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GROUP ACTION PRESERVING THE HAAGERUP PROPERTY OF C^*-ALGEBRAS

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Dedicated to Professor Li-xin Xuan for his encouragement

Abstract

From the viewpoint of C^*-dynamical systems, we define a weak version of the Haagerup property for the group action on a C^*-algebra. We prove that this group action preserves the Haagerup property of C^*-algebras in the sense of Dong ['Haagerup property for C^*-algebras', *J. Math. Anal. Appl.* **377** (2011), 631–644], that is, the reduced crossed product C^*-algebra $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ has the Haagerup property with respect to the induced faithful tracial state $\widetilde{\tau}$ if A has the Haagerup property with respect to τ.

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1. Introduction and preliminaries

A discrete group Γ is amenable if and only if its reduced group C^*-alegbra $C^*_{\mathrm{r}}(\Gamma)$ is nuclear (see [7]). Thus amenability in geometric group theory corresponds to nuclearity in operator algebra. A similar correspondence is expected for the Haagerup property [5], which is a weak version of amenability (for example, equivalent to Gromov's a-T-menability [4]). To this end, motivated by the definition of the Haagerup property of a von Neumann algebra [6], Dong recently defined the Haagerup property for a C^*-algebra as follows.

Definition 1.1 [2]. Let A be a unital C^*-algebra and τ a faithful tracial state on A. The pair (A, τ) is said to have the *Haagerup property* if there is a net $(\phi_i)_{i\in I}$ of unital completely positive maps from A to itself satisfying the following conditions:

(1) each ϕ_i decreases τ, that is, $\tau(\phi_i(a)) \le \tau(a)$ for any $a \in A^+$;
(2) for any $a \in A$, $\|\phi_i(a) - a\|_\tau \to 0$ as $i \to \infty$;
(3) each ϕ_i is L^2-compact, that is, from the first condition, ϕ_i extends to a compact bounded operator on its GNS space $L^2(A, \tau)$.

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With this definition, Dong [2] successfully set up the following correspondence that is similar to the case of amenability and nuclearity: a discrete group Γ has the Haagerup property if and only if its reduced group C^*-algebra $C^*_{\mathrm{r}}(\Gamma)$ has the Haagerup property with respect to the canonical tracial state. Moreover, Dong also explored the behaviour of the Haagerup property of a C^*-algebra in a C^*-dynamical system and obtained the following theorem.

Theorem 1.2 [2]. *Suppose that A is a unital separable C^*-algebra with a faithful tracial state τ and a τ-preserving action α of a countable discrete group Γ. If Γ is amenable and (A, τ) has the Haagerup property, then $(A \rtimes_{\alpha,\mathrm{r}} \Gamma, \tau')$ also has the Haagerup property, where τ' is the induced trace of τ on $A \rtimes_{\alpha,\mathrm{r}} \Gamma$.*

The proof of Theorem 1.2 relies on an approximation technique from [1, Lemma 4.3.3], where Følner sets play a crucial role. Therefore the group Γ is assumed to be amenable. We also note that [1, Definition 4.3.1] gives the concept of *amenable action* of a discrete group on a C^*-algebra and [1, Theorem 4.3.4] shows that, if $\alpha : \Gamma \curvearrowright A$ is amenable, then A is nuclear if and only if $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ is nuclear. Inspired by this theorem, we wish to define a natural Haagerup property for a discrete group action on a C^*-algebra which preserves the Haagerup property of C^*-algebras.

There is some related work along these lines. Dong and Ruan [3] defined the following Haagerup property for a discrete group action on a C^*-algebra. Recall that a map $h : \Gamma \longrightarrow A$ is called *positive definite* with respect to an action $\alpha : \Gamma \curvearrowright A$ if for any $s_1, \dots, s_n \in \Gamma$, we have $[\alpha_{s_j}(h_{s_i^{-1}s_j})] \geq 0$ in $M_n(A)$. According to [3], the action $\alpha : \Gamma \curvearrowright A$ has the *Haagerup property* if there exists a sequence of bounded maps $\{h_n : \Gamma \longrightarrow \mathcal{Z}(A)\} \subseteq C_0(\Gamma, A)$ which are positive definite with respect to the action α and such that $h_n \to 1$ pointwise on Γ.

The motivation for this definition comes from the Haagerup property of the group, with $h_n \in c_0(\Gamma)$ replaced by $h_n \in C_0(\Gamma, A)$. Analogous to [2, Theorem 2.6], Dong and Ruan proved that an action $\alpha : \Gamma \curvearrowright A$ has the Haagerup property if and only if the reduced crossed product $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ has the Hilbert A-module Haagerup property [3, Theorem 3.6]. Although this is a delicate generalisation of the Haagerup property of a discrete group, the C^*-algebra A in this definition only functions as a 'scalar' and little attention is paid to its role in the C^*-dynamical system. In order to relate the Haagerup property of $A \rtimes_{\alpha,\mathrm{r}} \Gamma$ with that of the C^*-algebra A, we need to modify the definition of Dong and Ruan so that information about the Haagerup property of the C^*-algebra A is engaged.

Let $\widetilde{\tau} = \tau \circ \mathcal{E}$ be the induced faithful tracial state of τ on $A \rtimes_{\alpha,\mathrm{r}} \Gamma$, where $\mathcal{E} : A \rtimes_{\alpha,\mathrm{r}} \Gamma \to A$ is the canonical faithful conditional expectation. We say that $h : \Gamma \longrightarrow A$ is vanishing at infinity with respect to a faithful tracial state τ on A, denoted as $h \in C_{0,\tau}(\Gamma, A)$, if for arbitrary $\varepsilon > 0$, there exists a finite subset $F \subseteq \Gamma$ such that $\|h_s\|_{2,\tau} < \varepsilon$ for all $s \notin F$, where $\|\cdot\|_{2,\tau}$ is the L^2-norm on the GNS space $L^2(A, \tau)$ induced by τ.

Definition 1.3. We say that an action $\alpha : \Gamma \curvearrowright A$ has the *Haagerup property with respect to a faithful tracial state* τ on A if there exists a sequence of bounded positive

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definite (with respect to the action α) maps $\{h_n : \Gamma \longrightarrow \mathcal{Z}(A)\} \subseteq C_{0,\tau}(\Gamma, A)$ such that $h_n \to 1$ pointwise with respect to τ (that is, $\|h_n(s) - 1\|_{2,\tau} \to 0$ as $n \to \infty$ for any $s \in \Gamma$).

With this definition, we are ready to present the main theorem of this paper.

THEOREM 1.4. *Let Γ be a countable discrete group and α be an action of Γ on a unital C^*-algebra A such that α has the Haagerup property with respect to a faithful tracial state τ on A. Then the reduced crossed product $A \rtimes_{\alpha,r} \Gamma$ has the Haagerup property with respect to the induced faithful tracial state $\widetilde{\tau}$ if A has the Haagerup property with respect to τ.*

REMARK 1.5. Suppose that a discrete group Γ has the Haagerup property, that is, there exists a sequence of positive definite functions $\{\varphi_i\}$ on Γ with $\varphi_i(e) = 1$, such that each φ_i vanishes at infinity and $\varphi_i \to 1$ pointwise, and $\alpha : \Gamma \curvearrowright A$ is an action of Γ on the C^*-algebra A. If we define $h_n : \Gamma \to A$ by $h_n = \varphi_n 1$, then it is trivial that, equipped with $\{h_n\}$, $\alpha : \Gamma \curvearrowright A$ has the Haagerup property with respect to any faithful tracial state τ on A, as we would expect. Hence, Theorem 1.2 becomes a simple corollary of our theorem, since an amenable group always has the Haagerup property. The proof of our theorem also contains a direct approximation technique, but it does not involve any intermediate constructions such as Følner sets.

2. Proof of the main theorem

Suppose that the C^*-algebra A has the Haagerup property with respect to a faithful tracial state τ. Then there exists a sequence of unital completely positive maps $\{\varphi_n\}$ from A into itself such that:

(1) for every n, $\tau \circ \varphi_n \le \tau$ and φ_n is $L_{2,\tau}$-compact;
(2) for every $a \in A$, $\|\varphi_n(a) - a\|_{2,\tau} \to 0$ as $n \to \infty$.

Here, a linear map φ from a C^*-algebra A to a C^*-algebra B is said to be *completely positive* if $\varphi_n : \mathbb{M}_n(A) \longrightarrow \mathbb{M}_n(B)$, defined by $\varphi_n([a_{i,j}]) = [\varphi(a_{i,j})]$, is positive (that is, maps positive matrices to positive matrices) for every n.

We will show that $\{\Phi_n\}$ defined by $\Phi_n(x) = \sum_{t\in\Gamma} \lambda_t \varphi_n(a_t) h_n(t)$, where $x = \sum_{t\in\Gamma} \lambda_t a_t \in C_c(\Gamma, A)$, fulfils the requirement for $A \rtimes_{\alpha,r} \Gamma$ to have the Haagerup property with respect to $\widetilde{\tau}$. Note that each φ_n is completely positive, hence $\{\Phi_n\}$ is also completely positive on $A \rtimes_{\alpha,r} \Gamma$ according to the proof of [3, Theorem 3.2]. Let $\widetilde{\Phi}_n$ denote the map on $L^2(A \rtimes_{\alpha,r} \Gamma, \widetilde{\tau})$ induced by Φ_n. We have $\|\widetilde{\Phi}_n\|_{2,\widetilde{\tau}} \le \|\Phi_n\|$ (corresponding to [3, Proposition 3.3], with the minor modification that $\tau \circ \mathcal{E}(\Phi^*(x)\Phi(x)) \le \|\Phi\|^2 \tau \circ \mathcal{E}(x^* x)$).

LEMMA 2.1. *Suppose that A is a unital C^*-algebra with a faithful tracial state τ, $\varphi : A \longrightarrow A$ is unital and $L_{2,\tau}$-compact, and $h : \Gamma \longrightarrow \mathcal{Z}(A)$ is a bounded positive definite map with respect to the discrete group action $\alpha : \Gamma \curvearrowright A$. Let $\Phi : A \rtimes_{\alpha,r} \Gamma \longrightarrow A \rtimes_{\alpha,r} \Gamma$ be defined by $\Phi(x) = \sum_{t\in\Gamma} \lambda_t \varphi(a_t) h(t)$, where $x = \sum_{t\in\Gamma} \lambda_t a_t \in C_c(\Gamma, A)$. Then h is vanishing at infinity with respect to τ if and only if the induced map $\widetilde{\Phi}$ is compact on $L^2(A \rtimes_{\alpha,r} \Gamma, \widetilde{\tau})$.*

PROOF. '⇒' Assume that h is vanishing at infinity with respect to τ. For any integer $k > 0$, we can find a finite subset $F_k \subseteq \Gamma$ such that $\|h(t)\|_{2,\tau} < 1/k$ for any $t \notin F_k$. Since $\varphi : A \longrightarrow A$ is $L_{2,\tau}$-compact, we can find a sequence of finite-rank maps $\{\varphi_k : A \longrightarrow A\}_{k\in\mathbb{N}}$ such that $\|\widetilde{\varphi} - \widetilde{\varphi}_k\|_{2,\tau} < 1/k$. Define $T_k(x) = \sum_{t\in F_k} \lambda_t \varphi_k(a_t)h(t)$, where $x = \sum_{t\in\Gamma} \lambda_t a_t$ ($\widetilde{T}_k$ is indeed a finite-rank map because so is φ_k). So, for any $x = \sum_{t\in\Gamma} \lambda_t a_t \in C_c(\Gamma, A)$, $\Phi(x) - T_k(x) = \sum_{t\in F_k} \lambda_t(\varphi - \varphi_k)(a_t)h(t) + \sum_{t\notin F_k} \lambda_t \varphi(a_t)h(t)$.

Let $E_1 = \sum_{t\in F_k} \lambda_t(\varphi - \varphi_k)(a_t)h(t)$ and $E_2 = \sum_{t\notin F_k} \lambda_t \varphi(a_t)h(t)$, so $\|\widetilde{\Phi}(x) - \widetilde{T}_k(x)\|^2_{2,\widetilde{\tau}} \le 2(\|E_1\|^2_{2,\widetilde{\tau}} + \|E_2\|^2_{2,\widetilde{\tau}})$. Now,

$$\begin{aligned}
\|E_1\|^2_{2,\widetilde{\tau}} &= \tau\Big(\sum_{t\in F_k} h^*(t)(\varphi - \varphi_k)^*(a_t)(\varphi - \varphi_k)(a_t)h(t)\Big)\\
&= \sum_{t\in F_k} \tau(h^*(t)(\varphi - \varphi_k)^*(a_t)(\varphi - \varphi_k)(a_t)h(t))\\
&\le \sum_{t\in F_k} \tau((\varphi - \varphi_k)^*(a_t)(\varphi - \varphi_k)(a_t))\tau(h^*(t)h(t))\\
&= \sum_{t\in F_k} \|(\widetilde{\varphi} - \widetilde{\varphi}_k)(a_t)\|^2_{2,\tau}\|h(t)\|^2_{2,\tau}\\
&< \sum_{t\in F_k} \frac{1}{k^2}\|a_t\|^2_{2,\tau} M \quad \text{(where } M = \max_{t\in\Gamma} \|h(t)\|^2_{2,\tau}\text{, since } h \text{ is bounded)}\\
&= \frac{M}{k^2}\sum_{t\in F_k} \tau(a_t^* a_t) \le \frac{M}{k^2}\sum_{t\in\Gamma} \tau(a_t^* a_t) = \frac{M}{k^2}\|x\|^2_{2,\widetilde{\tau}},\\
\|E_2\|^2_{2,\widetilde{\tau}} &= \tau\Big(\sum_{t\notin F_k} h^*(t)\varphi^*(a_t)\varphi(a_t)h(t)\Big)\\
&= \sum_{t\notin F_k} \tau(h^*(t)\varphi^*(a_t)\varphi(a_t)h(t))\\
&\le \sum_{t\notin F_k} \tau(\varphi^*(a_t)\varphi(a_t))\tau(h^*(t)h(t)) = \sum_{t\notin F_k} \|\widetilde{\varphi}(a_t)\|^2_{2,\tau}\|h(t)\|^2_{2,\tau}\\
&\le \sum_{t\notin F_k} \|\widetilde{\varphi}\|^2_{2,\tau}\|a_t\|^2_{2,\tau}\frac{1}{k^2} \le \frac{\|\widetilde{\varphi}\|^2_{2,\tau}}{k^2}\|x\|^2_{2,\widetilde{\tau}}.
\end{aligned}$$

Thus, $\|\widetilde{\Phi}(x) - \widetilde{T}_k(x)\|^2_{2,\widetilde{\tau}} \le k^{-1}\sqrt{2(M + \|\widetilde{\varphi}\|^2_{2,\tau})}\,\|x\|_{2,\widetilde{\tau}}$. This shows that $\widetilde{\Phi}$ is $L_{2,\widetilde{\tau}}$-compact on $L^2(A \rtimes_{\alpha,\mathrm{r}} \Gamma, \widetilde{\tau})$.

'⇐' Suppose that $\widetilde{\Phi}$ is compact on $L^2(A \rtimes_{\alpha,\mathrm{r}} \Gamma, \widetilde{\tau})$. For any $\varepsilon > 0$, there exists a finite-rank map T on $L^2(A \rtimes_{\alpha,\mathrm{r}} \Gamma, \widetilde{\tau})$ such that $\|\widetilde{\Phi} - T\|_{2,\widetilde{\tau}} < \varepsilon$. Without loss of generality, we can assume that T has the form $T = \sum_{s,t\in F} \lambda_s a_s\langle\lambda_t b_t, \cdot\rangle_{\widetilde{\tau}}$ for some finite subset $F \subseteq \Gamma$. For any $r \notin F$, we have $T(\lambda_r) = \sum_{s,t\in F} \lambda_s a_s\langle\lambda_t b_t, \lambda_r\rangle = 0$. Since $\widetilde{\Phi}(\lambda_r) = \lambda_r h(r)$,

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$$\begin{aligned}\|h(r)\|_{2,\tau} &= \|h^*(r)h(r)\|_{2,\tau}^{1/2} = (\tau\circ\mathcal{E}(\Phi^*(\lambda_r)\Phi(\lambda_r)))^{1/2}\\ &= \|\widetilde{\Phi}(\lambda_r)\|_{2,\widetilde{\tau}} = \|\widetilde{\Phi}(\lambda_r) - T(\lambda_r)\|_{2,\widetilde{\tau}} < \varepsilon\end{aligned}$$

for any $r \notin F$. This shows that $h \in C_{0,\tau}(\Gamma, A)$. □

LEMMA 2.2. *Suppose that A is a unital C^*-algebra with a faithful tracial state τ, $\{\varphi_n : A \longrightarrow A\}$ is a sequence of maps such that $\widetilde{\varphi}_n$ converges to $\mathbf{1}$ on $L_2(A,\tau)$ pointwise, and $\{h_n : \Gamma \longrightarrow \mathcal{Z}(A)\}$ is a sequence of bounded positive definite maps with respect to the group action $\alpha : \Gamma \curvearrowright A$. Let $\Phi_n : A \rtimes_{\alpha,\mathrm{r}} \Gamma \longrightarrow A \rtimes_{\alpha,\mathrm{r}} \Gamma$ be defined by $\Phi_n(x) = \sum_{t\in\Gamma} \lambda_t\varphi_n(a_t)h_n(t)$, where $x = \sum_{t\in\Gamma}\lambda_t a_t \in C_c(\Gamma, A)$. Then $\{h_n\}$ converges to the constant function 1 with respect to τ pointwise on Γ if and only if the induced maps $\{\widetilde{\Phi}_n\}$ converge pointwise to the identity map on $L^2(A \rtimes_{\alpha,\mathrm{r}} \Gamma, \widetilde{\tau})$.*

PROOF. '⇒' For any $x = \sum_{t\in F}\lambda_t a_t \in C_c(\Gamma, A)$, where $F \subseteq \Gamma$ is finite,

$$\begin{aligned}\|\widetilde{\Phi}_n(x) - x\|_{2,\widetilde{\tau}}^2 &= \Big\|\sum_{t\in F}(\lambda_t\varphi_n(a_t)h_n(t) - \lambda_t a_t)\Big\|_{2,\widetilde{\tau}}^2\\ &= \tau\Big(\sum_{t\in F}(\varphi_n(a_t)h_n(t) - a_t)^*(\varphi_n(a_t)h_n(t) - a_t)\Big)\\ &= \sum_{t\in F}\|\varphi_n(a_t)h_n(t) - a_t\|_{2,\tau}^2\\ &\le 2\sum_{t\in F}(\|\varphi_n(a_t)h_n(t) - \varphi_n(a_t)\|_{2,\tau}^2 + \|\varphi_n(a_t) - a_t\|_{2,\tau}^2).\end{aligned}$$

For any $\varepsilon > 0$, since F is finite, there is an integer $N > 0$ such that $\|h_n(t) - 1\|_{2,\tau} < \varepsilon$, $\|\varphi_n(a_t) - a_t\|_{2,\tau} < \varepsilon$ and $\|\varphi_n(a_t)\|_{2,\tau} < M$, for any $t \in F$ and $n > N$. Hence, if $n > N$, we have $\|\widetilde{\Phi}_n(x) - x\|_{2,\widetilde{\tau}}^2 < 2\sum_{t\in F}(M^2\varepsilon^2 + \varepsilon^2)$, that is, $\|\widetilde{\Phi}_n(x) - x\|_{2,\widetilde{\tau}} < \sqrt{2\sum_{t\in F}(M^2+1)}\,\varepsilon$.

'⇐' Observe that

$$\|h_n(s) - 1\|_{2,\tau}^2 = \tau((h_n(s) - 1)^*(h_n(s) - 1)) = \|\widetilde{\Phi}_n(\lambda_s) - \lambda_s\|_{2,\widetilde{\tau}}^2 \to 0.$$ □

According to Definitions 1.1 and 1.3, it is clear that Theorem 1.4 follows from Lemmas 2.1 and 2.2.

3. Concluding remarks

It is interesting to notice that, since we use a weak version (associated with a faithful tracial state) of the Haagerup property for a C^*-algebra, the corresponding definition of the Haagerup property for the action also turns out to be weak (both vanishing and converging to 1 *tracially*). Is it possible to define a strong version of the Haagerup property of a C^*-algebra that does not depend on the choice of faithful tracial states or is even not related to them at all? We know that this can be done for the case of a von Neumann algebra (see [6]), and this provided the motivation for Definition 1.1.

Regarding this question, Suzuki [8] recently presented both positive and negative facts about Definition 1.1. On the positive side, he proved that a nuclear C^*-algebra has the Haagerup property with respect to any possible faithful tracial state on it; on the negative side, he provided an example of a C^*-algebra that has the Haagerup property with respect to one faithful tracial state but not with respect to another. Without the help of tracial states (especially some canonical ones when dealing with group C^*-algebras and crossed product C^*-algebras), it seems rather difficult to measure the approximation of the finite-rank maps to the compact one. For example, in [2, Theorem 2.6], there is the following approximation estimation inequality: for each $x = \sum_{t\in\Gamma} a_t\lambda_\Gamma(t) \in \mathbb{C}[\Gamma] \subseteq C_r^*(\Gamma)$,

$$\|m_{\varphi_i}(x) - m_{\varphi_i^{(k)}}(x)\|_2^2 = \Big\| \sum_{t\in\Gamma\setminus F_i^{(k)}} \varphi_i(t)a_t\lambda_\Gamma(t)\Big\|_2^2 = \sum_{t\in\Gamma\setminus F_i^{(k)}} |\varphi_i(t)|^2|a_t|^2.$$

But it is rather difficult to estimate $\sum_{t\in\Gamma\setminus F_i^{(k)}} \varphi_i(t)a_t\lambda_\Gamma(t)$ by the C^*-norm $\|\cdot\|$ in $C_r^*(\Gamma)$. So it is not as simple as just requiring the approximation maps $\{\Phi_i\}$ to be compact on the *operator space A* (actually it is a C^*-algebra) under the C^*-norm.

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