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A note on rational approximation

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It is shown that the inequality

$$|e-(p/q)| < \tfrac{1}{2}\big((\log \log q)/(q^2 \log q)\big)$$

holds for an infinity of integers p, q and that here the factor $\tfrac{1}{2}$ may not be replaced by a smaller number.

Corresponding best possible inequalities are given for the numbers $e^{\pm 2/t}$, where t is a positive integer.

In a recent paper (Davis [2]), the author gave the following result on approximation by rationals to numbers of the form $e^{\pm 2/t}$, where t is a positive integer.

THEOREM. *If* $a = \pm 2/t$, *where* $t \in \mathbb{N}$, *and*

$$c = \begin{cases} 1/t, & t \text{ even}, \\ 1/(4t), & t \text{ odd}, \end{cases}$$

then, for any $\varepsilon > 0$, *the inequality*

(1) $$|e^a-(p/q)| < (c+\varepsilon)\big((\log \log q)/(q^2 \log q)\big)$$

has an infinity of solutions in integers p, q. *Further, there exists a number* q', *depending only on* ε *and* t, *such that*

$$|e^a-(p/q)| > (c-\varepsilon)\big((\log \log q)/(q^2 \log q)\big)$$

for all integers p, q *with* $q \geq q'$.

The second statement of the theorem shows that the constant c in the

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inequality (1) is 'best possible' in the sense that it can not be replaced by any smaller number. Nonetheless, the inequality (1) may be improved, in that $c + \varepsilon$ may be replaced by c, and it is the purpose of this note to establish this result, thus giving the

THEOREM. *If* $a = \pm 2/t$, *where* $t \in \mathbb{N}$, *and*

$$c = \begin{cases} 1/t , & t \text{ even}, \\ 1/(4t) , & t \text{ odd} , \end{cases}$$

then the inequality

$$|e^a-(p/q)| < c\big((\log \log q)/\big(q^2 \log q\big)\big)$$

has an infinity of solutions in integers p, q. *If* c *be replaced by any smaller number, the inequality has only a finite number of integer solutions.*

In the paper cited, details of the proof were given for the case $a = 1$ (in which case $c = \frac{1}{2}$). The inequality (1) was established by explicitly constructing integers P_n, Q_n, for each $n \in \mathbb{N}$, such that

$$|e-\big(P_n/Q_n\big)| = |J_n|/Q_n^2 ,$$

where $|J_n| \sim 1/2n$ and $Q_n \sim \sqrt{(2/e)}(4n/e)^n$ as $n \to \infty$. The result (1) of the theorem follows, on taking $p = P_n$, $q = Q_n$, and observing that $n \sim \big(\log Q_n\big)/\big(\log \log Q_n\big)$. However, in the course of proving the second statement of the theorem it is shown that P_n/Q_n is that convergent of the simple (or regular) continued fraction

$$e = [2, \overline{1, 2n, 1}]_{n=1}^{\infty}$$

which arises by terminating that fraction immediately before the partial quotient $2n$. Hence

$$|e-(p/q)| < 1/2nq^2 .$$

Now

$$(2) \qquad \log q = n \log n + O(n) = n \log n\big\{1+O\big(1/(\log n)\big)\big\} ,$$

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so

$$\log\log q = \log n + \log\log n + O\big(1/(\log n)\big)$$
$$= (\log q)/n + \log\log n + O(1) ,$$

and hence

(3) $$1/n < (\log\log q)/(\log q)$$

for all sufficiently large n. Thus

$$|e-(p/q)| < \tfrac{1}{2}\big((\log\log q)/\big(q^2 \log q\big)\big)$$

for an infinity of p, q, as asserted.

We observe here, for later use, that (3) may be replaced by

(4) $$1/(n-m) < (\log\log q)/(\log q) ,$$

for any bounded m, since, by (2),

$$\log q = (n-m)\log n + O(n) .$$

In order to complete the proof to cover other values of a, we quote relevant results from Davis [1]. We denote by a_n, p_n/q_n $(n = 0, 1, \ldots)$ respectively the partial quotients and convergents of the continued fractions in question. Further, we observe that our Q_n is the $B_{n,n}$ of the paper just cited and that hence

$$Q_n \sim (4n/ae)^n\sqrt{\big(2e^{-a}\big)} .$$

Thus if we take $q = Q_n$ (or $\tfrac{1}{2}Q_n$, if appropriate), the inequality (4) still holds.

For $a = 2/t$ with t even, say $t = 2k$, and $k > 1$, we have $a_{3n-2} = (2n-1)k - 1$ and take $q = q_{3n-3} = Q_n$. Noting that

$$a_{3n-2} = 2nk - (k+1) > 2nk - 2k = t(n-1) ,$$

we have

$$|e^a-(p/q)| < 1/\big(t(n-1)q^2\big)$$

and the result follows, on using (4).

The case $a = 2/t$ with t odd is a little more complicated in detail

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and, for simplicity, we write $3n+1 = N$. Then

(i) if $t = 1$, $a_{5n} = 6(2n+1) = 4N + 2$,

$$q = q_{5n-1} = \tfrac{1}{2}Q_{3n+1} = \tfrac{1}{2}Q_N \ ;$$

(ii) if $t > 1$, $a_{5n+2} = 6t(2n+1) = 4tN + 2t$,

$$q = q_{5n+1} = \tfrac{1}{2}Q_{3n+1} = \tfrac{1}{2}Q_N \ .$$

The result in this case follows as before.

Finally, the case of e^{-a} with $a > 0$ is essentially the same, since here we simply take $q = p_{K-1}$ instead of q_K (the notation referring to the continued fraction for the corresponding e^a).

References

[1] C.S. Davis, "On some simple continued fractions connected with e ", *J. London Math. Soc.* 20 (1945), 194-198.

[2] C.S. Davis, "Rational approximations to e ", *J. Austral. Math. Soc. Ser. A* 25 (1978), 497-502.

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