1 QFT in Zero Dimensions

1.1 Introduction

It makes no sense to talk about elementary particles without either special relativity or quantum mechanics. For now, we concentrate on the quantum-mechanical nature of nature[1]. The fundamental object associated with particles is a *quantum field*.[2] Such a field assigns one or more numbers to every point in spacetime. So it is a pretty complicated thing; its behavior cannot be described trivially, especially since it also undergoes quantum fluctuations. It is a good idea to first build up some expertise, in a more controllable situation. Therefore we shall simplify the whole four-dimensional spacetime arena of particle physics. We shall reduce spacetime to a single point, a *zero-dimensional* arena.[3] Now we have only a single point to assign numbers to, and the simplest quantum field is a single stochastic, or *random*, number. We can already learn many of the techniques of quantum field theory studying this simple case! We shall meet path integrals, Green's functions, the Schwinger–Dyson equation, Feynman diagrams and the effective action, in a quite natural way.

1.2 Probabilistic Considerations

1.2.1 Green's Functions and the Path Integral

Let us imagine a quantum field φ that can take on all real values from $-\infty$ to $+\infty$. Since it is a random variable, the most we can hope to specify about it is its probability density $P(\varphi)$,[4] which we write as[5]

$$P(\varphi) = N\ \exp\Big(-S(\varphi)\Big)\,,\quad N^{-1} = \int \exp\Big(-S(\varphi)\Big)\ d\varphi. \tag{1.1}$$

1 In a moment you will understand why relativity does not enter – yet.

2 In many treatments quantum fields are considered to be *distribution-valued operators*. In this book I am not interested in the internal details of quantum states, but rather in the scattering amplitudes; we shall adopt Feynman's approach and use what are called *c-number fields*.

3 That's why.

4 This is what it *means* to be a random variable.

5 If not explicitly indicated otherwise, integrals run from $-\infty$ to $+\infty$.

The function $S(\varphi)$ is called the *action* of the particular quantum field theory; it *defines* the theory. For the probability density to be acceptable, $S(\varphi)$ must go to infinity sufficiently fast as $|\varphi| \to \infty$.[6] Since the quantum field is a random variable, the most that can be known about it[7] is the collection of its moments, in the jargon called *Green's functions*:[8]

$$G_n \equiv \left\langle \varphi^n \right\rangle \equiv N \int \exp\Big(-S(\varphi)\Big)\ \varphi^n\, d\varphi, \quad n = 0, 1, 2, 3, \ldots . \tag{1.2}$$

We shall assume that G_n exists for all n. By construction, we must always have

$$G_0 = \left\langle \varphi^0 \right\rangle = \langle 1 \rangle = 1. \tag{1.3}$$

The most fruitful way[9] of discussing the set of all Green's functions is in terms of a *generating function*:

$$Z(J) = \sum_{n \geq 0} \frac{1}{n!} J^n\ G_n. \tag{1.4}$$

This is called the *path integral*, for reasons that will become clear later. It can be written as

$$Z(J) = N \int d\varphi\ P_J(\varphi), \quad P_J(\varphi) = \exp\Big(-S(\varphi) + J\varphi \Big). \tag{1.5}$$

The number J, which here serves purely as a device to distinguish the various Green's functions, is called a *source*, again for reasons that will become apparent later. Once $Z(J)$ is known, an individual Green's function is extracted by differentiation:

$$G_n = \left\lfloor \frac{\partial^n}{(\partial J)^n} Z(J) \right\rfloor_{J=0}. \tag{1.6}$$

The path integral $Z(J)$ contains all the information about the Green's functions, and hence about the probability density $P(\varphi)$. The same information is, therefore, *also* contained in its logarithm. We write

$$W(J) = \log Z(J) \equiv \sum_{n \geq 1} \frac{1}{n!} J^n\ C_n. \tag{1.7}$$

[6] Barring elaborate unnatural counter examples, of course; see the remarks on paralysis in the introduction.

[7] You are here approaching a **career decision**! You may decide simply to *measure* the value of φ: in that case you have decided to become an *experimentalist* rather than a *theorist*.

[8] We need to clear up, in advance, a possible confusion. In this book, the Green's functions are simply *defined* to be expectation values. This may appear to contrast with the use of Green's functions in the solution of inhomogeneous linear differential equations such as are encountered in classical electrodynamics, where we use them to compute the electromagnetic field configurations for given sources. The difference is only apparent since, as we shall recognize, the latter type of Green's functions are in our treatment simply the *two-point connected Green's functions*; and for theories such as electrodynamics, where the electromagnetic fields do not undergo self-interaction, the two-point functions are in fact the *only* nonzero connected Green's functions. Be not, therefore, misled into thinking that there are somehow two sorts of Green's functions. The Green's function formulation of electrodynamics will in fact appear as the classical limit of the Schwinger–Dyson equation discussed below.

[9] Kids! Do this at home. Whenever an infinite collection of objects with some kind of relation between them occurs, generating functions are *always* a good idea.

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The quantities C_n (with, obviously $C_0 = 0$ since $G_0 = 1$) are called the *connected Green's functions* of the theory, and will play an important rôle in what follows. The connected
1 Green's functions can be recognized to be the *cumulants* of the probability density:

$$\begin{aligned} &C_0 = 0 && \text{by normalization,} \\ &C_1 = \langle\varphi\rangle && \text{the mean,} \\ &C_2 = \left\langle(\varphi - \langle\varphi\rangle)^2\right\rangle && \text{the variance,} \\ &C_3 = \left\langle(\varphi - \langle\varphi\rangle)^3\right\rangle && \text{the skewness,} \end{aligned} \tag{1.8}$$

and so on. Since $W(0) = C_0 = 0$, all information about the probability density is *also* contained in its derivative, the *field function*:

$$\phi(J) \equiv \frac{\partial}{\partial J} W(J) = \sum_{n \geq 0} \frac{1}{n!} J^n \, C_{n+1}. \tag{1.9}$$

Since from its definition, we have

$$\phi(J) = \left[\int d\varphi \; \varphi \; P_J(\varphi)\right] \left[\int d\varphi \; P_J(\varphi)\right]^{-1}, \tag{1.10}$$

we can say that $\phi(J)$ is the expectation value of the quantum field φ in the presence of sources: to denote this, we might write

$$\phi(J) = \langle\varphi\rangle_J, \tag{1.11}$$

which explains the similar typographies for the quantum field and the field function.[10] We should not, however, forget the difference in status of these objects! φ is the *physical* entity, an unknowable, fluctuating *random* field; but $\phi(J)$ is a perfectly well-defined *function* that contains all the information about the *probability density* of φ and is *computable* once the action is given.[11]

1.2.2 The Free Theory

The simplest probability density is probably[12] the Gaussian one, given by the action

$$S(\varphi) = \frac{1}{2} \mu \, \varphi^2, \tag{1.12}$$

with μ a positive real number. For any action, we shall call the part quadratic in the fields (or bilinear in the case of several fields) the *kinetic part*. This action, called the *free action*, consists of *only* a kinetic part. The path integral is now simply computed by

$$\begin{aligned} Z(J) &= N \int \exp\left(-\frac{1}{2}\mu\varphi^2 + J\varphi\right) \; d\varphi \\ &= N \int \exp\left(-\frac{1}{2}\mu \left(\varphi - \frac{J}{\mu}\right)^2 + \frac{J^2}{2\mu}\right) \; d\varphi = \exp\left(\frac{J^2}{2\mu}\right). \end{aligned} \tag{1.13}$$

10 Try this out: "phi of J equals phi with J".
11 In principle, if not in practice easily or completely.
12 A uniform density may be thought even simpler, but then it cannot run from $\varphi = -\infty$ to $\varphi = +\infty$. As a matter of fact, ask a mathematician or physicist to name you a nice probability density over the whole real line, and she will almost certainly suggest the Gaussian.

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It is not even necessary[13] to actually calculate the value of N. By Taylor expansion of the exponential, we immediately find that

$$G_{2n} = \frac{(2n)!}{(2\,\mu)^n n!}, \quad G_{2n+1} = 0, \quad n = 0, 1, 2, \ldots . \tag{1.14}$$

The connected Green's functions follow from

$$W(J) = \log Z(J) = \frac{J^2}{2\mu}, \quad \phi(J) = \frac{J}{\mu}, \tag{1.15}$$

so that the only nonvanishing connected Green's function is $C_2 = 1/\mu$. The fact that here only the two-point connected Green's function is nonvanishing is the reason for calling this model the free theory. Again, things will become clearer later on, in a more realistic spacetime – after all, what does "freedom" mean if you are confined to a single point?

1.2.3 The φ^4 Model and Perturbation Theory

An action $S(\varphi)$ may contain other terms than just the quadratic one. Such terms are called *interaction terms*: they may be linear, but more usually they are of higher power in the field φ. The simplest acceptable interacting theory in our probabilistic setting is therefore given by the action

$$S(\varphi) = \frac{1}{2}\mu\varphi^2 + \frac{1}{4!}\lambda_4\varphi^4. \tag{1.16}$$

The (nonnegative!) real number λ_4 is called a *coupling constant*: this model is called the
2 φ^4 theory.[14] Computing the path integral is now a much less trivial matter. A possible approach is to assume that, in *some* sense, the φ^4 theory is close to a free theory, that is, in the same *some* sense, λ_4 is a small number. We can then expand the probability density in powers of λ_4:

$$\exp(-S(\varphi)) = \exp\left(-\frac{1}{2}\mu\varphi^2\right) \sum_{k\geq 0} \frac{1}{k!} \left(-\frac{\lambda_4}{24}\right)^k \varphi^{4k}. \tag{1.17}$$

This procedure is called *perturbation theory*. Having thus reduced the problem to the previous case of the free theory, we cavalierly[15] interchange the series expansion in λ_4 with the integration over φ and arrive at the following expression for the Green's functions:

$$\begin{aligned} G_{2n} &= H_{2n}/H_0, \\ H_{2n} &= \frac{1}{\mu^n} \sum_{k\geq 0} \frac{(4k+2n)!}{2^{2k+n}(2k+n)!\,k!} \left(-\frac{\lambda_4}{24\mu^2}\right)^k . \end{aligned} \tag{1.18}$$

[13] Because we must always have $Z(0) = 1$.

[14] An action in which φ^3 is the highest power does not lead to a convergent integral over the real axis (see, however, Appendix F). Of course, an action of the form $S(\varphi) = \mu\varphi^2/2 + \lambda_3\varphi^3/3! + \lambda_4\varphi^4/4!$ is perfectly acceptable, and we shall consider this "$\varphi^{3/4}$ model" later on.

[15] This interchange does not come without its price: see Appendix A. *Nemo me impune lacessit*.

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For example, we have

$$
\begin{aligned}
H_0 &= 1 - \frac{1}{8}u + \frac{35}{384}u^2 - \frac{385}{3072}u^3 + \cdots, \\
1/H_0 &= 1 + \frac{1}{8}u - \frac{29}{384}u^2 + \frac{107}{1024}u^3 + \cdots, \qquad (1.19)
\end{aligned}
$$

3 with $u \equiv \lambda_4/\mu^2$. In this theory, also the normalization N has to be treated perturbatively, which explains the expression for $1/H_0$. For the first few nonvanishing Green's functions we find

$$
\begin{aligned}
G_0 &= 1, \\
G_2 &= \frac{1}{\mu}\left(1 - \frac{1}{2}u + \frac{2}{3}u^2 - \frac{11}{8}u^3 + \cdots\right), \\
G_4 &= \frac{1}{\mu^2}\left(3 - 4u + \frac{33}{4}u^2 - \frac{68}{3}u^3 + \cdots\right), \\
G_6 &= \frac{1}{\mu^3}\left(15 - \frac{75}{2}u + \frac{445}{4}u^2 - \frac{1585}{4}u^3 + \cdots\right). \qquad (1.20)
\end{aligned}
$$

The corresponding nonzero connected Green's functions are given by

$$
\begin{aligned}
C_2 &= \frac{1}{\mu}\left(1 - \frac{1}{2}u + \frac{2}{3}u^2 - \frac{11}{8}u^3 + \cdots\right), \\
C_4 &= \frac{1}{\mu^2}\left(-u + \frac{7}{2}u^2 - \frac{149}{12}u^3 + \cdots\right), \\
C_6 &= \frac{1}{\mu^3}\left(10u^2 - 80u^3 + \cdots\right). \qquad (1.21)
\end{aligned}
$$

Whereas the Green's functions all have a perturbation expansion starting with terms containing no λ_4, the connected Green's functions of increasing order are also of increasingly high order in λ_4: the higher connected Green's functions need more interactions than the lower ones.

1.2.4 The Schwinger–Dyson Equation

Although the path integral is, generally, a very complicated function of J, we can easily find an equation that describes it completely. This is the *Schwinger–Dyson equation* (SDe), which we construct as follows. Let the action be given by the general expression

$$
S(\varphi) = \sum_{k\geq 1} \frac{1}{k!}\,\lambda_k\,\varphi^k, \qquad (1.22)
$$

where $\lambda_2 = \mu$.[16] Now, from the observation that

$$
\frac{\partial^p}{(\partial J)^p}Z(J) = N\int \exp\Big(-S(\varphi) + J\varphi\Big)\;\varphi^p\,d\varphi, \quad p = 0,1,2,3,\ldots, \qquad (1.23)
$$

[16] The sum starts at 1 since a constant, φ-independent term in the action is always immediately swallowed up by the normalization factor N.

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we immediately deduce that

$$\left[-J+\sum_{k\geq 0}\frac{\lambda_{k+1}}{k!}\frac{\partial^k}{(\partial J)^k}\right]Z(J)$$
$$= N\int \exp\Big(-S(\varphi)+J\varphi\Big)\left[-J+\sum_{k\geq 0}\frac{\lambda_{k+1}}{k!}\varphi^k\right]d\varphi$$
$$= N\int \exp\Big(-S(\varphi)+J\varphi\Big)\Big[S'(\varphi)-J\Big]\,d\varphi \;=\; 0, \tag{1.24}$$

where in the last lemma we have recognized a total derivative, and used the fact that the integrand vanishes at the endpoints. Symbolically, we may write the SDe as

$$\left\lfloor \frac{\partial}{\partial\varphi}S(\varphi)\right\rfloor_{\varphi=\partial/\partial J} Z(J) = S'\left(\frac{\partial}{\partial J}\right)Z(J) = JZ(J). \tag{1.25}$$

4 For our sample model, the φ^4 theory, the SDe reads[17]

$$\frac{1}{6}\lambda_4 Z'''(J)+\mu Z'(J)-JZ(J)=0. \tag{1.26}$$

Using the series expansion of the path integral, we can express this as a relation between different Green's functions:

$$\frac{\lambda_4}{6}G_{n+3}+\mu G_{n+1}-nG_{n-1}=0,\quad n\geq 1. \tag{1.27}$$

This relation may usefully be rewritten as follows:

$$G_n=\frac{1}{\mu}\left((n-1)G_{n-2}-\frac{\lambda_4}{6}G_{n+2}\right),\quad n\geq 2. \tag{1.28}$$

If we start by assigning to the Green's functions the values $G_n = \delta_{0,n}$, then repeated applications of Eq. (1.28) will precisely reproduce the Green's functions of Eq. (1.20).[18]

1.2.5 The Schwinger–Dyson Equation for the Field Function

From the definition of $\phi(J)$ as the derivative of the logarithm of the path integral, we can infer that

$$\frac{1}{Z(J)}\frac{\partial^p}{(\partial J)^p}Z(J)=\left(\phi(J)+\frac{\partial}{\partial J}\right)^p e(J). \tag{1.29}$$

Here, $e(J)$ is the unit function: $e(J)\equiv 1$. We immediately arrive at the form of the SDe for
5 the field function:

[17] The SD equation is, in general, of higher than the first order. It therefore has several independent solutions, only *one* of which corresponds to the usual perturbative expansion. The nature of the other solutions is discussed in Appendix F.

[18] The correct way to do this is to subsequently evaluate $G_2, G_4, G_6, \ldots$. On the first iteration, the lowest-order expressions are obtained. Each subsequent iteration gives one higher order in perturbation theory. If we want to obtain the kth order term in G_n, the $(k+1)$th order term in G_{n+2} is needed, and so on. It is therefore necessary to compute the lower-order terms for more G_ns.

$$S'\left(\phi(J) + \frac{\partial}{\partial J}\right) e(J) = J. \tag{1.30}$$

For the $\varphi^{3/4}$ theory, it reads

$$\begin{aligned}\phi(J) = \frac{J}{\mu} - \frac{\lambda_3}{2\mu}\left(\phi(J)^2 + \frac{\partial}{\partial J}\phi(J)\right) \\ -\frac{\lambda_4}{6\mu}\left(\phi(J)^3 + 3\phi(J)\frac{\partial}{\partial J}\phi(J) + \frac{\partial^2}{(\partial J)^2}\phi(J)\right).\end{aligned} \tag{1.31}$$

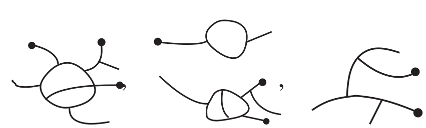
6 Although this leads to very nonlinear relations between the various connected Green's
functions this form of the SD equation is actually even simpler to apply with $\phi(J) = 0$ as
7 a starting point, iterating the assignment (1.31) then results[19] in the correct form of $\phi(J)$,
8 giving the connected Green's functions of Eq. (1.21).

1.3 Diagrammatics

1.3.1 Feynman Diagrams

There exists an extremely useful toolbox for computing Green's functions and connected Green's functions: *Feynman diagrams*. In this section we shall first introduce these diagrams and their concomitant *Feynman rules*. Only after that shall we prove that these diagrams do, indeed, correctly describe Green's functions.

Feynman diagrams are constructs of *lines* and *vertices*. A vertex is a meeting point for one or more lines.[20] Diagrams are allowed in which one or more lines do not end in a vertex but, in a sense move off toward infinity: such lines are called *external lines*. Lines that are not external lines, and end up at vertices at both ends, are called *internal lines*. Diagrams may be *connected*, in which case one can move between any two points in the diagram following lines of that diagram; or they may be *disconnected*, in which case it consists of two or more disjoint pieces that are themselves connected. Any graph[21] consists of a finite number of connected subgraphs. The "empty" graph $\mathcal{E}$, containing no lines or vertices whatsoever, also exists; it does not count as connected.[22] Diagrams containing one or more closed loops are perfectly allowed. Diagrams with no closed loops are called *tree diagrams*. A few examples are



[19] For this approach to work in practice, it turns out to be useful to truncate $\phi(J)$ as a power series in J, the truncation order increasing by one with each iteration. If you don't do this, each iteration *triples* the highest power in J, leading to very unwieldy expressions with only the first few terms being actually correct.

[20] In the mathematical world of *graph theory*, lines are often called *edges* for some reason.

[21] For *us*, the terms "diagram" and "graph" are interchangeable.

[22] Sophistry alert: it has no points between which to move.

where you see, respectively, a connected graph, a *dis*connected graph, and a connected *tree* graph. The precise *shape* of the lines and the precise *position* of the vertices are irrelevant. The important thing is the way in which the lines are connected to the vertices.[23]

1.3.2 Feynman Rules

The noteworthy thing about Feynman diagrams is that they have an algebraic interpretation; that is, they correspond to *numbers* that may be added and multiplied. The assignment of a number to a Feynman diagram is governed by the *Feynman rules*, which postulate a numerical object for every ingredient of a Feynman graph. In the simple zero-dimensional theories that we consider here the Feynman rules are just numbers. We will use the following rules:[24]

 $\leftrightarrow \dfrac{1}{\mu}$,  $\leftrightarrow -\lambda_3$,  $\leftrightarrow -\lambda_4$, $\leftrightarrow +J$. (1.32)

A vertex at which a single line ends (and which carries a Feynman rule factor $+J$) is called a *source vertex*. A disconnected diagram evaluates to the product of the values of its disjunct connected pieces. Because of this multiplicative rule, the value of the empty diagram $\mathcal{E}$ is taken to be unity. In addition, we assign to every Feynman diagram a *symmetry factor*. The symmetry factor is the single most nontrivial ingredient of the diagrammatic approach, so it deserves its own section.

1.3.3 Symmetries and Multiplicities

Feynman diagrams have, in general, an "inner" and an "outer" part. The "inner" part consists of the various vertices and internal lines: the "outer" part is made up from the external lines (if any). The inner part concomitates with the *symmetry factor* of the diagram, and for the outer part we have what may be called the *multiplicity*, to be discussed below. Let us first turn to the symmetry factor. The rules are as follows:

- for every set of k lines that may be permuted without changing the diagram, there will be a factor $1/k!$;
- for every set of m vertices that may be permuted without changing the diagram, there will be a factor $1/m!$;
- for every set of p disjunct connected pieces that maybe interchanged without changing the diagram, there will be a factor $1/p!$;
- a factor $1/k$ for every k-fold rotational symmetry;[25]
- a factor 1/2 for every mirror symmetry.

[23] Throughout this book I try to avoid drawing Feynman diagrams with straight lines, or to draw blobs or closed loops as circles. Many texts *do* employ only straight lines and circles. This not only leads to awfully unæsthetic-looking pictures, but is also deeply misleading. There is a (natural) tendency to look at Feynman diagrams with the idea that the lines represent "particles moving freely through space' so that the lines "ought" to be straight according to Newton's first law. This is completely wrong! In the zero-dimensional world we are dealing with for now, there cannot be any notion of movement yet, let alone any Newton to pronounce on it. In fact, Newton's first law ought to be *derived* from our theory, and we shall do so in due course.

[24] These will make the diagrams do just what we want, see below.

[25] Note: $1/k$, *not* $1/(k!)$.

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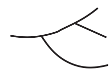
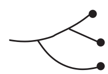
External lines *cannot* be permuted without changing the diagram.[26] Therefore only *vacuum diagrams*, that is diagrams without any external lines, can have a rotational symmetry. The symmetry factor cannot be read off from the individual components of the diagram, but depends on the topology of the whole diagram.[27] As our universe grows from zero to more dimensions, and as the particles considered acquire more properties, the Feynman rules will grow in complication; but the symmetry factors remain the same.[28]

Let us look at a few examples of diagram values. First, consider the diagram



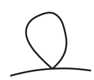
$$= \frac{\lambda_3{}^2}{\mu^5}. \tag{1.33}$$

In this case, the symmetry factor is 1, since for a tree diagram, no internal lines or vertices can be interchanged with impunity. The similar-looking diagram



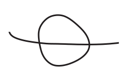
$$= \frac{1}{2}\frac{\lambda_3{}^2}{\mu^5}J^3. \tag{1.34}$$

has a symmetry factor 1/2! since the upper two one-point vertices are interchangeable. Then, there is the graph



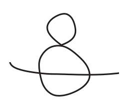
$$= -\frac{1}{2}\frac{\lambda_4}{\mu^3}.$$

Here, there is a symmetry factor 1/2 because the "leaf" can be flipped over without changing the diagram.[29] The diagram



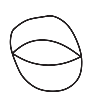
$$= \frac{1}{6}\frac{\lambda_4{}^2}{\mu^5}$$

carries a symmetry factor of 1/3! because the three internal lines are interchangeable. The graph



$$= -\frac{1}{4}\frac{\lambda_4{}^3}{\mu^7}$$

carries a symmetry factor (1/2!)(1/2!) since there are now only *two* interchangeable internal lines, and a single "leaf." Finally, the diagram



$$= \frac{1}{48}\frac{\lambda_4{}^2}{\mu^4}$$

9 has a symmetry factor (1/4!)(1/2!) since there are four equivalent internal lines, and moreover the diagram can be "flipped over" without changing it.

[26] Think of them as anchored somewhere very far away.

[27] This is what makes the automated evaluation of diagrams a nontrivial task: component factors of diagrams can be easily assigned, but working out the symmetry factor of a diagram calls for very complicated computer algorithms indeed.

[28] This is only modified if we include lines of different types, or *oriented lines*, that is lines that are deemed to run in a particular direction. Then again, the more-dimensional diagrams have the same symmetry factors as their zero-dimensional siblings.

[29] This is due to the fact that the line in the loop is not oriented: for oriented lines it will no longer hold.

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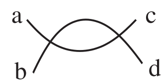
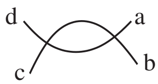
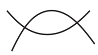
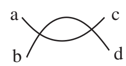
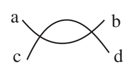
Next, we address the multiplicity. This is the number of different ways the external lines (that each have their own "individuality") can be attached. To determine the multiplicity we must imagine that the whole diagram, or a part of it, can be "flipped over" while retaining the same attachment of the external lines. To illustrate this, we temporarily denote the external lines with a letter, and then notice that the two diagrams

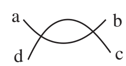


and


are, in fact, identical; the multiplicity of this graph is therefore 3, since there are three ways to group four letters into two pairs without regard to ordering:

 = 

\+ 

\+ 

. (1.35)

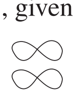
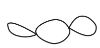
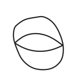
We shall often use the product of the symmetry and multiplicity, which factor we shall denote by $\mathfrak{sm}$.

We see that the diagram of Eq. (1.33) has, also, multiplicity 3, while that of Eq. (1.34) has multiplicity 1. We see that replacing p external lines with p one-point source vertices reduces $\mathfrak{sm}$ by a factor of $1/p!$; that will become important later on.[30]

The determination of symmetry factors may appear somewhat fanciful,[31] but of course it has a solid and unambiguous basis; the symmetry factor (and the multiplicity) can always be computed. The procedure is somewhat involved and is outlined in Appendix B.2.

1.3.4 Vacuum Bubbles

There are Feynman diagrams that contain neither external lines nor source vertices. These are called *vacuum bubbles*; the empty graph $\mathcal{E}$ is obviously a vacuum bubble. We may consider the set of *all* vacuum bubbles, which we denote by $\mathcal{H}_0$. Let us assume that only four-point vertices occur. Then, $\mathcal{H}_0$, given by

$\mathcal{H}_0 = \mathcal{E} +$  $+$  $+$  $+$  $+ \cdots$ (1.36)

(where the ellipsis denotes diagrams with more four-vertices) evaluates to

$$\begin{aligned}\mathcal{H}_0 &= 1 - \frac{1}{8}\frac{\lambda_4}{\mu^2} + \frac{1}{2}\left(\frac{1}{8}\frac{\lambda_4}{\mu^2}\right)^2 + \frac{1}{16}\frac{{\lambda_4}^2}{\mu^4} + \frac{1}{48}\frac{{\lambda_4}^2}{\mu^4} + \cdots \\ &= 1 - \frac{1}{8}\frac{\lambda_4}{\mu^2} + \frac{35}{384}\frac{{\lambda_4}^2}{\mu^4} + \cdots,\end{aligned} \tag{1.37}$$

which, indeed, looks suspiciously like H_0 for the φ^4 theory.

[30] In higher dimensions the symmetry factor is unchanged, but you may wonder what happens to the multiplicity. Essentially, that is also unchanged although less visible: the multiplicity tells us *how many* diagrams of a certain type there are, as in Eq. (1.35). Of course the *values* of these diagrams are generally no longer the same since the external lines carry their own momentum.

[31] The discussion of symmetry factors of Feynman diagrams goes, in practice, with a lot of remarks like "so you flip over this leaf, you wriggle this set of internal lines, you shove these vertices back and forth ... see?" Although the symmetry factor is totally unambiguous, the *arguments* for a symmetry factor often come with a lot of prestidigitatorial arguments accompanied by hand-waving and finger-wriggling in front of a blackboard.

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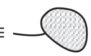
1.3.5 An Equation for Connected Graphs

We shall now construct an equation for a special set of diagrams. We do this for the set of Feynman rules of Section 1.3.2. First, let us denote by $\mathcal{C}_n$ the set of *all connected* graphs with *no* source vertices and precisely n external lines. Clearly this is a enumerably infinite set. Next, we define the object $\Psi(J)$, denoted by the symbol

$$\Psi(J) \equiv \tag{1.38}$$



to be the set of *all connected* diagrams with precisely *one* external line, and any number of source vertices. The shading indicates that *all* the diagrams in the blob must be *connected*. Clearly, then, we have

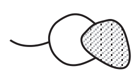
$$\Psi(J) \;=\; \sum_{n\geq 0} \frac{1}{n!} J^n \, \mathcal{C}_{n+1}, \tag{1.39}$$

where the extra factor $1/n!$ is the additional $\mathfrak{sm}$ factor for n source vertices.

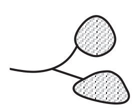
Let us now consider what can happen if we enter the blob of Eq. (1.38) along the single external line. In the first place, we can simply encounter a source vertex, so that we have *Game Over!*, and the diagram is

$$= \; J/\mu. \tag{1.40}$$

Alternatively, we may encounter another vertex. If this is a three-point vertex, the line splits into two. Taking one of these branches, we *may* be able to come back to the vertex via the other branch. In that case, the diagram has the form



On the other hand, it may happen that the two branches end up in disjunct connected pieces of the diagram, which then looks like



These two alternative cases can be unambiguously distinguished because we have restricted ourselves to using only *connected* graphs. Another important insight is that, in this last diagram, the two final blobs (with their attached lines) are both *exactly identical* to the original $\Psi(J)$ of Eq. (1.38), and therefore also to each other.[32] In contrast, the closed-loop blob of the first alternative is *not* equal to $\Psi(J)$ since it has not one but two lines sticking out; but then again these two lines are *completely* equivalent. If we encounter a four-point rather than a three-point vertex, the line splits into three, with three alternatives: no branches meeting again further on, all three meeting again, or only two out of the three.

[32] This is of course only possible because the blobs represent *infinite* sets of diagrams

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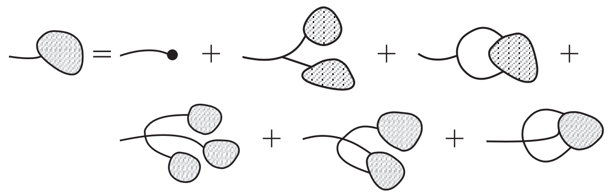
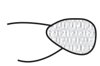
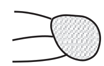
We find the diagrammatic equation



$$(1.41)$$

Now, realize that

 $= \sum_{n\geq 0} \frac{J^n\, \mathcal{C}_{n+2}}{n!} = \frac{\partial \Psi(J)}{\partial J}$,  $= \sum_{n\geq 0} \frac{J^n\, \mathcal{C}_{n+3}}{n!} = \frac{\partial^2 \Psi(J)}{(\partial J)^2}$, (1.42)

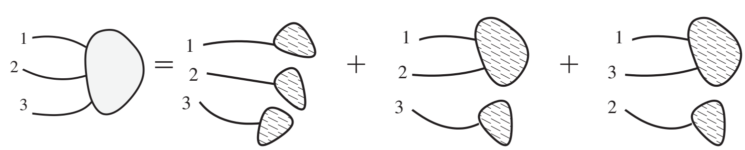
so that we can translate the diagrammatic equation (1.41) into an algebraic equation for $\Psi(J)$ by carefully implementing the correct Feynman rules, including the symmetry factors for equivalent blobs and lines:

$$\Psi(J) = \frac{J}{\mu} - \frac{\lambda_3}{\mu}\left(\frac{1}{2}\Psi(J)^2 + \frac{1}{2}\frac{\partial}{\partial J}\Psi(J)\right)$$
$$-\frac{\lambda_4}{\mu}\left(\frac{1}{6}\Psi(J)^3 + \frac{1}{2}\Psi(J)\frac{\partial}{\partial J}\Psi(J) + \frac{1}{6}\frac{\partial^2}{(\partial J)^2}\Psi(J)\right). \tag{1.43}$$

Now Eq. (1.43), obtained from the Feynman diagrams via the Feynman rules, has exactly the same form as Eq. (1.31), valid for the field function $\phi(J)$ – now you see how important the symmetry factors are! Moreover, the iterative solution for $\phi(J)$ starts with $\phi(J) = J/\mu$, also identical to the diagrammatic starting point —•. We therefore conclude that $\Psi(J) =$
10 $\phi(J)$, in other words $\mathcal{C}_n = C_n$ $(n \geq 1)$. This proves that connected Green's functions can be
11 obtained by the following recipe: **to obtain C_n $(n \geq 1)$, write out *all* connected Feynman**
12 **diagrams with no source vertices and precisely n external lines. Evaluate the diagrams using the Feynman rules, and sum them**.[33]

1.3.6 Semiconnected Graphs and the SDe

A useful notion, which allows us to write SDes more compactly, is that of *semi-connected graphs*. We shall denote these with a lightly shaded blob, and they are defined as follows: a semiconnected graph with $n \geq 1$ lines at the left is a general graph with n lines on the left (and *any* number of other external lines), with the constraint that each connected piece of the semiconnected graph is attached to at least one of the lines indicated on the left. This may sound more intimidating that is actually is: an example is



[33] Of course we can also obtain the G_n, by allowing disconnected diagrams.

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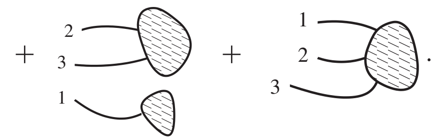
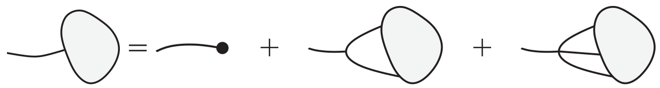


(1.44)

A single semiconnected graph with n indicated lines stands for $B(n)$ diagrams with explicit connected graphs, where $B(n)$ is the so-called Bell number: the number of ways to divide n distinct objects into nonempty groups.[34] For $\varphi^{3/4}$ theory, the SDe then becomes simply

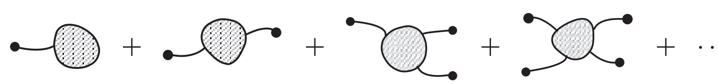


(1.45)

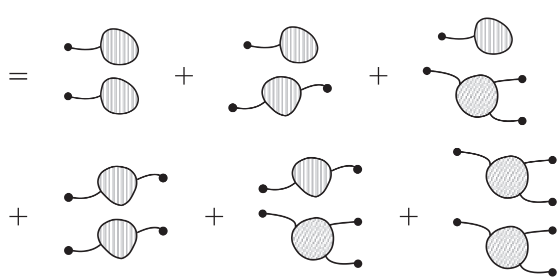
You should keep in mind that a semiconnected graph acts as a *symmetrizer*: the lines entering it on the left are to be connected to whatever happens inside the blob in *all admissible ways*. We shall use semiconnected diagrams to good effect in later chapters. Note that the sum of the symmetry factors of all connected diagrams arising from a φ^p vertex must be equal to $B(p-1)/(p-1)!$, which may serve as a check on your SDes.

1.3.7 The Path Integral as a Set of Diagrams

By affixing a source vertex to the single external line of $\Psi(J)$, we immediately have the result that **the generating function** $W(J)$ **is the sum of all connected Feynman diagrams without external lines and at least one source vertex**. If we explicitly indicate the source vertices, and recall that n source vertices in a diagram imply a factor $1/n!$, we can write

$W(J) =$  $+ \cdots,$ (1.46)

where the ellipsis contains connected contributions with more source vertices. Vacuum bubbles do *not* contribute to $W(J)$. By taking careful account of the symmetry factor assigned to identical connected parts of a disconnected diagram, we can see that

$\frac{1}{2!}W(J)^2 =$ 

$+$ (lots of other diagrams). (1.47)

Similar arguments hold for higher powers of $W(J)$. In addition, $W(J)^0 = \mathcal{E} = 1$. From this it easy to see that **the path integral** $Z(J)$ **consists of all Feynman diagrams without external lines, and without vacuum bubbles, but including the empty diagram**.

[34] For small n we have $B(0) = 1$, $B(1) = 1$, $B(2) = 2$, $B(3) = 5$, $B(4) = 15$, and $B(5) = 52$; more general values can be obtained from the generating function derived in Appendix T.7.

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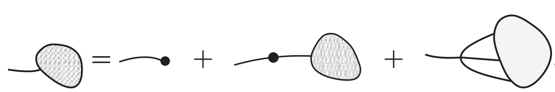
Why are the vacuum bubbles so conspicuously absent? Suppose that we would allow the inclusion of arbitrary numbers of vacuum bubbles in $Z(J)$. Then the Green's function $G_0 = 1$ would be represented not by the single empty graph but by the whole set $\mathcal{H}_0$ discussed before: indeed, $\mathcal{H}_0$ is proportional to H_0. In fact, *any* Green's function G_n would acquire exactly the same additional factor $\mathcal{H}_0$. The normalization factor N, that must be chosen such as to make G_0 equal to unity, therefore extracts exactly the factor $\mathcal{H}_0$ from any Green's function. In the jargon, the vacuum bubbles "disappear into the normalization of the path integral." This is not to say that vacuum diagrams are never important; but in our approach to computing Green's functions and connected Green's functions they are indeed irrelevant. Another way of seeing this is very simple: if we take our diagrammatic prescription of $Z(J)$ and then take $J = 0$, all diagrams disappear except the empty one, and we find $Z(0) = \mathcal{E} = 1$, just as we must.

1.3.8 Dyson Summation

Why is the Feynman rule for lines, stemming from the quadratic part of the action, so different from those for the vertices, that come from the nonquadratic terms? To see that our treatment is actually a consistent one, let us consider the action

$$S(\varphi) = \frac{1}{2}\mu\varphi^2 + \frac{1}{2}\lambda_2\varphi^2 + \frac{1}{4!}\lambda_4\varphi^4. \tag{1.48}$$

If we wish, we may treat the λ_2 term as an interaction, described by a vertex with two legs. the SDe is then seen to be


(1.49)

corresponding to

$$\phi(J) = \frac{J}{\mu} - \frac{\lambda_2}{\mu}\phi(J) - \frac{\lambda_4}{6\mu}\left(\phi(J)^3 + 3\phi(J)\frac{\partial}{\partial J}\phi(J) + \frac{\partial^2}{(\partial J)^2}\phi(J)\right). \tag{1.50}$$

Multiplying the equation by μ and transposing the λ_2 term to the left, we obtain

$$\phi(J) = \frac{J}{\mu+\lambda_2} - \frac{\lambda_4}{6(\mu+\lambda_2)}\left(\phi(J)^3 + 3\phi(J)\frac{\partial}{\partial J}\phi(J) + \frac{\partial^2}{(\partial J)^2}\phi(J)\right), \tag{1.51}$$

precisely what we would have obtained by taking the combination $(\mu + \lambda_2)$ as the kinetic part from the start! This procedure, by which the effect of two-point (effective) vertices is subsumed in a redefinition of the kinetic part, is called *Dyson summation*. In the present example, the summation is of course trivial; but we shall see that two-point interactions can also arise from more complicated Feynman diagrams corresponding to higher orders in perturbation theory. The manner in which Dyson summation is usually treated is by explicitly writing out the propagator, "dressed" with two-point vertices in all possible ways:

$$= \frac{1}{\mu} - \frac{1}{\mu}\lambda_2\frac{1}{\mu} + \frac{1}{\mu}\lambda_2\frac{1}{\mu}\lambda_2\frac{1}{\mu} - \frac{1}{\mu}\lambda_2\frac{1}{\mu}\lambda_2\frac{1}{\mu}\lambda_2\frac{1}{\mu} + \cdots$$

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$$= \frac{1}{\mu} \sum_{k \geq 0} \left(- \frac{\lambda_2}{\mu} \right)^k = \frac{1}{\mu} \frac{1}{1 + \lambda_2/\mu} = \frac{1}{\mu + \lambda_2}, \tag{1.52}$$

where, no surprise, we cheerfully ignore all issues about convergence, in the spirit of per-
14 turbation theory. Every propagator line can (and must) be dressed in this way once any two-point vertex (either elementary, or effective, that is, as the result of a collection of closed loops with two legs sticking out) occurs.

1.4 Synopsis of Feynman Rules

The Feynman rules for zero-dimensional $\varphi^{3/4}$ theory found in this chapter are

 $\leftrightarrow \dfrac{1}{\mu}$,  $\leftrightarrow -\lambda_3$,

 $\leftrightarrow -\lambda_4$, $\leftrightarrow +J$. (1.53)

1.5 Exercises

Exercise 1 Green's functions and connected Green's functions

We have

$$Z(J) = \sum_{n \geq 0} G_n \frac{J^n}{n!}, \quad W(J) = \sum_{n \geq 0} C_n \frac{J^n}{n!}, \quad W(J) = \log\left(Z(J)\right)$$

1. Using that $G_0 = 1$, and the expansion

$$- \log(1 - x) = \sum_{n \geq 1} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \cdots,$$

   express $C_{0,\ldots,5}$ in terms of the G's.
2. From $Z'(J) = W'(J)Z(J)$, prove the recursion relation

$$G_n = C_n + \sum_{m=1}^{n-1} \binom{n-1}{m-1} C_m G_{n-m}, \quad n \geq 1$$

3. Prove this last result using Feynman diagrams. Show how the factor $\binom{n-1}{m-1}$ avoids double counting.

Exercise 2 The problem with φ^4 theory

Prove that in φ^4 theory any *positive* real value for λ_4 leads to a well-defined theory, while a *negative* value of λ_4 leads to an undefined theory. The limit $\lambda_4 \to 0$ is, actually, an essentially singular situation!

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Exercise 3 Actually doing it for φ^4

This exercise shows how to express the path integral for the φ^4 theory in terms of "known functions." We consider

$$H = \int d\varphi \; \exp\left(-\frac{\mu}{2\hbar}\varphi^2 - \frac{\lambda}{24\hbar}\varphi^4\right).$$

1. Show that the combination

$$g = \frac{\hbar\lambda}{\mu^2}$$

is dimensionless.

2. Perform the substitution

$$\varphi = \sqrt{\frac{3\hbar}{\mu g}}\left(\psi^{1/4} - \psi^{-1/4}\right)$$

to derive

$$H = \sqrt{\frac{3\hbar}{4\mu g}}\; e^{3/(4g)} \int\limits_0^\infty d\psi \; \psi^{-3/4} \exp\left(-\frac{3}{8g}\left(\psi + \frac{1}{\psi}\right)\right).$$

3. The so-called *modified Bessel functions of the second kind*, denoted by $K_\nu(z)$, are discussed in Appendix T.8. Use this to express H in terms of these Bessel functions.
4. The function $K_\nu(z)$ has two expansions: one, an unattractive-looking but convergent series in positive powers of z, and a nonconvergent, asymptotic series in terms of negative powers of z, given in Eq. (T.54). Show that H therefore has a regular series expansion in $1/g$ and an asymptotic one in g. Show that the leading term in the asymptotic expansion is independent of g (but not of μ), and compute the next two terms. Compare your result with Eq. (1.19).

Exercise 4 The SDe for Z in another action

Find the SDe for the path integral $Z(J)$ for the action

$$S(\varphi) = \frac{\mu}{2!}\varphi^2 + \sum_{k=3}^{6}\frac{\lambda_k}{k!}\varphi^k.$$

Exercise 5 Writing the SDe for ϕ

Prove Eq. (1.29). Do this using the fact that

$$Z(J) = \exp\left(\frac{1}{\hbar}\int dJ \; \phi(J)\right),$$

and then considering Z', Z'', and so on.

Exercise 6 The SDe for ϕ in another action

For the action of exercise 4, derive the SDe for $\phi(J)$.

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Exercise 7 A programming exercise

For this exercise you have to employ computer-algebra code that is at least capable of series expansions[35] (or, if you want to do it by hand, you have to *really* want to). Starting with $\phi(J) = 0$, iterate the SDe for φ^4 theory a number of times, judiciously truncating the series expansions at every iteration. I find it useful to replace the first term, J/μ, by zJ/μ and expand in powers of z. That way, high powers of both J and $\hbar$ are truncated simultaneously. Try to at least reproduce Eq. (1.21).

Exercise 8 Stepping equations

Consider the path integral for the $\varphi^{3/4}$ theory:

$$Z(J) = Z(J, \mu, \lambda_3, \lambda_4) = N \int d\varphi \, \exp\left(-\frac{\mu}{2}\varphi^2 - \frac{\lambda_3}{6}\varphi^3 - \frac{\lambda_4}{24}\varphi^4 + J\varphi\right),$$

where we have indicated all parameters this once.

1. Show that Z obeys the equation
$$\mu\frac{\partial}{\partial J}Z - \lambda_3\frac{\partial}{\partial \mu}Z - \lambda_4\frac{\partial}{\partial \lambda_3}Z - JZ = 0.$$
2. Differentiating this equation with respect to J, show that the field function $\phi = \phi(J, \mu, \lambda_3, \lambda_4)$ obeys the *stepping equation*
$$\frac{\partial}{\partial J}\phi = \frac{1}{\mu} + \frac{\lambda_3}{\mu}\frac{\partial}{\partial \mu}\phi + \frac{\lambda_4}{\mu}\frac{\partial}{\partial \lambda_3}\phi.$$
If we are given $C_1 = \phi(0, \mu, \lambda_3, \lambda_4)$, the stepping equation allows us to compute the higher connected Green's functions.
3. Prove the stepping equation diagrammatically.

Exercise 9 The symmetry factor of life, the universe, and everything

Devise a diagram (connected or disconnected) that has a symmetry factor of 1/42.

Exercise 10 Diagrammatic SDe for φ^6 theory

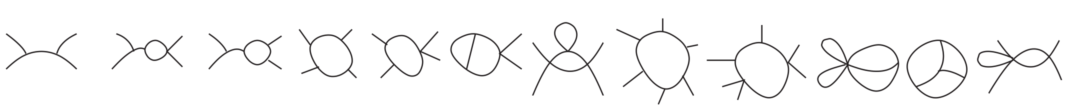
Give the diagrammatic SDe for φ^6 theory, and write it out algebraically.

Exercise 11 Actually doing it

Consider the diagrammatic SDe of Eq. (1.41). Starting with the empty diagram, iterate this diagrammatic equation two or three times, and write down the resulting set of diagrams.

Exercise 12 Some diagrams to consider

Of the following 12 diagrams, determine the multiplicity factor, the symmetry factor, and the number of loops:



[35] I usually rely on Waterloo's `MAPLE` package, but most other algebraic codes will serve as well.

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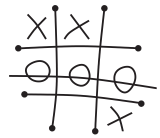
Exercise 13 A multi-loop, multi-leg diagram

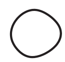
Consider the following diagram:



As before, determine the number of loops of this diagram, its symmetry factor, and its multiplicity factor.

Exercise 14 The One Ring

The One Ring diagram



might be argued to evaluate to zero because of its ∞-fold rotational symmetry. In this exercise we shall try to be more careful.

1. We consider a theory without any sources, with action

$$S(\varphi) = \frac{\mu_0}{2}\varphi^2 + \frac{\delta\mu}{2}\varphi^2, \quad \delta\mu = \mu - \mu_0,$$

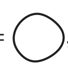
   where the second term is, for the moment, considered an interaction. The Feynman rules are therefore

    $= 1/\mu_0,$  $= -\delta\mu.$

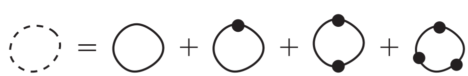
   Prove that the Dyson summation gives for the full propagator:

    $+ \cdots = 1/\mu.$

2. Let us denote the two One Ring diagrams by

   $\mathcal{R}(\mu_0) =$ , $\mathcal{R}(\mu) =$ 

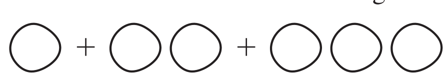
   From

    $+\cdots$

   prove that

$$\mathcal{R}(\mu) = \mathcal{R}(\mu_0) - \frac{1}{2}\log\left(\frac{\mu}{\mu_0}\right) \Rightarrow \mathcal{R}(\mu) = \frac{1}{2}\log\left(\frac{c}{\mu}\right)$$

   for some μ-independent constant c.
3. Introduce a φ^3 interaction, and show that the stepping equation (exercise 8) gives the correct one-loop tadpole.
4. Choose $c = 2\pi$. Show that for the sum of all vacuum diagrams:

   $\mathcal{E}$ +  $+\cdots$

   we obtain precisely the correct unnormalized free-action path integral.

https://doi.org/10.1017/9781108665209.002 Published online by Cambridge University Press