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ON THE STRUCTURE OF TAME NEAR-RINGS

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Abstract

Tame near-rings form an important class of near-rings. They have the common feature that all N-subgroups in a faithful N-group are ideals. Tame near-rings can be very close to and very far from rings. Most of the important classes of distributively generated near-rings and all 2-semisimple near-rings are examples of tame near-rings.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*) (1985 *Revision*): 16 A 76.

If a near-ring N is tame on some N-group T then T is of type 2 if and only if it is of type 0. Hence it might make sense to ask if for tame near-rings these types 0 and 2 coincide on all N-groups. The answer is "yes", if a quite general condition is fulfilled, but we give a counterexample in the other case.

DEFINITIONS. Let N be a right zero-symmetric near-ring, and let T be an N-group. Then T is said to be *tame* if every N-subgroup of T is an ideal of T. If, moreover, T is faithful then N is said to be *tame on* T.

For more on tame near-rings, see the papers cited in the references. Now we are going to study the behaviour of a tame near-ring on some (other) N-group Γ. For this, we need a slight generalization of a result of G. Betsch [1, 3.4(i)].

LEMMA. *Let N be a zero-symmetric near-ring, Γ an N-group with $\Gamma = N\gamma_0$ for some γ_0 in Γ, and L_1, L_2 be left ideals of N such that $L_1 + (0:\gamma_0) = L_2 + (0:\gamma_0) = N$, but $L_1 \cap L_2 \subseteq (0:\gamma_0)$. Then $N/(0:\Gamma)$ is a ring.*

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This follows easily from [1, 3.14(a) and 3.4(i)], applied to Γ as an $N/(0:\Gamma)$-group.

DEFINITION. Let T and Γ be N-groups. Then T is said to be *subversive* to γ_0 in Γ if $N\gamma_0 = \Gamma$ and there is a family $(\tau_\alpha)_{\alpha\leq\lambda}$ in T such that λ is a limit ordinal and $\bigcap_{\alpha\leq\beta}(0:\tau_\alpha) \nsubseteq (0:\gamma_0)$ for all $\beta < \lambda$, but $\bigcap_{\alpha\leq\lambda}(0:\tau_\alpha) \subseteq (0:\gamma_0)$.

Since all $(0:\tau_\alpha)$ (and their intersections) are left ideals in N, we get

PROPOSITION. *If N has the descending chain condition on left ideals then no N-group can be subversive.*

THEOREM. *Let N be a zero-symmetric near-ring, T be a tame N-group and let $\Gamma = N\gamma_0$ be of type* 0. *If $(0:T) \subseteq (0:\gamma_0)$ and if T is not subversive to γ_0 then Γ is of type* 2.

PROOF. *Case* (i): $(0:\tau_0) \subseteq (0:\gamma_0)$ holds for some $\tau_0 \in T^* = T - \{0\}$. Then $h: N\tau_0 \to \Gamma = N\gamma_0$, $n\tau_0 \to n\gamma_0$, is a well-defined N-epimorphism. Also, $N\tau_0$ is tame. By [1, 9.171], $N\tau_0/\operatorname{Ker} h$ (which is N-isomorphic to Γ) is tame. Hence Γ is tame and thus of type 2.

Case (ii): not (i). Then $(0:\tau) \nsubseteq (0:\gamma_0)$ holds for all $\tau \in T^*$.

Case (a): there is no $\Sigma \subset T^*$ such that $(0:\Sigma) \subseteq (0:\gamma_0)$,

$$L_1 := (0:\ T - \{\tau\}) \nsubseteq (0:\ \gamma_0), \quad L_2 := (0:\ \tau) \nsubseteq (0:\ \gamma_0),$$

but

$$L_1 \cap L_2 = (0:\ T) \subseteq (0:\ \gamma_0) \quad \text{for each} \tau \in T^*.$$

Since Γ is of type 0 and $N/(0:\gamma_0) \cong_N \Gamma$, $(0:\gamma_0)$ is a maximal left ideal in N. Hence $L_i + (0:\gamma_0) = N$ for $i = 1, 2$. By the lemma, $N/(0:\Gamma)$ is a ring.

Case (b): there exists some $\Sigma \subset T^*$ with $(0:\Sigma) \subseteq (0:\gamma_0)$. Let $\Sigma = \{\sigma_\alpha | \alpha \in A\}$ and let A be well-ordered. Hence $\Sigma = \{\sigma_0, \sigma_1, \dots\}$. Let ω be the ordinal of A. Take some $\tau \in T^* - \Sigma$ and add τ to Σ as $\tau = \sigma_\omega$ (=last element of $\overline{\Sigma} := \Sigma \cup \{\tau\}$). Then again $(0:\overline{\Sigma}) \subseteq (0:\gamma_0)$ and $\overline{\Sigma} = \{\sigma_\alpha | \alpha \leq \omega\}$. Take $B := \{\alpha \in A | \bigcap_{i\leq\alpha}(0:\sigma_i) \subseteq (0:\gamma_0)\}$. Since $\omega \in B$, $B \neq \emptyset$ and hence B contains a smallest element β_0. Take $L_1 := \bigcap_{\alpha<\beta_0}(0:\sigma_\alpha)$ and $L_2 := (0:\sigma_{\beta_0}) \nsubseteq (0:\gamma_0)$. If β_0 is not a limit ordinal then $L_1 = \bigcap_{\alpha\leq\beta_0-1}(0:\sigma_\alpha) \nsubseteq (0:\gamma_0)$ and we might proceed as in (a) to show that $N/(0:\Gamma)$ is a ring. If β_0 is a limit ordinal, however, the family $(\sigma_\alpha)_{\alpha\leq\beta_0}$ is subversive to γ_0, a contradiction.

Hence in both cases (a) and (b), we know that $N/(0:\Gamma)$ must be a ring. Let Δ be an N-subgroup of Γ. We want to show that Δ is an ideal of Γ. For this, take $\delta = d\gamma_0 \in \Delta$, $\gamma = n\gamma_0 \in \Gamma$ and $A := (0:\Gamma)$.

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(1) $(N/A, +)$ is abelian, so $n + d - n \equiv d \pmod{A}$. Thus

$$\begin{aligned}\gamma + \delta - \gamma &= n\gamma_0 + d\gamma_0 - n\gamma_0 \\ &= (n + d - n)\gamma_0 = d\gamma_0 + a\gamma_0 \quad (\text{for some } a \in A) \\ &= d\gamma_0 = \delta.\end{aligned}$$

Hence Γ is abelian and Δ is normal.

If $n' \in N$ then a similar argument shows that

$$n'(\gamma + \delta) - n\gamma = (n'(d + n) - n'n)\gamma_0 = n'd\gamma_0 = n'\delta \in \Delta.$$

Hence Δ is an ideal in Γ and hence trivial.

This shows that Γ is of type 2.

The subversity-condition in the theorem cannot be omitted.

EXAMPLE. Let N be the near-ring of all analytic functions from $\mathbb{C}$ into $\mathbb{C}$ which have an (everywhere) convergent power series expansion at 0 with only real coefficients. Take $\Gamma = \mathbb{C}$ and $\Delta = \mathbb{R}$.

(i) N can easily be seen to be 2-primitive on Δ (1 can serve as a generator; if E would be a non-trivial N-subgroup, take $\varepsilon \in E^*$, $\delta \notin E$, $n = \delta z/\varepsilon$, then $n(\varepsilon) = \delta \notin E$). Also, if n is zero on Δ, $n = 0$.

(ii) Therefore, N is tame on Δ, too.

(iii) Γ is an N-group of type 0: as γ_0 we might take $i = \sqrt{-1}$, since if $a + bi \in \mathbb{C}$ and $n = bz - az^2$ then $n(i) = a + bi$. If Θ is an ideal of Γ then all $n(\gamma + \Theta) - n(\gamma)$ must be contained in Θ for $\gamma \in \Gamma$ and $\theta \in \Theta$. Take $n = e^z$. Then

$$n(\gamma + \theta) - n(\gamma) = e^{\gamma+\theta} - e^\gamma = e^\gamma(e^\theta - 1) \in \Theta.$$

If $\theta \neq 0$, it is always possible to get $e^\gamma(e^\theta - 1)$ to be any value in $\mathbb{C}$. Hence θ must be trivial.

(iv) N is even 0-primitive on Γ, but Γ is not of type 2, since it has Δ as a non-trivial N-subgroup.

(v) $T = \Delta$ is subversive to $\gamma_0 = i$. To see this, take $\tau_k = k^{-1}$ $(k \in \mathbb{N})$. Each $\bigcap_{k=1}^m (0 : \tau_k) \nsubseteq (0 : i)$, since, for example, $n = \prod_{k=1}^m (z - \tau_k)$ belongs to $\bigcap_{k=1}^m (0 : \tau_k)$, but not to $(0 : i)$. But $\bigcap_{k=1}^\infty (0 : \tau_k) = 0$, since any n in this intersection has a non-isolated pole, something an analytic function cannot have.

(vi) $(0 : \gamma_0) = (0 : i) \neq 0$, since, for example, $n = z^2 + 1 \in (0 : i)$.

COROLLARY. *Let N be zero-symmetric and tame on T. If T is not subversive to any γ_0 in some N-group Γ then Γ is of type 0 if and only if it is of type 2.*

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COROLLARY. *Let N be zero-symmetric, tame, and with the descending chain condition on left ideals. Then any N-group of type* 0 *is of type* 2, *the* J_0- *and the* J_2-*radicals coincide, and* $J_2(N)$ *is nilpotent.*

Compare this corollary to [1, 9.188]. Some interesting problems remain open.

Open problems. (i) Find a tame near-ring such that its J_0- and J_2-radicals do not coincide.

(ii) If I is an ideal of N, such that I and N/I are tame, is N then necessarily tame, too?

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