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MAXIMUM GENUS EMBEDDINGS OF LATIN SQUARES

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Abstract. It is proved that every non-trivial Latin square has an upper embedding in a non-orientable surface and every Latin square of odd order has an upper embedding in an orientable surface. In the latter case, detailed results about the possible automorphisms and their actions are also obtained.

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1. Introduction. A triangular embedding of a complete tripartite graph $K_{n,n,n}$ is face two-colourable if and only if the supporting surface is orientable [**1**]. In such a case, the faces of each colour class can be regarded as the triples of a transversal design TD(3, n), of order n and block size 3. A TD(3, n) determines a Latin square of order n by identifying the three groups of the design as the rows, the columns, and the entries of the Latin square. When $n = 1$, the Latin square will be termed *trivial*. The two Latin squares formed by such a triangular embedding are said to be *biembedded* in the surface. Whenever such a biembedding exists, it represents a face two-colourable embedding of $K_{n,n,n}$ in a surface of minimum genus.

In this paper, we investigate the opposite case, namely the embeddings of Latin squares in surfaces of maximum genus. To be precise, we seek a face two-colourable embedding of $K_{n,n,n}$ in a surface in which the faces in one of the two colour classes are triangles and so determine a Latin square of order n, while there is just one face in the second colour class and the interior of that face is homeomorphic to an open disc. We call this an *upper embedding* of the Latin square. These types of embeddings have already been investigated for Steiner triple systems in [**2**] where it was shown that for $n > 3$, every Steiner triple system STS(n) has both an orientable and a non-orientable embedding in which the triples of the STS(n) appear as triangular faces and there is

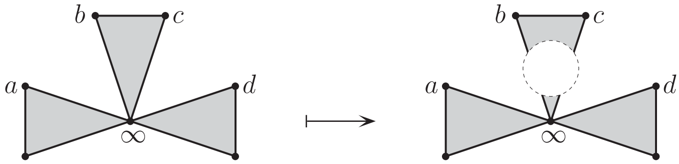


Figure 1. Joining two white faces by adding a crosscap.

just one additional large face. Detailed results about the possible automorphisms of such embeddings were also obtained.

Here, in results which parallel those for Steiner triple systems, we first prove that every non-trivial Latin square has an upper embedding in a non-orientable surface. For orientable surfaces, a necessary condition obtained from Euler's formula ($V + F - E = 2 - 2g$) is that the order of the Latin square is odd and we prove that this is also sufficient. In Section 3, we investigate possible automorphisms of the orientable embeddings and show that if $n \geq 3$, these must always be orientation preserving. With respect to the three sets of row points, column points, and entry points, it is shown that the automorphism group and its action is one of two possibilities. Either the group is cyclic with order dividing the order of the Latin square and the sets are preserved or the group is the cyclic group of order 3 and permutes the three sets. Thus in general, automorphisms of a Latin square are not respected by any upper embedding. This is well illustrated by Example 3.9. The trivial case where $n = 1$ is exceptional because there are just two faces both of which are triangles and the automorphism group is the symmetric group $\mathcal{S}_3$.

2. Existence of upper embeddings. Consider an upper embedding of a Latin square of order n. In the corresponding embedding of the complete tripartite graph $K_{n,n,n}$, the number of vertices (V) is $3n$, the number of edges (E) is $3n^2$ and the number of faces (F) is $n^2 + 1$. So $V + F - E = 1 + 3n - 2n^2$. For a non-orientable upper embedding, the genus $\gamma = (2n - 1)(n - 1)$, whilst for an orientable upper embedding, the genus $g = (2n - 1)(n - 1)/2$ which requires that in this case n must be odd. We first consider the non-orientable case and prove the following theorem.

THEOREM 2.1. *Every non-trivial Latin square has an upper embedding in a non-orientable surface.*

Proof. Begin with any face two-colourable embedding of $K_{n,n,n}$ in which the black faces are triangles representing the Latin square. If there is just one white face, then we have an upper embedding. Otherwise, there exists at least one black triangle that is incident to two white faces. With the addition of a crosscap across the black triangle, we join these two white faces together, reducing the number of faces by one and increasing the non-orientable genus by one as shown in Figure 1. By repetition of this procedure, we obtain a non-orientable upper embedding of the Latin square. □

For the remainder of this paper, we focus our attention on orientable surfaces. As we showed earlier, orientable embeddings require the order of the Latin square to be odd. As we will see and is to be expected, the proof for the existence of orientable upper embeddings is much more involved compared to the non-orientable case.

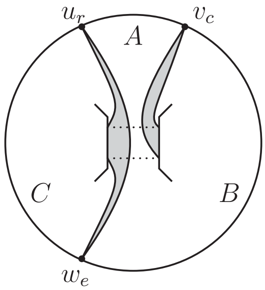


Figure 2. Adding a black triangle.

THEOREM 2.2. *Every Latin square of odd order n has an upper embedding in an orientable surface.*

Proof. There are two parts to the proof. The first of these is to construct an initial configuration of black triangles and one white face embedded on the sphere so that every point occurs on the boundary of the white face. The second part is then to add triangles, one at a time, increasing the genus by one at each step.

Denote the row points, the column points, and the entry points of the Latin square L by i_r, j_c, k_e, respectively. The black triangles of the embedding are then the triples $\{i_r, j_c, k_e\}$, where $k = L(i, j)$. Choose a fixed row point x_r and a fixed column point y_c. Take the triangle T containing both of these points together with a further $(n-1)/2$ triangles containing x_r and a further $(n-1)/2$ triangles containing y_c such that, together with T, these n triangles contain all n entry points. These triangles can be represented on a sphere giving n black triangles and one white face containing all entry points. Now take the remaining $(n-1)/2$ row points and the remaining $(n-1)/2$ column points and pair them arbitrarily. Attach the triangles containing these pairs to the spherical embedding at the appropriate entry points. This procedure gives a spherical embedding containing $(3n-1)/2$ black triangles and one white face with every row, column, and entry point occurring at least once on its boundary. Note also that the black triangles can be oriented in such a way that the points on the boundary follow the sequence $i_r\ j_c\ k_e \ldots$ where the suffices r, c, and e are always followed, respectively, by the suffices c, e, and r.

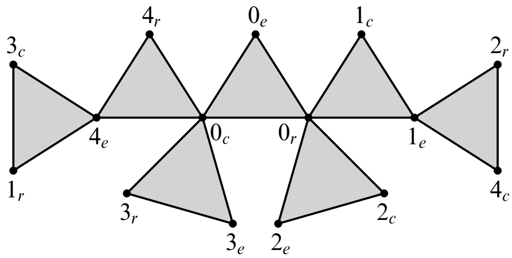
We now proceed to add the remaining $(2n^2 - 3n + 1)/2$ triples of the Latin square. Consider at any stage the boundary of the white face. We will use the fact that every point of the Latin square appears on the boundary at least once. This assumption is certainly true for the initial embedding described above. If the next triple to be added is $\{u_r, v_c, w_e\}$, then we locate one occurrence of each of these points on the boundary of the white face, add a handle to the white face, and paste on the triangle $\{u_r, v_c, w_e\}$.

If the points u_r, v_c, w_e originally divided the boundary of the white face into three sections A, B, and C, e.g. $Av_cBw_eCu_r$, then it is easy to see that, after the addition of the black triangle $\{u_r, v_c, w_e\}$ there still remains only one white face with boundary $Av_cw_eCu_rv_cBw_eu_r$ (see Figure 2). This face has three more edges than at the previous stage and every point of the Latin square still appears on the boundary. It is also clear that since the interior of the white face was homeomorphic to an open disc prior to the addition of the black triangle, then it will remain so after this addition. $\square$

EXAMPLE 2.3. Consider the following Latin square of order 5:

	0	1	2	3	4
0	0	1	2	3	4
1	1	0	3	4	2
2	2	3	4	0	1
3	3	4	1	2	0
4	4	2	0	1	3

First, we take the initial configuration to be the triangles $\{0_r, 0_c, 0_e\}$, $\{0_r, 1_c, 1_e\}$, $\{0_r, 2_c, 2_e\}$, $\{3_r, 0_c, 3_e\}$, $\{4_r, 0_c, 4_e\}$, $\{1_r, 3_c, 4_e\}$, $\{2_r, 4_c, 1_e\}$, and represent them on a sphere as shown below:



Then the large face of the initial embedding is

$$0_r\ \underline{1_c\ 1_e\ 2_r\ 4_c\ 1_e\ 0_r\ 2_c\ 2_e\ 0_r\ 0_c}\ 3_e\ \underline{3_r\ 0_c\ 4_e\ 1_r}\ 3_c\ \underline{4_e\ 4_r\ 0_c\ 0_e}$$

The second part is then to add the remaining 18 triples of the Latin square, one at a time. We will just illustrate the addition of the first triple. Choose the triangle $\{0_r, 3_c, 3_e\}$. The underlined sections in the large face above are those that divide the boundary in order to accommodate the addition of the new triangle.

The large face is now

$$0_r\ 0_e\ 0_c\ 4_r\ 4_e\ 3_c\ 3_e\ 0_c\ 0_r\ 2_e\ 2_c\ 0_r\ 1_e\ 4_c\ 2_r\ 1_e\ 1_c\ 0_r\ 3_c\ 1_r\ 4_e\ 0_c\ 3_r\ 3_e$$

By repetition of this technique, an upper embedding of the Latin square is obtained whose large face is

$4_r\ 4_e\ 2_c\ 0_e\ 1_c\ 2_e\ 3_c\ 1_e\ 2_c\ 0_r\ 1_e\ 1_r\ 2_e\ 0_c\ 0_r\ 2_e\ 4_c\ 3_e\ 0_r\ 4_c\ 1_r\ 3_e\ 1_c\ 0_r\ 3_c\ 2_r\ 4_e\ 3_c\ 3_r\ 1_e\ 1_c\ 1_r\ 4_e$
$1_c\ 2_r\ 2_e\ 4_r\ 2_c\ 3_r\ 4_e\ 0_r\ 0_e\ 4_r\ 3_c\ 3_e\ 2_c\ 2_r\ 3_e\ 0_c\ 2_r\ 1_e\ 4_r\ 4_c\ 2_r\ 0_e\ 4_c\ 4_e\ 0_c\ 3_r\ 0_e\ 0_c\ 1_e\ 4_c\ 3_r\ 2_e\ 2_c$
$1_r\ 0_e\ 3_c\ 1_r\ 0_c\ 4_r\ 1_c\ 3_r\ 3_e$

The rotation scheme (we assume that the reader is familiar with this concept, see, for example, [**3**] or [**5**]) is

$0_r : 0_c0_e\ 4_e4_c\ 3_e3_c\ 1_c1_e\ 2_c2_e$	$0_c : 0_e0_r\ 2_e2_r\ 3_e3_r\ 4_e4_r\ 1_r1_e$	$0_e : 0_r0_c\ 3_r4_c\ 2_r3_c\ 1_r1_c\ 2_c4_r$
$1_r : 1_e0_c\ 3_c4_e\ 1_c0_e\ 2_c3_e\ 4_c2_e$	$1_c : 0_e1_r\ 1_e0_r\ 3_e2_r\ 4_e3_r\ 4_r2_e$	$1_e : 0_r1_c\ 3_r2_c\ 3_c4_r\ 2_r4_c\ 0_c1_r$
$2_r : 0_c2_e\ 1_c3_e\ 2_c4_e\ 3_c0_e\ 4_c1_e$	$2_c : 4_r0_e\ 4_e2_r\ 3_e1_r\ 2_e0_r\ 1_e3_r$	$2_e : 0_r2_c\ 3_r3_c\ 1_c4_r\ 2_r0_c\ 1_r4_c$
$3_r : 0_c3_e\ 1_c4_e\ 2_c1_e\ 3_c2_e\ 4_c0_e$	$3_c : 0_e2_r\ 0_r3_e\ 4_r1_e\ 2_e3_r\ 4_e1_r$	$3_e : 3_c0_r\ 4_c4_r\ 3_r0_c\ 2_r1_c\ 1_r2_c$
$4_r : 0_c4_e\ 3_e4_c\ 1_e3_c\ 0_e2_c\ 2_e1_c$	$4_c : 0_e3_r\ 1_e2_r\ 4_r3_e\ 2_e1_r\ 0_r4_e$	$4_e : 4_c0_r\ 3_r1_c\ 1_r3_c\ 2_r2_c\ 4_r0_c$

Of course, many other upper embeddings can be obtained by making different choices to the order in which the triples are added.

3. Automorphisms. Throughout the remainder of this paper, we investigate possible automorphisms of an orientable upper embedding of a Latin square. By such an automorphism, we mean a permutation of the vertex set which is an automorphism of $K_{n,n,n}$ and preserves the faces of the embedding. Each of the three sets of row points, column points, and entry points will be called a *part*. We prove a number of propositions, the first of which is fairly easy.

PROPOSITION 3.1. *Let ϕ be an automorphism of an orientable upper embedding of a Latin square of order n. If ϕ is not the identity automorphism, then it can have fixed points from only one part.*

Proof. Suppose that ϕ has two fixed points, a and b, each from different parts. Since ϕ must preserve the large face and the edge ab appears somewhere on the boundary of this face, it must fix the points adjacent to the edge ab on this boundary. By repetition of this argument, ϕ fixes every point and is the identity. □

Automorphisms may be either orientation preserving or orientation reversing. We will first show that the latter do not exist.

PROPOSITION 3.2. *Orientation-reversing automorphisms of an orientable upper embedding of a Latin square of order $n \geq 3$ do not exist.*

Proof. Assume that such an automorphism does exist. Then it will act on the boundary of the large face as a reflection across an axis. The number of edges on the boundary of the large face is $3n^2$, which since n is odd, is also odd. Therefore, this axis will pass through exactly one point, say 0_r, and exactly one edge; thus, the automorphism will be an involution having a single fixed point 0_r. Now consider the triangles containing the point 0_r. There is an odd number of these and so one of them, without loss of generality $\{0_r, 0_c, 0_e\}$, must be fixed. Hence, the transposition $(0_c\ 0_e)$ is part of the involution. Consequently, since every automorphism of $K_{n,n,n}$ must preserve the tripartition, the automorphism will map a row point to a row point, a column point to an entry point and an entry point to a column point. Therefore, such an automorphism will be of the form

$$(0_r)\,((x_1)_r\,(x_2)_r)\,\ldots\,((x_{n-2})_r\,(x_{n-1})_r)\,(0_c\,0_e)\,((y_1)_c\,(z_1)_e)\,\ldots\,((y_{n-1})_c\,(z_{n-1})_e)$$

It further follows that the edge through which the axis passes is of the form $\{\alpha_c, \beta_e\}$.

But if $n \geq 3$, such an automorphism cannot exist. Assume that this automorphism maps u_c to v_e and vice versa, where $u \neq \alpha$ and $v \neq \beta$. Then the edge u_cv_e exists somewhere on the boundary of the large face on one side of the axis. Since this automorphism is a reflection, the edge v_eu_c must also exist on the other side of the axis. This means the same edge appears twice on the boundary of the large face, a contradiction. □

So if $n \geq 3$, all automorphisms are orientation-preserving and we now consider these. Since the action of any such automorphism on the boundary of the large face is a rotation, the group is cyclic and its order must divide $3n^2$, the number of edges in the large face. Orientation-preserving automorphisms will be of three types:

(1) those that preserve all three parts,
(2) those that fix one part and interchange the other two,
(3) those that cyclically permute all three parts.

Consider first orientation-preserving automorphisms that preserve all three parts (the other two types will be dealt with later). Let G be the group of these automorphisms.

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Then as observed above, $G = \mathbb{Z}_m$ and $m \mid 3n^2$. However, since G preserves all three parts, it follows that $m \mid n^2$. But in fact, we can prove that $m \mid n$.

PROPOSITION 3.3. *Let $G = \mathbb{Z}_m$ be the group of orientation and part-preserving automorphisms of the orientable upper embedding of a Latin square of order n. Then $m \mid n$.*

Proof. Let $n > 1$ and denote the orientable upper embedding of the Latin square by M. To obtain further restrictions on m, we will replace M with a related map on which G will act freely and use the elementary theory of regular coverings.

Let T be the truncation of M. *Truncation* refers to the substitution of every vertex v of the embedding by a cycle of order $deg(v)$. This truncation has $3n$ *yellow* faces of length $2n$ that arise by truncating each of the $3n$ vertices of the original map, n^2 *green* faces of length 6 that arise from the n^2 triangular faces of M, and one *white* face of length $6n^2$ arising from the large face of M. The cyclic group G clearly acts freely on the vertex set of T. Since G preserves each part of $K_{n,n,n}$ and no triangle of $K_{n,n,n}$ has two vertices from the same part, no two distinct vertices of a yellow face in T can be mapped onto each other by the action of G on T.

Consider now the quotient map $M' = T/G$ whose vertices, edges, and faces are G-orbits of the vertices, edges, and faces of T. The conclusion of the previous paragraph implies that M' has n^2/m *green* hexagonal faces arising from the n^2 green faces of T. The action of G on the white face of T leaves one *white* face of length $6n^2/m$ in M'. The next step is to determine what happens to the $3n$ yellow faces of length $2n$ in T when passing to the quotient M'. For each vertex v of $K_{n,n,n}$, let G_v be the stabilizer of v in the action of G on vertices of $K_{n,n,n}$ and let $|G_v| = m_v$. Being a subgroup of a cyclic group, each G_v must be cyclic, and the natural covering $T \to M'$ maps a G-orbit consisting of m/m_v yellow faces in T onto a single *yellow* face in M' of length $2n/m_v$. As an aside, observe that m_v must be a divisor of n, since G_v acts freely as a cyclic group of order m_v on the n triangular faces of the original map M incident with v.

By Theorem 2.2.2 of [**3**], we know that the regular covering $T \to M'$ induced by the free action of G can be reconstructed by means of a lift with the help of an ordinary voltage assignment α on the darts of M' in the group G. In the reconstruction process, we will use elementary properties of regular coverings as listed in [**3**]. The net voltage on each of the n^2/m green faces of M' must be zero, as each of them lifts onto m green faces of T of the same length. For each vertex v of $K_{n,n,n}$, the yellow face of M' of length $2n/m_v$ lifts onto m/m_v yellow faces of T of length $2n$. Therefore, the net voltage on such a yellow face of M' must be an element of G of order m_v. Finally, the net voltage on the white face of M' must be an element of G of order m because this face of length $6n^2/m$ lifts onto the white face of T of length $6n^2$. Here and in what follows, we assume that all the net voltages are calculated with respect to a fixed orientation of the supporting surface of the map M'. Of course, the net voltages in our case do not depend on choosing the initial point on a cycle because our voltage group G is Abelian.

Since the sum of the net voltages on all faces of M' is zero, the above analysis implies that the negative of the net voltage w on the white face is equal to the sum S of the net voltages on all yellow faces of M'. The element w has order m in G, hence so has S. Observe that all summands in S have orders m_v where v ranges over a set O of representatives of the orbits of G on the vertex set of $K_{n,n,n}$. But the elements of $G \cong \mathbb{Z}_m$ of order m_v have precisely the form $(m/m_v)t_v$ where $\gcd(m_v, t_v) = 1$ and $1 \le t_v < m_v$.

It follows that S can be expressed in the form

$$S = \sum_{v \in O} \frac{m}{m_v} t_v \text{ in } \mathbb{Z}_m.$$

In the field of rationals, this means that $S = rm + s$ where r and s are integers and $\gcd(s, m) = 1$. Let $m = ac$ and $n = bc$ where a, b, and c are positive integers and $\gcd(a, b) = 1$. Recalling that $m_v \mid n$ for each $v \in O$, we have

$$S = \sum_{v \in O} \frac{m}{m_v} t_v = \frac{m}{n} \sum_{v \in O} \frac{n}{m_v} t_v = \frac{a}{b} j,$$

where j is an integer. Since a and b are relatively prime, it follows that j/b is an integer and so $S = rm + s$ is divisible by a. But also $a \mid m$. Hence, $a \mid s$ but since s and m are relatively prime this is possible only if $a = 1$. Consequently, $m \mid n$, as claimed. □

PROPOSITION 3.4. *The cyclic group $\mathbb{Z}_m$, where $m \mid n$, does not act freely on all three parts of the embedding.*

Proof. Suppose that $\mathbb{Z}_m$ does act freely on each part of the embedding. The quotient of the embedding under the action of $\mathbb{Z}_m$ is $K^m_{n/m,n/m,n/m}$ where the superscript m denotes that every edge has multiplicity m., i.e. embedded with n^2/m triangles and one large face of length $3n^2/m$. It follows from Theorem 2.2.2 of [**3**] that the original upper embedding of the Latin square of order n can be reconstructed by lifting this quotient embedding. To do so, we need a voltage assignment on $K^m_{n/m,n/m,n/m}$ in $\mathbb{Z}_m$ such that the voltages on the triangles sum to zero while the voltages on the large face sum to an element relatively prime to m. Since the large face consists of all the edges in the embedding which also form the triangles, the voltage sum will always be zero. Contradiction. □

However, the group can act freely on two of the parts. The following is an example. In it, both m and n are equal to 3 and the cyclic group $\mathbb{Z}_n$ acts freely on two of the parts and fixes the third.

EXAMPLE 3.5. Consider the cyclic Latin square of order 3.

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

An upper embedding of the above Latin square has the following rotation scheme:

$$\begin{array}{lll}
0_r : 0_c0_e\ 1_c1_e\ 2_c2_e & 0_c : 0_e0_r\ 1_e1_r\ 2_e2_r & 0_e : 0_r0_c\ 1_r2_c\ 2_r1_c \\
1_r : 0_c1_e\ 1_c2_e\ 2_c0_e & 1_c : 0_e2_r\ 1_e0_r\ 2_e1_r & 1_e : 0_r1_c\ 1_r0_c\ 2_r2_c \\
2_r : 0_c2_e\ 1_c0_e\ 2_c1_e & 2_c : 0_e1_r\ 1_e2_r\ 2_e0_r & 2_e : 0_r2_c\ 2_r0_c\ 1_r1_c
\end{array}$$

The large face is

$$0_e\ 0_r\ 1_c\ 2_e\ 0_r\ 0_c\ 1_e\ 2_r\ 0_c\ 0_e\ 1_r\ 0_c\ 2_e\ 1_r\ 2_c\ 1_e\ 0_r\ 2_c\ 0_e\ 2_r\ 2_c\ 2_e\ 2_r\ 1_c\ 1_e\ 1_r\ 1_c$$

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The action of the automorphism group isomorphic to $\mathbb{Z}_3$ is

$$i_e \mapsto i_e, i_r \mapsto (i+1)_r, i_c \mapsto (i+2)_c, 0 \leq i \leq 2.$$

We can generalize the above example to any cyclic Latin square of odd order n. The rotation scheme and the large face of the upper embedding will be as follows:

$i_r : \ldots\ j_c(i+j)_e\ (j+1)_c(i+j+1)_e\ \ldots$

$j_c : \ldots\ k_e(k-j)_r\ (k+1)_e(k+1-j)_r\ \ldots$

$k_e : \ldots\ i_r(k-i)_c\ (i+1)_r(k-i-1)_c\ \ldots, k \neq -1$

$(-1)_e : \ldots\ i_r(-i-1)_c\ (i+t)_r(-i-1-t)_c\ \ldots, t \neq 1, (t,n)=1, (t-1,n)=1$

Such a value of t always exists. For example, we can take $t = 2$.

The large face is

$$\begin{array}{ccccccccccccc}
(-1)_e & t_r & (-t)_c & 1_e & (t+2)_r & (-t)_c & 3_e & (t+4)_r & (-t)_c & \ldots & (-3)_e & (t-2)_r & (-t)_c \\
(-1)_e & (2t-1)_r & (-2t+1)_c & 1_e & (2t+1)_r & (-2t+1)_c & 3_e & (2t+3)_r & (-2t+1)_c & \ldots & (-3)_e & (2t-3)_r & (-2t+1)_c \\
(-1)_e & (3t-2)_r & (-3t+2)_c & 1_e & (3t)_r & (-3t+2)_c & 3_e & (3t+2)_r & (-3t+2)_c & \ldots & (-3)_e & (3t-4)_r & (-3t+2)_c \\
 & \vdots & & & \vdots & & & \vdots & & & & \vdots & \\
(-1)_e & 1_r & (-1)_c & 1_e & 3_r & (-1)_c & 3_e & 5_r & (-1)_c & \ldots & (-3)_e & (-1)_r & (-1)_c
\end{array}$$

The action of the automorphism group isomorphic to $\mathbb{Z}_n$ is

$$i_e \mapsto i_e, i_r \mapsto (i-t)_r, i_c \mapsto (i+t), 0 \leq i \leq n-1.$$

So to summarize the results so far, we have shown that the group G of orientation-preserving and part-preserving automorphisms of an orientable upper embedding of a Latin square of order n is cyclic $\mathbb{Z}_m$ where $m \mid n$. Further the case $m = n$ is achieved. The cyclic group $\mathbb{Z}_m$ cannot act freely on all three parts of the embedding but can act freely on two of the three parts. In the construction given, the action of the group $\mathbb{Z}_n$ can be described by the notation $n^1 n^1 1^n$ and when $n = p$ is prime, this is the only possibility.

Now consider automorphisms of the other two types. We prove two further results.

PROPOSITION 3.6. *Automorphisms which fix one part and interchange the other two do not exist.*

Proof. Let ϕ be such an automorphism. Then ϕ^2 fixes all three parts. It follows that ϕ has even order. But all automorphisms are of odd order, a contradiction. $\square$

PROPOSITION 3.7. *Automorphisms which cyclically permute all three parts have order 3.*

Proof. Let θ be such an automorphism. Then θ^3 fixes all three parts. Suppose that θ^3 has an orbit of length i in one part and of length j in a second part where $j < i$. If x is an element of the orbit of length j in the second part, then $\theta^{3j}(x) = x$. Further $\theta^{3j}(\theta(x)) = \theta(\theta^{3j}(x)) = \theta(x)$. So θ^{3j} stabilizes vertices in different parts. But θ^{3j} is not the identity because $j < i$. This proves that all orbits of θ^3 have the same length, say m, which must be the order of θ^3. Thus, the group generated by θ^3 acts freely on all three parts which is a contradiction by Proposition 3.4, unless θ^3 is the identity. Hence, any automorphism which permutes the parts cyclically must have order 3. $\square$

We assume, without loss of generality, that the automorphism θ which permutes the parts cyclically is of the form $\prod_{i=0}^{n-1}(i_r\ i_c\ i_e)$ since any other automorphism will give a Latin square isotopic to the Latin square obtained by θ. Note that in a Latin square with such an automorphism, if $\{x_r, y_c, z_e\}$ is a triple, then $\{x_c, y_e, z_r\}$ and $\{x_e, y_r, z_c\}$ must also be triples. This is equivalent to a Latin square obtained from a quasi-group having the *semi-symmetric property*, i.e. $xy = z \implies yz = x \implies zx = y$. An example for $n = 5$ is the following.

EXAMPLE 3.8. Consider the Latin square of order 5.

	0	1	2	3	4
0	0	3	4	1	2
1	3	2	1	0	4
2	4	1	3	2	0
3	1	0	2	4	3
4	2	4	0	3	1

An upper embedding of the above Latin square with an automorphism of order 3 which permutes the parts cyclically has the following rotation scheme:

$$
\begin{array}{lll}
0_r : 0_c0_e\ 4_c2_e\ 3_c1_e\ 2_c4_e\ 1_c3_e & 0_c : 0_e0_r\ 4_e2_r\ 3_e1_r\ 2_e4_r\ 1_e3_r & 0_e : 0_r0_c\ 4_r2_c\ 3_r1_c\ 2_r4_c\ 1_r3_c \\
1_r : 0_c3_e\ 4_c4_e\ 3_c0_e\ 1_c2_e\ 2_c1_e & 1_c : 0_e3_r\ 4_e4_r\ 3_e0_r\ 1_e2_r\ 2_e1_r & 1_e : 0_r3_c\ 4_r4_c\ 3_r0_c\ 1_r2_c\ 2_r1_c \\
2_r : 0_c4_e\ 4_c0_e\ 2_c3_e\ 3_c2_e\ 1_c1_e & 2_c : 0_e4_r\ 4_e0_r\ 2_e3_r\ 3_e2_r\ 1_e1_r & 2_e : 0_r4_c\ 4_r0_c\ 2_r3_c\ 3_r2_c\ 1_r1_c \\
3_r : 0_c1_e\ 4_c3_e\ 2_c2_e\ 1_c0_e\ 3_c4_e & 3_c : 0_e1_r\ 4_e3_r\ 2_e2_r\ 1_e0_r\ 3_e4_r & 3_e : 0_r1_c\ 4_r3_c\ 2_r2_c\ 1_r0_c\ 3_r4_c \\
4_r : 0_c2_e\ 4_c1_e\ 3_c3_e\ 2_c0_e\ 1_c4_e & 4_c : 0_e2_r\ 4_e1_r\ 3_e3_r\ 2_e0_r\ 1_e4_r & 4_e : 0_r2_c\ 4_r1_c\ 3_r3_c\ 2_r0_c\ 1_r4_c
\end{array}
$$

The large face is
$0_e\ 0_r\ 4_c\ 1_e\ 3_r\ 4_c\ 2_e\ 4_r\ 4_c\ 0_e\ 1_r\ 1_c\ 0_e\ 2_r\ 2_c\ 1_e\ 2_r\ 0_c\ 3_e\ 3_r\ 2_c\ 3_e\ 1_r\ 4_c\ 3_e\ 0_r\ 0_c\ 4_e\ 1_r\ 3_c\ 4_e\ 2_r\ 4_c$
$4_e\ 0_r\ 1_c\ 1_e\ 0_r\ 2_c\ 2_e\ 1_r\ 2_c\ 0_e\ 3_r\ 3_c\ 2_e\ 3_r\ 1_c\ 4_e\ 3_r\ 0_c\ 0_e\ 4_r\ 1_c\ 3_e\ 4_r\ 2_c\ 4_e\ 4_r\ 0_c\ 1_e\ 1_r\ 0_c\ 2_e\ 2_r\ 1_c$
$2_e\ 0_r\ 3_c\ 3_e\ 2_r\ 3_c\ 1_e\ 4_r\ 3_c\ 0_e\ 0_r$

Possibly more instructive and interesting is the example below. We need a further definition. If the Latin square is also idempotent, i.e. $xx = x$, the quasi-group corresponds to a *Mendelsohn triple system*, MTS(n). This is an ordered pair $(V, \mathcal{B})$ where V is a base set of cardinality n and $\mathcal{B}$ is a collection of cyclically ordered triples (x, y, z) which have the property that every ordered pair of distinct elements of V is contained in precisely one triple. Such systems exist if and only if $v \equiv 0, 1 \pmod 3$, $v \neq 6$, [**4**]. The quasi-group is defined by the operation $xx = x$ and $xy = z$ where (x, y, z) is a cyclically ordered triple and is called a *Mendelsohn quasigroup*.

EXAMPLE 3.9. Let $V = \mathbb{Z}_7$ and $\mathcal{B} = \{(i, 1+i, 3+i),\ (i, 3+i, 2+i) : 0 \le i \le 6\}$. Then $(V, \mathcal{B})$ is an MTS(7). The Latin square obtained from the Mendelsohn quasi-group has an upper embedding with an automorphism of order 3 which permutes the parts cyclically, defined by the following rotation scheme:

$$
\begin{array}{l}
0_r : 0_c0_e\ 6_c4_e\ 5_c1_e\ 4_c5_e\ 3_c2_e\ 2_c6_e\ 1_c3_e \\
1_r : 1_c1_e\ 0_c5_e\ 6_c2_e\ 5_c6_e\ 4_c3_e\ 3_c0_e\ 2_c4_e \\
2_r : 2_c2_e\ 1_c6_e\ 0_c3_e\ 6_c0_e\ 5_c4_e\ 4_c1_e\ 3_c5_e \\
3_r : 3_c3_e\ 2_c0_e\ 1_c4_e\ 0_c1_e\ 6_c5_e\ 5_c2_e\ 4_c6_e \\
4_r : 4_c4_e\ 3_c1_e\ 2_c5_e\ 1_c2_e\ 0_c6_e\ 6_c3_e\ 5_c0_e \\
5_r : 5_c5_e\ 4_c2_e\ 3_c6_e\ 2_c3_e\ 1_c0_e\ 0_c4_e\ 6_c1_e \\
6_r : 6_c6_e\ 1_c5_e\ 2_c1_e\ 4_c0_e\ 0_c2_e\ 5_c3_e\ 3_c4_e
\end{array}
$$

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The rotations about the points i_c and i_e, $0 \leq i \leq 6$, are obtained by applying the automorphism. Note that the cyclic automorphism of order 7 of the Mendelsohn triple system does not extend to the embedding. In fact, no orientable upper embedding of a Latin square can have a non-trivial part-preserving automorphism as well as an automorphism which permutes the parts cyclically. The reason for this is as follows. First recall that, since the action of the automorphism group on the boundary of the large face is a rotation, the automorphism group G of the upper embedding is cyclic. Let α be an automorphism which permutes the parts cyclically. By Proposition 3.7, $\alpha^3 = i$ (the identity) and thus α and α^2 are the only elements of order 3 in G. Now let β be a non-trivial part preserving automorphism. Then $\alpha\beta$ also permutes the parts cyclically. So $\alpha\beta = \alpha^2$, a contradiction.

We summarize the results in this section by the theorem below.

THEOREM 3.10. *Let ϕ be a non-trivial automorphism of an orientable upper embedding of a Latin square of order n. Then ϕ is orientation-preserving and either preserves all three parts of the vertex set partition or cyclically permutes them. In the former case, the group G of all such automorphisms is equal to $\mathbb{Z}_m$ where $m \mid n$ and does not act freely on all three parts, though it can do so on two of the parts. In the latter case, $G = \mathbb{Z}_3$.*

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