p-5$ entries, respectively. Since Lemma 1 implies $[a,(p-1)b]\in G_2^p$, it follows that

$$[a,(p-1)b,\dots]\in[G_2^p,\underbrace{G,\dots,G}_{2p-4}]=G_{2p-2}^p\leqslant\left(G_2^{p^2}\right)^p=G_2^{p^3}=1.$$

The Jacobi identity then gives

$$[a,c,(p-1)b,\dots]=[c,(p-1)b,a,\dots]^{-1}[a,(p-1)b,c,\dots]=1.$$

Thus every three-generator subgroup of G has nilpotency class at most $3p-5$ as required.

(d) $e>3$: the induction hypothesis applied to $G/G_2^{p^{e-2}}$ yields

$$G_{2+(e-2)(p-2)}\leqslant G_2^{p^{e-2}},$$

and it follows that

$$\begin{aligned}G_{2+e(p-2)}&=[G_{2+(e-2)(p-2)},\underbrace{G,\dots,G}_{2p-4}]\\&\leqslant[G_2^{p^{e-2}},\underbrace{G,\dots,G}_{2p-4}]\\&=G_{2p-2}^{p^{e-2}}\\&\leqslant G_2^{p^e}\\&=1,\end{aligned}$$

the desired result.

Meier-Wunderli [5] constructed three-generator metabelian groups of exponent p with nilpotency class p. To complete the proof we construct a two-generator metabelian group of exponent p^e with nilpotency class $e(p-2)+1$ which is regular.

Let H be the direct product of $p-1$ cyclic groups of order p^e with generating set $\{c_0,\dots,c_{p-2}\}$. Clearly H has an automorphism α such that

$$c_i\alpha=c_ic_{i+1}\qquad\text{for } i \text{ in } \{0,\dots,p-3\}$$

and

$$c_{p-2}\alpha=c_{p-2}c_1^p.$$

For $i>p-2$ put $c_i=c_{i-p+2}^p$; then c_i is not the identity for $i\leqslant e(p-2)$ and c_i is the identity for $i>e(p-2)$. It is routine to check that

$$c_i\alpha^t=\prod_{j=0}^{t}c_{i+j}^{b(t,j)}$$

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where $b(t,j)$ is the binomial coefficient $t!/(j!(t-j)!)$ and therefore that α has order p^e. Let G be the semi-direct product of H by $\langle\alpha\rangle$. Clearly G is metabelian and generated by $\{c_0, \alpha\}$. Also $[c_0, e(p-2)\alpha] = c_{e(p-2)}$, so G has nilpotency class $e(p-2)+1$. Moreover $G_p \leqslant G_2^p$ and it follows from Lemma 1 that G is regular and hence has exponent p^e. ∎

References

[1] W. Brisley and I.D. Macdonald, 'Two classes of metabelian groups', *Math. Z.* **112** (1969), 5–12.

[2] N.D. Gupta and M.F. Newman, 'On metabelian groups', *J. Austral. Math. Soc.* **6** (1966), 362–368.

[3] N.D. Gupta, M.F. Newman and S.J. Tobin, 'On metabelian groups of prime-power exponent', *Proc. Roy. Soc. London Ser. A* **302** (1968), 237–242.

[4] B. Huppert, *Endliche Gruppen I.: Die Grundlehren der Mathematischen Wissenschaften* 134 (Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[5] H. Meier-Wunderli, 'Metabelischen Gruppen', *Comment. Math. Helv.* **25** (1951), 1–10.

[6] M.F. Newman, 'Metabelian groups of prime-power exponent', Groups–Korea 1983, in *Lecture Notes in Mathematics* **1098**, pp. 87–98 (Springer-Verlag, Berlin, Heidelberg, New York, 1984).

[7] M.Y. Xu, *On finite regular p-groups*, Graduation Thesis at Peking University, 1964.

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