CELL GROWTH PROBLEMS

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In memory of Eoin L. Whitney, a friend and teacher

1. Introduction. The *square lattice* is the set of all points of the plane whose Cartesian coordinates are integers. A *cell* of the square lattice is a point-set consisting of the boundary and interior points of a unit square having its vertices at lattice points. An *n-omino* is a union of n cells which is connected and has no finite cut set.

The set of all n-ominoes, R_n, is an infinite set for each n; however, we are interested in the elements of two finite sets of equivalence classes, S_n and T_n, which are defined on the elements of R_n as follows: Two elements of R_n belong to the same equivalence class (i) in S_n, or (ii) in T_n, if one can be transformed into the other by (i) a translation or (ii) by a translation, rotation, and reflection of the plane. An element of R_n is in *standard position* if it is above the x-axis with a cell at the origin and all cells in the first row are to the right of the y-axis.

There is exactly one element in each equivalence class in S_n which is in standard position, while an equivalence class in T_n may contain as many as eight elements all in standard position. Thus, if $s(n)$ and $t(n)$ denote the number of elements in S_n and T_n, then

$$\tag{1} \tfrac{1}{8}s(n) \leqslant t(n) \leqslant s(n).$$

Harary **(3)** has listed the cell growth problem as an unsolved problem in the enumeration of graphs. Stated in the terms we have just defined, Harary's formulation of the cell growth problem is to find $t^*(n)$, the number of equivalence classes of T_n which contain simply connected n-ominoes. If an n-omino is not simply connected, in the sense that it has "holes," then it is said to be multiply connected. Harary **(4)** later reported that a computer had been programmed to find $t(n)$ for $n \leqslant 12$, and he listed these values. Evidently this work was carried out independently by Stein, Walden, and Williamson at the Los Alamos Scientific Laboratories and by Lander and Parkin at the Aerospace Corporation. This and more detailed information about the counts were communicated to the author by Mr. Parkin.

Read **(8)** calls representative elements of the equivalence classes of S_n and T_n *fixed* and *free animals with n cells* respectively; also, he gave a method for finding the number of equivalence classes of S_n which contain n-ominoes in standard position having cells in exactly k rows above the x-axis. He calculated

Received December 9, 1965.

these numbers for $k \leqslant 5$ and $n \leqslant 10$ and used these results to find $t(n)$ and $t^*(n)$ for $n \leqslant 10$; an error in his calculations involving $t(10)$ was discovered by Parkin and subsequently corrected by Read. The known values of $t(n)$ and $t^*(n)$ are as follows:

(2)

n	1	2	3	4	5	6	7	8	9	10	11	12
$t(n)$	1	1	2	5	12	35	108	369	1285	4655	17073	63600
$t^*(n)$	1	1	2	5	12	35	107	363	1248	4271	—	—

Eden **(1)** seems to have been the first person to give upper and lower bounds for $t(n)$; his bounds are

$$(3.14)^n < t(n) < 4^n, \tag{3}$$

for sufficiently large n. The proof for this upper bound is questionable, however.

We shall show that $\alpha = \lim_{n\to\infty} (s(n))^{1/n}$ exists, so that from (1) we can conclude that $\lim_{n\to\infty} (t(n))^{1/n}$ exists and is equal to α; to do this, we shall require a lemma due to Fekete (see Pólya and Szego **(7**, p. 171**)** for similar results).

LEMMA 1. *If $\{U_n\}$ is a sequence of natural numbers such that $\{(U_n)^{1/n}\}$ is bounded and $U_m U_n \leqslant U_{m+n}$, then $\lim_{n\to\infty} (U_n)^{1/n}$ exists.*

To show that $\{s(n)\}$ satisfies the conditions of Lemma 1 we prove two more lemmas; in the proof of Lemma 3 we indicate the method Eden used in his attempt to establish the upper bound given in (3).

LEMMA 2. $s(m)\, s(n) \leqslant s(m + n)$, $m, n = 1, 2, \ldots$.

Proof. Let X and Y be representative elements from equivalence classes in S_m and S_n respectively, such that the lower edge of the first cell in the bottom row of Y is joined to the upper edge of the last cell in the top row of X. The $(m + n)$-omino just described is a representative element of an equivalence class in S_{m+n}. The existence of this one–one correspondence between $S_m \times S_n$ and a subset of S_{m+n} implies the lemma.

LEMMA 3. $s(n) < (27/4)^n$.

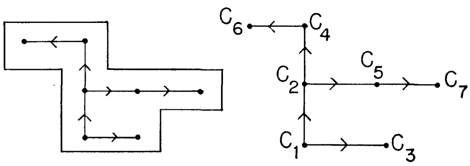
Proof. Following Eden, we assign a unique sequence of binary digits to each element X of S_n which will be denoted by $W(X) = \{(\alpha_1\, \beta_1)(\alpha_2\, \beta_2\, \gamma_2)\ldots(\alpha_n\, \beta_n\, \gamma_n)\}$. To do this, we assume X is in standard position and draw a labelled, directed tree over the cells of X; the nodes of the tree will be the centres of the cells of X and the directed edges will be drawn between some of the nodes in connected cells. The centre of the cell located at the origin is given the label C_1, and directed edges are drawn to the centres of the cells connected to it; these nodes are called $C_2, \ldots$, proceeding clockwise from the y-axis around C_1. Now we describe a process that must be carried out at the nodes $C_2, C_3, \ldots, C_n$, the nodes being taken in this order. Suppose the process has been carried out at $C_1, C_2, \ldots, C_{i-1}$, and that a directed edge has been drawn from C_j to C_i.

Beginning with the side of C_i to the right of C_j and proceeding clockwise around C_i we draw a directed edge to any cell connected to C_i which has not already been labelled, giving the labels $C_k, C_{k+1}, \ldots$, to the new nodes, where k is the smallest index not used previously. Now α_i, β_i, and γ_i are assigned the binary digits 1 or 0 if an edge going from C_i cuts the first, second, or third side encountered, proceeding clockwise from C_j around C_i.

The units which appear as digits in $W(X)$ correspond to the edges of the tree drawn over the cells of X; thus, the binary digits of $W(X)$ must sum to $n-1$, since a tree with n nodes has exactly $n-1$ edges. The number of binary sequences of length $3n-1$ that contain exactly $n-1$ ones is $\binom{3n-1}{n-1}$; since different elements of S_n give rise to different binary sequences, it follows that

$$s(n) \leqslant \binom{3n-1}{n-1} < \left(\frac{27}{4}\right)^n. \tag{4}$$

This completes the proof of Lemma 3; in Figure 1 we show a 7-omino in standard position with its labelled tree and corresponding binary sequence.



(11)(011)(000)(100)(010)(000)(000)

FIGURE 1

Combining Lemmas 1, 2, and 3 we have the following result.

THEOREM 1. $\lim_{n\to\infty} (s(n))^{1/n} = \alpha$ *exists, and* $\alpha \leqslant 27/4$.

If there is a directed edge from C_i to C_j, and from C_j to C_k, then the choice of the binary digits in $(\alpha_i \beta_i \gamma_i)$ and $(\alpha_j \beta_j \gamma_j)$ usually imposes restraints on the choice of $(\alpha_k \beta_k \gamma_k)$. Using this idea, Eden claims to have shown that fewer than 4^n of the binary sequences of length $3n-1$ have digits which satisfy these restraints, but his argument is thought to be incomplete.

If an n-omino in standard position has exactly a_j cells in the jth row above the x-axis, for $j = 1, 2, \ldots, i$, we say the equivalence class it represents in S_n belongs to the set $F_{a_1a_2\ldots a_i}$. Eden observed that if $f(a_1, a_2, \ldots, a_i)$ denotes the number of elements in $F_{a_1a_2\ldots a_i}$, then

$$f(a_1, a_2, \ldots, a_i) \geqslant (a_1 + a_2 - 1)(a_2 + a_3 - 1)\ldots(a_{i-1} + a_i - 1). \tag{5}$$

The product in the right member of (5) gives the number of equivalence classes in $F_{a_1a_2\ldots a_i}$ which contain n-ominoes that have a connected strip of a_j

https://doi.org/10.4153/CJM-1967-080-4 Published online by Cambridge University Press

cells in the jth row, for $j = 1, 2, \ldots, i$, when the n-omino is in standard position. This follows from the fact that a strip of r cells can be joined above a strip of s cells in $r + s - 1$ ways. Since there is a set $F_{a_1 a_2 \ldots a_i}$ corresponding to every composition of n, we have

$$(6) \qquad s(n) \geqslant b(n) = \sum (a_1 + a_2 - 1)(a_2 + a_3 - 1) \ldots (a_{i-1} + a_i - 1),$$

where the sum extends over all compositions of n into an unrestricted number of positive parts.

Eden was able to show that for sufficiently large n, $b(\mathrm{n}) > (3.14)^n$ and in this way proved the lower bound given in (3). Klarner **(5)** later improved this estimate to $(3.20)^n < b(n) < (3.21)^n$, for sufficiently large n; in a subsequent paper **(6)** a theory for sums like (6) was given. We shall now apply this theory to find the generating function for $\{b(n)\}$ as defined by (6), and at the same time prepare the machinery for treating similar problems.

2. Generating functions. Suppose $\{f(m, n)\colon m, n = 1, 2, \ldots\}$ and $\{g(n)\colon m = 1, 2, \ldots\}$ are given sets of numbers and consider the set of numbers $\{b(\mathrm{n})\colon n = 1, 2, \ldots\}$ defined by

$$(7) \qquad b(n) = \sum f(a_1, a_2)f(a_2, a_3)\ldots f(a_{i-1}, a_i)g(a_i),$$

where the sum extends over all compositions $(a_1, a_2, \ldots, a_i)$ of n into an unrestricted number of positive parts, and $g(n)$ is the contribution to the sum when the number of parts of the composition is one. The symbol $b_k{}^j(a, n)$, used with all or only some of the suffixes, denotes the partial sum obtained from (7) when the index of summation has been restricted to those compositions of n which have exactly k parts, no part greater than j, and the first part equal to a; if a suffix is dropped, the corresponding restriction on the index of summation is dropped as well.

In **(6)** we showed that if

$$(8) \qquad F(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) x^m y^n$$

is an analytic function of x for fixed y and of y for fixed x in neighbourhoods of $x = 0$ and $y = 0$ respectively, and if

$$(9) \qquad G(x) = \sum_{n=1}^{\infty} g(n) x^n$$

is an analytic function in a neighbourhood of $x = 0$, then each of the functions found by deleting any of the subscripts from

$$(10) \qquad B_k{}^j(x, y) = \sum_{n=1}^{\infty} \sum_{a=1}^{n} b_k{}^j(a, n) y^a x^n$$

is an analytic function of x for fixed y in a neighbourhood of $x = 0$. Furthermore,

$$(11) \qquad B_1(x, y) = G(xy),$$

$$(12) \qquad B_{k+1}(x, y) = \frac{1}{2\pi i}\int_c F(xy, 1/s)B_k(x, s)\frac{ds}{s}, \qquad k = 1, 2, \ldots,$$

$$(13) \qquad B(x, y) = G(xy) + \frac{1}{2\pi i}\int_c F(xy, 1/s)B(x, s)\frac{ds}{s},$$

where c is a contour in the s-plane which includes the singularities of $F(xy, 1/s)/s$ but excludes those of $B_k(x, s)$ or $B(x, s)$ respectively. Since $B(x, y) = B_1(x, y) + B_2(x, y) + \ldots$, (13) follows from (11) and (12); actually, (13) is a special case of the Fredholm integral equation.

Now (6) has the form of (7) with $g(n) = 1$ and $f(m, n) = m + n - 1$. If we suppose that $f(m, n) = w(m) + t(n)$ and that

$$W(x) = w(1)x + w(2)x^2 + \ldots, \qquad T(x) = t(1)x + t(2)x^2 + \ldots,$$

then

$$(14) \qquad F(x, y) = \frac{yW(x)}{1-y} + \frac{xT(y)}{1-x}.$$

In general, if the kernel of (13) has the special form

$$(15) \qquad F(x, y) = \sum_{k=1}^{r} R_k(x)S_k(y),$$

the integral equation is said to be of finite rank because its solution reduces to a system of linear algebraic equations. (For more details on this point, see Riesz and Sz-Nagy (**9**, p. 161).) Substituting the expression for $F(xy, 1/s)$ given by (14) into (13), we obtain an integral equation having a kernel with finite rank:

$$(16) \qquad B(x, y) = G(xy) + W(xy)B(x, 1) + \frac{1}{2\pi i}\int_c \frac{xyT(1/s)B(x, s)ds}{(1-xy)s}.$$

Multiplying equation (16) by $T(1/y)/y$ and integrating, we obtain a relation which implies that

$$(17) \qquad \frac{1}{2\pi i}\int_c T(1/y)B(x, y)\frac{dy}{y} = \frac{P(x) + B(x, 1)Q(x)}{1 - T(x)},$$

where

$$(18) \qquad P(x) = \frac{1}{2\pi i}\int_c T(1/y)G(xy)\frac{dy}{y} = \sum_{n=1}^{\infty} t(n)g(n)x^n,$$

$$(19) \qquad Q(x) = \frac{1}{2\pi i}\int_c T(1/y)W(xy)\frac{dy}{y} = \sum_{n=1}^{\infty} t(n)w(n)x^n.$$

The integral representation of sums such as appear in (18) and (19) was probably first discovered by Hadamard (see for example, Titchmarsh (**10**,

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pp. 157–159)). Substituting the expression for the integral given by (17) into (16) and setting $y = 1$, we obtain a linear equation in $B(x, 1)$; solving this we obtain

$$B(x, 1) = \frac{(1-x)G(x)(1-T(x)) + xP(x)}{(1-x)(1-T(x))(1-W(x)) - xQ(x)}. \tag{20}$$

The relations in (16), (17), and (20) can be combined to find $B(x, y)$ in closed form in terms of W, T, G, P, and Q. When $G(x) = x/(1-x)$, then $P(x) = T(x)$, and (20) reduces to

$$B(x, 1) = x/\{(1-x)(1-T(x))(1-W(x)) - xQ(x)\}. \tag{21}$$

An elementary proof of (21) is given in the appendix of this paper.

3. Some lower bounds. Now to find the generating function of $\{b(n)\}$ as defined in (6), we put

$$W(x) = T(x) = \sum_{n=1}^{\infty}\left(n - \frac{1}{2}\right)x^n = \frac{x(1+x)}{2(1-x)^2}, \tag{22}$$

$$Q(x) = \sum_{n=1}^{\infty}\left(n - \frac{1}{2}\right)^2 x^n = \frac{x(1+6x+x^2)}{4(1-x)^3}. \tag{23}$$

Substituting these functions into (21) gives

$$B(x, 1) = \frac{x(1-x)^3}{1-5x+7x^2-4x^3} = \sum_{n=1}^{\infty} b(n)x^n. \tag{24}$$

Multiplying equation (24) by $1 - 5x + 7x^2 - 4x^3$ and equating coefficients in the resulting identity gives $b(1) = 1$, $b(2) = 2$, $b(3) = 6$, $b(4) = 19$, and

$$b(n+3) = 5b(n+2) - 7b(n+1) + 4b(n), \tag{25}$$

for $n = 2, 3, \ldots$. Since the largest real root of the auxiliary equation for the difference equation in (25) lies between 3.20 and 3.21, we conclude that $(3.20)^n < b(n) < (3.21)^n$ for sufficiently large n.

The method just used to obtain a lower bound for n-ominoes also applies to animals with cells of different shapes. For example, animals with connected strips of hexagons in each row are enumerated by

$$b(n) = \sum (a_1 + a_2)(a_2 + a_3)\ldots(a_{t-1} + a_t), \tag{26}$$

where the sum extends over all compositions of n. This follows since a strip of r hexagons can be connected along the upper edge of a strip of s hexagons in $r + s$ ways.

To find the generating function for the sequence $\{b(n)\}$ defined by (26), we substitute

$$W(x) = T(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \tag{27}$$

and

$$Q(x) = \sum_{n=1}^{\infty} n^2x^n = \frac{x(1+x)}{(1-x)^3} \tag{28}$$

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into equation (21) to obtain

$$B(x, 1) = \frac{x(1-x)^3}{1 - 6x + 10x^2 - 7x^3 + x^4} = \sum_{n=1}^{\infty} b(n)x^n. \tag{29}$$

The relation in (29) implies that $b(1) = 1$, $b(2) = 3$, $b(3) = 11$, $b(4) = 42$, and

$$b(n+4) = 6b(n+3) - 10b(n+2) + 7b(n+1) - b(n), \tag{30}$$

for $n = 1, 2, \ldots$. Furthermore, the largest real root of the auxiliary equation for the difference equation in (30) lies between 3.87 and 3.88, so that for sufficiently large n, $(3.88)^n > b(n) > (3.87)^n$.

Golomb **(2)** suggested the problem of determining the number of incongruent n-celled animals with hexagonal cells; these numbers are 1, 1, 3, 7, 22, and 83 for $n = 1, 2, 3, 4, 5$, and 6 respectively. They correspond to free animals while the numbers defined in (26) correspond to fixed animals of a certain type. Since a hexagon has 12 symmetries in the plane, the numbers of fixed and free animals with hexagonal cells differ by a factor of 12 at most.

A rhombus is formed when two equilateral triangles are joined along an edge; thus, a lower bound for the number of animals with n rhomboidal cells can be used to find a lower bound for the number of animals with $2n$ triangular cells. A connected strip of r rhombuses can be joined above a strip of s rhombuses in $2(r + s - 1)$ ways, since the strip of r rhombuses has two orientations with respect to a reflection about its mid-section. Thus, a lower bound for the number of fixed animals with n rhomboidal cells is

$$b(n) = \sum (2a_1 + 2a_2 - 2)(2a_2 + 2a_3 - 2)\ldots(2a_{i-1} + 2a_i - 2), \tag{31}$$

where the sum extends over all compositions of n. Now to find the generating function of $\{b(n)\}$ as defined by (31), we can obtain the appropriate expression for $W(x)$ and $Q(x)$ by multiplying equations (22) and (23) by 2 and 4 respectively. Substituting these functions into (21) gives

$$B(x, 1) = \frac{x(1-x)^3}{1 - 6x + 8x^2 - 6x^3 - x^4} = \sum_{n=1}^{\infty} b(n)x^n. \tag{32}$$

Of course, (32) implies $b(1) = 1$, $b(2) = 3$, $b(3) = 13$, $b(4) = 59$, and

$$b(n+4) = 6b(n+3) - 8b(n+2) + 6b(n+1) + b(n), \tag{33}$$

for $n = 1, 2, \ldots$. The auxiliary equation for (33) has its largest real root between 4.54 and 4.55, so for sufficiently large n, $(4.55)^n > b(n) > (4.54)^n$; from this we can conclude that the number of fixed animals with n triangular cells is greater than $(2.13)^n$ for sufficiently large n.

4. Improved lower bounds. We have defined $F_{a_1 a_2 \ldots a_i}$ to be the subset of equivalence classes in S_n which contain n-ominoes with exactly a_j cells in the

jth row of the n-omino for $j = 1, 2, \ldots, i$, and in (5) gave a lower bound for $f(a_1, a_2, \ldots, a_i)$, the number of equivalence classes in $F_{a_1a_2\ldots a_i}$. Now we are going to show that there is a one–one correspondence between the elements of $F_{a_1a_2} \times F_{a_2a_3} \times \ldots \times F_{a_{i-1}a_i}$ and a subset of $F_{a_1a_2\ldots a_i}$.

THEOREM 2. $f(a_1, a_2, \ldots, a_i) \geqslant f(a_1, a_2)f(a_2, a_3) \ldots f(a_{i-1}, a_i)$.

Proof. We call an $r \times 1$ rectangle located in a column of the square lattice and a non-empty subset of the r cells contained in this rectangle an *r-component*. For example, an n-omino in an equivalence class in $F_{a_1a_2\ldots a_r}$ is a sequence of r-components contained in an $r \times j$ rectangle.

Now suppose an $r \times j$ rectangle contains a sequence R of r-components such that the top row of the rectangle contains exactly c cells belonging to the components of R; also, suppose an $s \times k$ rectangle contains a sequence S of s-components such that the bottom row of the rectangle contains exactly c cells belonging to the components of S. Clearly, we can translate the components to the right in either or both of the rectangles leaving gaps between the components so that the c cells in the bottom row of S can be made to cover the c cells in the top row of R. The sequence $R * S$ of $(r + s - 1)$-components which results when the components of R and S are joined in this way is called the *sum* of R and S.

Let $(Y_1, Y_2, \ldots, Y_{i-1})$ be a given element of $F_{a_1a_2} \times F_{a_2a_3} \times \ldots \times F_{a_{i-1}a_i}$ and suppose y_j is a representative element of Y_j, for $j = 1, 2, \ldots, i - 1$. We are going to construct a representative element y of an equivalence class $Y = Y(Y_1, Y_2, \ldots, Y_{i-1})$ of $F_{a_1a_2\ldots a_i}$ such that the sequence of 2-components in the jth and $(j + 1)$st rows of y is the same as the sequence of 2-components of y_j, for $j = 1, 2, \ldots, i - 1$.

Consider the sequence of i-components given by

$$((\ldots((y_1 * y_2) * y_3) * \ldots) * y_{i-1}) = y_1 * y_2 * \ldots * y_{i-1};$$

if this sequence is not an n-omino, 2-components containing disconnected cells can be translated to the left and joined to cells they were joined to formerly in $y_1, y_2, \ldots, y_{i-1}$, it being understood that overlapping 2-components must be translated simultaneously. The n-omino formed in this way has the desired properties; since, different sequences $(Y_1, Y_2, \ldots, Y_{i-1})$ of $F_{a_1a_2} \times F_{a_2a_3} \times \ldots \times F_{a_{i-1}a_i}$ give rise to different elements Y of $F_{a_1a_2\ldots a_i}$, the theorem is proved. An example of the construction is given in Figure 2.

We note that when $y_1 * y_2 * \ldots * y_{i-1}$ (described in the proof of Theorem 2) is disconnected, there may be many ways in which the cells can be translated to the right or the left to eventually form an n-omino; thus, n-ominoes in different equivalence classes in $F_{a_1a_2\ldots a_i}$ may have the same sequence of 2-components in their jth and $(j + 1)$st rows, for $j = 1, 2, \ldots, i - 1$. An example of this is given in Figure 3. Since $f(m, n)$ is generally larger than

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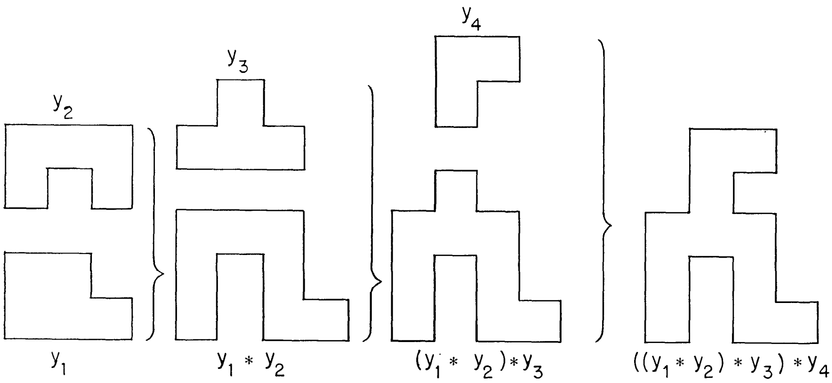

FIGURE 2

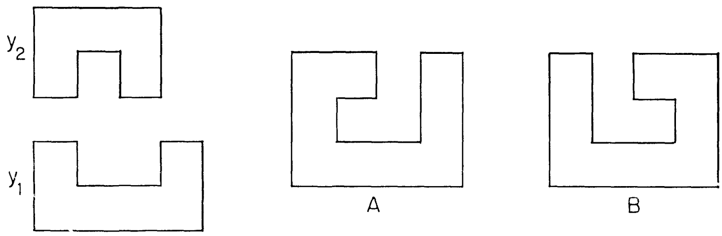

FIGURE 3. Both A and B correspond to (y_1, y_2); A corresponds to $y_1 * y_2$.

$m + n - 1$, Theorem 2 leads to a lower bound for $s(n)$ which is larger than the one given by (5). Thus, if

$$b(n) = \sum f(a_1, a_2) f(a_2, a_3) \ldots f(a_{i-1}, a_i), \tag{34}$$

where the sum extends over all compositions $(a_1, a_2, \ldots, a_i)$ of n, then $s(n) \geqslant b(n)$, for $n = 1, 2, \ldots$. Once we have the generating function of $\{f(m, n)\}$, the methods given in **(6)** can be applied to the sum in (34).

THEOREM 3.

$$H(x, y) = \frac{1 - xy}{1 - x - y + x^2 y^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m, n) x^m y^n. \tag{35}$$

Proof. First, we observe that

$$f(n, 0) = f(0, n) = 1, \qquad n = 0, 1, \ldots,$$

and

$$f(n, 1) = f(1, n) = n, \qquad n = 1, 2, \ldots.$$

Next, we show that

$$f(m + 2, n + 2) = f(m + 2, n + 1) + f(m + 1, n + 2) - f(m, n), \tag{36}$$

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for $m, n = 0, 1, \ldots$. There is a way to construct a representative element of each equivalence class in $F_{m+2,n+2}$ by adding a cell to representative elements in each of the equivalence classes in $F_{m+2,n+1} \cup F_{m+1,n+2}$ so that exactly $f(m, n)$ elements are duplicated in the process. To do this suppose x is in standard position and is an element of an equivalence class in $F_{m+2,n+1}$ (or in $F_{m+1,n+2}$), and join a new cell to x in the second row (or first row) so that the new cell is a maximum distance to the right of the origin. It is easy to see that the $(n + m + 4)$-ominoes obtained in this way are distinct except those that can be obtained by connecting a 2×2 block to the right end of elements in standard position selected from the equivalence classes of F_{mn}; these $(m + n + 4)$-ominoes will be constructed once in connection with an element of $F_{m+1,n+2}$ and again in connection with an element of $F_{m+2,n+1}$.

In order to verify that $H(x, y)$ generates $\{f(m, n): m, n = 0, 1, \ldots\}$ we check that $H(x, y)$ and $\partial H(x, y)/\partial y$ respectively generate $\{f(m, 0)\}$ and $\{f(m, 1)\}$ at $y = 0$; similarly, we see that $H(x, y)$ and $\partial H(x, y)/\partial x$ generate $\{f(0, n)\}$ and $\{f(1, n)\}$ at $x = 0$. Multiplying relation (35) by $(1 - x - y + x^2y^2)$ and equating coefficients of $x^m y^n$, we see that (35) implies (36). This completes the proof of Theorem 3.

Theorem 3 implies that

$$F(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m, n) x^m y^n = H(x, y) - \frac{(1 - xy)}{(1 - x)(1 - y)}; \tag{37}$$

furthermore, we know that $B(x, y)$, the generating function of $\{b(a, n)\}$ as defined by (34), satisfies the integral equation (13) with the kernel defined by (37) and $G(x) = x/(1 - x)$.

Since

$$b(n) \geqslant b^j(n) = \sum f(a_1, a_2) f(a_2, a_3) \ldots f(a_{i-1}, a_i), \tag{38}$$

where the sum extends over all compositions $(a_1, a_2, \ldots, a_i)$ of n into positive parts less than or equal to j, a lower bound for the Taylor coefficients of $B^j(x, 1)$ is also a lower bound for $b(n)$. In **(6)** we showed that $B^j(x, 1)$ is a rational function whose denominator is given by $\det \Delta_j(x)$, where $\Delta_j(x) = [a_{rs}]$, and

$$a_{rs} = \begin{cases} f(r, r)x^r - 1 & \text{if } r = s, \\ f(r, s)x^r & \text{if } r \neq s, \end{cases} \tag{39}$$

$r, s = 1, 2, \ldots, j$. Thus, if θ_j is the largest real root of $\det \Delta_j(1/x) = 0$, we have

$$\lim_{n\to\infty} (b^j(n))^{1/n} = \theta_j < \lim_{n\to\infty} (b(n))^{1/n},$$

and $\theta_1 < \theta_2 < \ldots$.

For small j it is feasible to calculate the polynomials $x^{j(j+1)/2} \det \Delta_j(1/x)$, and the corresponding polynomial equations can be used to find $\theta_1 = 1.00\ldots$,

$\theta_2 = 2.41\ldots, \theta_3 = 3.04\ldots, \theta_4 = 3.34\ldots, \theta_5 = 3.50\ldots, \theta_6 = 3.59\ldots$ Calculations made by T. R. Parkin and associates at the Aerospace Corporation on the CDC 6600 computer show that $\theta_j = 3.72\ldots$, for $j = 15, 16, \ldots, 23$, with $\theta_{23} = 3.72274322\ldots$ Thus, we have the following theorem.

THEOREM 4. *There are more than* $(3.72)^n$ *n-ominoes for all sufficiently large* n.

The method we have used to prove the last theorem can be applied to problems involving other types of animals. For example, if regular hexagons with unit area are used as cells of an animal, the number of "two-rowed animals" of this type with m cells in the first row and n cells in the second is exactly $\binom{m+n}{n}$. By modifying the proof of Theorem 2 slightly we can show that there are more than

$$(40) \qquad b(n) = \sum \binom{a_1 + a_2}{a_2}\binom{a_2 + a_3}{a_3}\ldots\binom{a_{i-1} + a_i}{a_i}$$

"fixed animals" with n hexagonal cells, where the sum in (40) extends over the compositions of n. We can estimate $b(n)$ as defined in (40) just as before, and obtain the following result.

THEOREM 5. *There are more than* 4^n *hexagonal celled animals for all sufficiently large* n.

Appendix. Let $b(n, n) = 1$, and for $1 \leqslant a < n$, $n = 1, 2, \ldots$, define

$$(41) \qquad b(a, n) = \sum \{w(a) + t(a_1)\}\{w(a_1) + t(a_2)\}\ldots,$$

where the sum extends over all compositions $(a_1, a_2, \ldots)$ of $n - a$; now (41) implies that

$$(42) \qquad \begin{aligned} b(a, n) &= w(a)\sum_{v=1}^{n-a} b(v, n-a) + \sum_{v=1}^{n-a} t(v)b(v, n-a) \\ &= w(a)b(n-a) + \sum_{v=1}^{n-a} t(v)b(v, n-a). \end{aligned}$$

Writing $a - k$ and $n - k$ in place of a and n in (42) we obtain a similar expression for $b(a - k, n - k)$; taking the difference $b(a, n) - b(a - k, n - k)$ and transposing a term gives

$$(43) \qquad b(a, n) = b(a-k, n-k) + [w(a) - w(a-k)]b(n-a).$$

When $k = a - 1$ in (43), we find that each of the numbers $b(a, n)$ can be written in terms of $b(1, v)$ and $b(v)$, $v = 1, 2, \ldots$; thus, for $a < n$,

$$(44) \qquad b(a, n) = b(1, n-a+1) + [w(a) - w(1)]b(n-a).$$

Using the fact that $b(n, n) = g(n) = 1$, we substitute expressions for $b(a, n)$ given by (44) into $b(n) = b(1, n) + b(2, n) + \ldots + b(n, n)$ to obtain

$$(45) \qquad b(n) = 1 + \sum_{a=1}^{n-1} b(1, n-a+1) + [w(a) - w(1)]b(n-a).$$

https://doi.org/10.4153/CJM-1967-080-4 Published online by Cambridge University Press

This can be used to show that

$$(46)\quad b(n) - b(n-1) = b(1, n) + \sum_{a=1}^{n-2} [w(n-a) - w(n-a-1)]b(a),$$

for $n > 2$; when $n = 1$ and 2 we have

$$(47)\qquad b(1) = b(1, 1) \quad\text{and}\quad b(2) - b(1) = b(1, 2).$$

Equations (46) and (47) imply the following relationship between the generating series:

$$(48)\quad \sum_{n=1}^{\infty} b(n)x^n - \sum_{n=1}^{\infty} b(n)x^{n+1} = \sum_{n=1}^{\infty} b(1, n)x^n$$
$$+ \sum_{n=3}^{\infty} \sum_{a=1}^{n-2} w(n-a)b(a)x^n - \sum_{n=3}^{\infty} \sum_{a=1}^{n-2} w(n-a-1)b(a)x^n.$$

Each of the series in (48) can be replaced by the function it represents. Writing ∂B for the partial derivative of $B(x, s)$ with respect to s at $s = 0$, the result after collecting terms is

$$(49)\qquad \partial B = \{w(1) + (1-x)(1 - \mathrm{W}(x))\}B(x, 1).$$

Now we eliminate ∂B from (49). First, setting $a = 1$ in (42) gives

$$(50)\qquad b(1, n) = w(1)b(n-1) + \sum_{v=1}^{n-1} t(v)b(v, n-1),$$

and substituting expressions for $b(v, n-1)$ given by (44) into the sum in the right member of (50) gives

$$(51)\quad b(1, n) = w(1)b(n-1) + \sum_{v=1}^{n-1} t(v)b(1, n-v)$$
$$+ \sum_{v=1}^{n-2} w(v)t(v)b(n-v-1) - w(1)\sum_{v=1}^{n-2} t(v)b(n-v-1);$$

for $n = 1$ and 2, the relations corresponding to (51) are

$$(52)\qquad b(1, 1) = 1 \quad\text{and}\quad b(1, 2) = w(1)b(1) + t(1).$$

Equations (51) and (52) imply the following relationship between the generating functions:

$$(53)\quad \partial B = x + w(1)xB(x, 1) + T(x)\partial B$$
$$- w(1)xT(x)B(x, 1) + xQ(x)B(x, 1),$$

where $Q(x)$ is the function defined in (19). Taken together, the relations in (49) and (53) imply (21).

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