THE EULER-LAGRANGE EXPRESSION AND DEGENERATE LAGRANGE DENSITIES

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1. Introduction and motivation

It is well known that many of the field equations from theoretical physics (e.g. Einstein field equations, Maxwell's equations, Klein-Gordon equation) can be obtained from a variational principle with a suitably chosen Lagrange density. In the case of the Einstein equations the corresponding Lagrangian is degenerate (*i.e.*, the associated Euler-Lagrange equations are of second order whereas in general these would be of fourth order), while in the cases of the Maxwell and Klein-Gordon equations the Lagrangian usually used is not degenerate. However, it is not generally realized that there exist degenerate Lagrange densities which also give rise to these last two field equations. In this note the general structure of this type of degenerate Lagrange density is examined.

We shall concentrate our attention on m quantities ρ^A $(A = 1, \ldots, m)$ which in general are each functions of position *i.e.*

$$\rho^A = \rho^A(x^j).$$

Under transformations of the type

$$\bar{x}^a = \bar{x}^a(x^j) \tag{1.1}$$

we shall assume that the ρ^A transform according to the law[1]

$$\bar{\rho}^A = C^A_B \rho^B, \tag{1.2}$$

where the C^A_B are functions of $\bar{x}^a$ (or x^j) and are completely determined by the transformation (1.1). To fix ideas we cite four examples all of which fall into the above category.

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[1] Unless otherwise noted the summation convention will apply, whereby repeated capital indices $A, B, \ldots$ will be summed from 1 to m and repeated latin indices $a, b, \ldots, i, j, \ldots$ from 1 to n.

(i) *Scalar field* ϕ. If ϕ is a scalar field the counterpart of (1.2) reads

$$\bar{\Phi} = \phi,$$

in which case $m = 1$, $\rho^A = \phi$ and $C^A_B = 1$.

(ii) *Vector field* ψ_i. In this case $m = n$, $\rho^A = \psi_i$ and $C^A_B = B^i_a \,(= \partial x^i/\partial \bar{x}^a)$ since a vector field transforms according to the law[1]

$$\bar{\Psi}_a = B^i_a \psi_i.$$

(iii) *Tensor field* g_{ij}. Here $m = n^2$, $\rho^A = g_{ij}$ and $C^A_B = B^i_a B^j_b$ corresponding to

$$\bar{g}_{ab} = B^i_a B^j_b g_{ij}.$$

(iv) *Non-tensorial field* α_{ij}. If α_{ij} are n^2 quantities which transform according to[2]

$$\bar{\alpha}_{ab} = \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\sum_{c=1}^{n} \sum_{d=1}^{n} B^i_{ac} B^j_{bd} A^c_i A^d_j + B^i_a B^j_b \right] \alpha_{ij}$$

where we have put

$$B^i_{ac} = \partial B^i_a / \partial \bar{x}^c \quad \text{and} \quad A^c_i = \partial \bar{x}^c / \partial x^i,$$

then under these circumstances

$$C^A_B = \sum_{c=1}^{n} \sum_{d=1}^{n} B^i_{ac} B^j_{bd} A^c_i A^d_j + B^i_a B^j_b.$$

From these examples it is evident that ρ^A may represent, on the one hand, the components of an arbitrary relative tensor field and, on the other hand, certain quantities which are manifestly non-tensorial in character.

We now assume that we are given a quantity L–the Lagrangian. It is furthermore supposed that L is a function of ρ^A and its first M partial derivatives together with q arbitrary preassigned functions of position λ^α ($\alpha = 1, \dots, q$) and the first Q partial derivatives of λ^α, *i.e.*

(1.3) $$L = L(\rho^A; \rho^A{}_{,i_1}; \dots; \rho^A{}_{,i_1 \dots i_M}; \lambda^\alpha; \lambda^\alpha{}_{,i_1}; \dots; \lambda^\alpha{}_{,i_1 \dots i_Q}),$$

where a comma denotes partial differentiation.

With L we can always associate the Euler-Lagrange expression $E_A(L)$ defined by

(1.4) $$E_A(L) = \frac{\partial L}{\partial \rho^A} + \sum_{r=1}^{M} (-1)^r \frac{\partial^r}{\partial x^{i_1} \dots \partial x^{i_r}} \left(\frac{\partial L}{\partial \rho^A{}_{,i_1 \dots i_r}} \right),$$

where, as mentioned above, the λ^α are not varied but are assumed to be preas-

[2] The summation convention does not apply to this example.

signed functions of position. In general the Euler-Lagrange expression (1.4) will be of order $2M$ in ρ^A (and order $M+Q$ in λ^α). If the order of this expression in ρ^A is less than $2M$ the corresponding Lagrangian is called *degenerate*. For examples of this see [2], [3] and [4].

In order to ensure that the so-called action integral corresponding to (1.3) *viz.*

$$\int \cdots \int L \; dx^1 \ldots dx^n$$

is an invariant, we assume that L is a scalar density, *i.e.* under (1.1)

$$\bar{L} = BL \tag{1.5}$$

where

$$B = \det \left| B^i_a \right| .$$

In theoretical physics the role played by the Euler-Lagrange equations

$$E_A(L) = 0 \tag{1.6}$$

is well known. It is usually possible to derive the field equations of physics from a variational principle with a suitably chosen Lagrangian L. To illustrate this we briefly discuss three important cases.

(a) *Symmetric tensor field: Einstein vacuum field equations.*

Consider the Lagrangian L given by

$$L(g_{ij};\, g_{ij,k};\, g_{ij,kh}) = \sqrt{g}\, g^{ij} R^h_{i\,jh}, \tag{1.7}$$

where $g = \det \left| g_{ij} \right|$, g^{ij} are characterised by

$$g^{ij} g_{kj} = \delta^i_k$$

and

$$R^h_{i\,jk} = \left\{ {h \atop i\;j} \right\}_{,k} - \left\{ {h \atop i\;k} \right\}_{,j} + \left\{ {r \atop i\;j} \right\} \left\{ {h \atop r\;k} \right\} - \left\{ {r \atop i\;k} \right\} \left\{ {h \atop r\;j} \right\}$$

with

$$\left\{ {h \atop i\;j} \right\} = \frac{1}{2}\, g^{hk} (g_{ik,j} + g_{jk,i} - g_{ij,k}).$$

With the correspondence

$$\rho^A = g_{ij}, \; \lambda^\alpha = 0,$$

in (1.3) the associated Euler-Lagrange expression (1.4) is[3]

$$E^{ij}(L) = -\sqrt{g} \left(g^{ih} g^{jk} R^l_{h\,kl} - \frac{1}{2} g^{ij} g^{hk} R^l_{h\,kl} \right), \tag{1.8}$$

[3] See *e.g.* [5] p. 258.

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and the Euler-Lagrange equations (1.6) are just the Einstein field equations in vacuo.

(b) *Scalar field*: *Klein-Gordon Equation.*

If we consider the Lagrangian

$$L(\phi; \phi_{,i}; g_{ij}) = \frac{1}{2}\sqrt{g}\,(g^{ij}\phi_{,i}\phi_{,j} + k^2\phi^2), \tag{1.9}$$

(where k = constant), and use the correspondence

$$\rho^A = \phi, \ \lambda^\alpha = g_{ij},$$

then (1.4) becomes

$$E(L) = \sqrt{g}(g^{ij}\phi_{|ij} - k^2\phi), \tag{1.10}$$

where the vertical bar denotes covariant differentiation with respect to g_{ij}. The Euler-Lagrange equation in this case is the Klein-Gordon equation.

(c) *Vector field*: *Maxwell's equation*

By choosing

$$L(\psi_1; \psi_{i,j}; g_{ij}) = \frac{1}{2}\sqrt{g}\,F_{ij}F_{hk}g^{ih}g^{jk} \tag{1.11}$$

where

$$F_{ij} = \psi_{i,j} - \psi_{j,i}$$

and by identifying

$$\rho^A = \psi_i, \ \lambda^\alpha = g_{ij},$$

we find that (1.4) reads

$$E^i(L) = 2\sqrt{g}\,g^{ih}g^{jk}F_{hk|j}. \tag{1.12}$$

The corresponding Euler-Lagrange equations are Maxwell's equations in the absence of sources.

Of these three Lagrangians *i.e.* (1.7), (1.9) and (1.11), only one is degenerate *viz.* (1.7). However, it is not generally realized that in the other two cases it is possible to choose Lagrange densities which are degenerate but which still yield (1.10) and (1.12) as the corresponding Euler-Lagrange expressions. These Lagrangians are

$$L(\phi; \phi_{,i}; \phi_{,ij}; g_{ij}; g_{ij,k}) = \frac{1}{2}\sqrt{g}\,\phi(g^{ij}\phi_{|ij} - k^2\phi), \tag{1.13}$$

and

$$L(\psi_i; \psi_{i,j}; \psi_{i,jk}; g_{ij}; g_{ij,k}) = \sqrt{g}\,\psi_i g^{ih} g^{jk} F_{hk|j}, \tag{1.14}$$

respectively. Aside from their degeneracy, the Lagrangians (1.7), (1.13) and (1.14) have something else in common — they all have the same structure *viz.*

$$a\rho^A E_A(L),$$

where a is a constant, in the general case. This raises three obvious questions:

1. *If L is a scalar density and ρ^A transform according to* (1.2) *is $\rho^A E_A(L)$ a scalar density?*

2. *If $\rho^A E_A(L)$ is a scalar density and we regard it as a new Lagrangian, is it always degenerate?*

3. *If $\rho^A E_A(L)$ is a degenerate Lagrange density are its Euler-Lagrange equations always $E_A(L) = 0$, up to a constant?*

These three questions will be answered in the next sections by means of Theorems 2, 4 and 6 respectively.

2. Certain properties of the Euler-Lagrange expression

The main purpose of this section is to establish the transformation law for $E_A(L)$ under (1.1) where it is assumed that ρ^A and L transform according to (1.2) and (1.5) respectively.

To this end we introduce the following notation. If $F^{\cdots}_{\cdots}$ is any function of x^i then

$$F^{\cdots}_{\cdots},_{i_0} = F^{\cdots}_{\cdots}$$

and

$$F^{\cdots}_{\cdots},_{i_0 i_1 \dots i_r} = \partial F^{\cdots}_{\cdots},_{i_0 i_1 \dots i_{r-1}}/\partial x^{i_r}$$

for $r = 1, 2, \dots$. Similarly if $G^{\cdots}_{\cdots}$ is any function of $\bar{x}^a$ then

$$G^{\cdots}_{\cdots},_{a_0} = G^{\cdots}_{\cdots}$$

and

$$G^{\cdots}_{\cdots},_{a_0 a_1 \dots a_r} = \partial G^{\cdots}_{\cdots},_{a_0 a_1 \dots a_{r-1}}/\partial \bar{x}^{a_r}$$

for $r = 1, 2, \dots$. In view of this the summation convention will *not* apply to the indices i_0 or a_0.

It is possible to rewrite (1.4) in a slightly more concise form by introducing the following definitions:

$$L_A^{i_0} = \partial L/\partial \rho^A,_{i_0} (= \partial L/\partial \rho^A),$$

$$L_A^{i_0 i_1 \dots i_r} = \partial L/\partial \rho^A,_{i_0 i_1 \dots i_r}$$

for $1 \leqq r \leqq M$. In this case (1.4) becomes

$$E_A(L) = \sum_{r=0}^{M} (-1)^r L_A{}^{i_0 i_1 \dots i_r}{}_{,i_0 i_1 \dots i_r}.$$

In order to obtain the transformation law for $E_A(L)$ we will require the

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transformation properties of $\bar{\rho}^A{}_{,a_0a_1\ldots a_k}$ for all k, $0 \leqq k \leqq M$. We shall obtain these properties in the form of a recurrence relation. In view of (1.2) it is clear that $\bar{\rho}^A{}_{,a_0a_1\ldots a_k}$ will be linear in $\rho^B{}_{,i_0i_1\ldots i_j}$ for all j, $0 \leqq j \leqq k$, and can thus be expressed in the form

$$\bar{\rho}^A{}_{,a_0a_1\ldots a_k} = \sum_{j=0}^{k} \alpha^{Ai_0i_1\ldots i_j}_{Ba_0a_1\ldots a_k} \rho^B{}_{,i_0i_1\ldots i_j}, \tag{2.1}$$

where $\alpha^{Ai_0i_1\ldots i_j}_{Ba_0a_1\ldots a_k}$ are functions of $\bar{x}^a$. Differentiation of (2.1) with respect to $\bar{x}^{a_{k+1}}$ yields

$$\bar{\rho}^A{}_{,a_0a_1\ldots a_{k+1}} = \sum_{j=1}^{k+1} \left[\alpha^{Ai_0i_1\ldots i_{j-1}}_{Ba_0a_1\ldots a_k} B^{i_j}_{a_{k+1}} + \alpha^{Ai_0i_1\ldots i_j}_{Ba_0a_1\ldots a_k,a_{k+1}} \right] \rho^B{}_{,i_0i_1\ldots i_j} + \tag{2.2}$$
$$+ \alpha^{A}_{Ba_0a_1\ldots a_k,a_{k+1}} \rho^B,$$

with the understanding that

$$\alpha^{Ai_0i_1\ldots i_{k+1}}_{Ba_0a_1\ldots a_k} = 0.$$

A comparison of (2.2) with (2.1) [with k replaced by $(k+1)$ in the latter] yields the following. If k is any integer, $0 \leqq k+1 \leqq M$,

$$\alpha^{Ai_0i_1\ldots i_j}_{Ba_0a_1\ldots a_{k+1}} = \begin{cases} \alpha^{Ai_0}_{Ba_0a_1\ldots a_k,a_{k+1}} & \textit{if } j=0 \textit{ and } k \geqq 0, \\ \alpha^{Ai_0i_1\ldots i_{j-1}}_{Ba_0a_1\ldots a_k} B^{i_j}_{a_{k+1}} + \alpha^{Ai_0i_1\ldots i_j}_{Ba_0a_1\ldots a_k,a_{k+1}} & \textit{if } 1 \leqq j \leqq k+1, \\ 0 & \textit{if } j > k+1. \end{cases} \tag{2.3}$$

Obviously

$$\alpha^{Ai_0}_{Ba_0} = C^A_B. \tag{2.4}$$

By means of (2.3) we can prove the

LEMMA. *If under* (1.1) ρ^A *and* L *transform according to* (1.2) *and* (1.5) *respectively then for* $1 \leqq k \leqq M-1$

$$\sum_{p=0}^{k-1} (-1)^p L_B{}^{i_0i_1\ldots i_{M-p}}{}_{,i_{M-p}i_{M-p-1}\ldots i_{M-k+1}} + (-1)^k L_B{}^{i_0i_1\ldots i_{M-k}} =$$

$$= \sum_{j=0}^{k-1} (-1)^j \left\{ \sum_{p=0}^{k-j-1} (-1)^p \bar{L}_A^{a_0a_1\ldots a_{M-p}}{}_{,a_{M-p}a_{M-p-1}\ldots a_{M-k+j+1}} + \right. \tag{2.5}$$
$$\left. + (-1)^{k-j} \bar{L}_A^{a_0a_1\ldots a_{M-k+j}} \right\} \alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0a_1\ldots a_{M-k+j-1}} B^{i_{M-k}}_{a_{M-k+j}} / B +$$
$$+ (-1)^k \bar{L}_A^{a_0a_1\ldots a_M} \alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0a_1\ldots a_{M-1}} B^{i_{M-k}}_{a_M} / B.$$

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PROOF. We shall prove this by induction over k. With $k = 1$ (2.5) reads

$$L_B^{i_0 i_1 \dots i_M}{}_{,i_M} - L_B^{i_0 i_1 \dots i_{M-1}} = (\bar{L}_A^{a_0 a_1 \dots a_M}{}_{,a_M} - \bar{L}_A^{a_0 a_1 \dots a_{M-1}}) \times \tag{2.6}$$
$$\times\, \alpha^{A i_0 i_1 \dots i_{M-2}}_{B a_0 a_1 \dots a_{M-2}} B^{i_{M-1}}_{a_{M-1}}/B - \bar{L}_A^{a_0 a_1 \dots a_M} \alpha^{A a_0 i_1 \dots i_{M-2}}_{B 1 0 a_1 \dots a_{M-1}} B^{i_{M-1}}_{a_M}/B.$$

From (1.5) and (2.1) we find

$$L_B^{i_0 i_1 \dots i_M} = \bar{L}_A^{a_0 a_1 \dots a_M} \alpha^{A i_0 i_1 \dots i_M}_{B a_0 a_1 \dots a_M}/B,$$

which, by virtue of (2.3), can be written as

$$L_B^{i_0 i_1 \dots i_M} = \bar{L}_A^{a_0 a_1 \dots a_M} \alpha^{A i_0 i_1 \dots i_{M-1}}_{B a_0 a_1 \dots a_{M-1}} B^{i_M}_{a_M}/B.$$

In view of the fact that
$$\frac{\partial}{\partial x^i}(B^i_a/B) = 0,$$
we thus have

$$L_B^{i_0 i_1 \dots i_M}{}_{,i_M} = (\bar{L}_A^{a_0 a_1 \dots a_M}{}_{,a_M} \alpha^{A i_0 i_1 \dots i_{M-1}}_{B a_0 a_1 \dots a_{M-1}} + \bar{L}_A^{a_0 a_1 \dots a_M} \alpha^{A i_0 i_1 \dots i_{M-1}}_{B a_0 a_1 \dots a_{M-1}, a_M})/B. \tag{2.7}$$

From (1.5) and (2.1) we also see that

$$L_B^{i_0 i_1 \dots i_{M-1}} = (\bar{L}_A^{a_0 a_1 \dots a_M} \alpha^{A i_0 i_1 \dots i_{M-1}}_{B a_0 a_1 \dots a_M} + \bar{L}_A^{a_0 a_1 \dots a_{M-1}} \alpha^{A i_0 i_1 \dots i_{M-1}}_{B a_0 a_1 \dots a_{M-1}})/B. \tag{2.8}$$

Subtraction of (2.8) from (2.7) yields

$$L_B^{i_0 i_1 \dots i_M}{}_{,i_M} - L_B^{i_0 i_1 \dots i_{M-1}} = (\bar{L}_A^{a_0 a_1 \dots a_M}{}_{,a_M} - \bar{L}_A^{a_0 a_1 \dots a_{M-1}}) \alpha^{A i_0 i_1 \dots i_{M-1}}_{B a_0 a_1 \dots a_{M-1}}/B +$$
$$+ \bar{L}_A^{a_0 a_1 \dots a_M} (\alpha^{A i_0 i_1 \dots i_{M-1}}_{B a_0 a_1 \dots a_{M-1}, a_M} - \alpha^{A i_0 i_1 \dots i_{M-1}}_{B a_0 a_1 \dots a_M})/B,$$

which is (2.6) when account is taken of (2.3). Hence (2.5) is valid for $k = 1$.

We now assume (2.5) to be true for fixed k and establish the validity of (2.5) with k replaced by $k+1$, *i.e.* we wish to show that (2.5) implies

$$\sum_{p=0}^{k} (-1)^p L_B^{i_0 i_1 \dots i_{M-p}}{}_{,i_{M-p} i_{M-p-1} \dots i_{M-k+1} i_{M-k}} + (-1)^{k+1} L_B^{i_0 i_1 \dots i_{M-k-1}} =$$
$$= \sum_{j=0}^{k} (-1)^j \left\{ \sum_{p=0}^{k-j} (-1)^p \bar{L}_A^{a_0 a_1 \dots a_{M-p}}{}_{,a_{M-p} a_{M-p-1} \dots a_{M-k+j+1} a_{M-k+j}} + \right. \tag{2.9}$$
$$\left. + (-1)^{k-j+1} \bar{L}_A^{a_0 a_1 \dots a_{M-k+j-1}} \right\} \alpha^{A i_0 i_1 \dots i_{M-k-2}}_{B a_0 a_1 \dots a_{M-k-2+j}} B^{i_{M-k-1}}_{a_{M-k+j-1}}/B +$$
$$+ (-1)^{k+1} \bar{L}_A^{a_0 a_1 \dots a_M} \alpha^{A i_0 i_1 \dots i_{M-k-2}}_{B a_0 a_1 \dots a_{M-1}} B^{i_{M-k-1}}_{a_M}/B.$$

We see that the first term on the left hand side of (2.9) is the derivative of the left hand side of (2.5) with respect to $x^{i_{M-k}}$. Consequently the left hand side of (2.9) can be expressed as the sum of six expressions, *viz.*

$$\sum_{j=0}^{k-1}(-1)^j\sum_{p=0}^{k-j}(-1)^p\bar{L}_A^{a_0a_1\ldots a_{M-p}}{}_{,a_{M-p}a_{M-p-1}\ldots a_{M-k+j+1}a_{M-k+j}}\alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0a_1\ldots a_{M-k+j-1}}/B+$$

$$+\sum_{j=0}^{k-1}(-1)^j\sum_{p=0}^{k-j-1}(-1)^p\bar{L}_A^{a_0a_1\ldots a_{M-p}}{}_{,a_{M-p}\ldots a_{M-k+j+1}}\alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0a_1\ldots a_{M-k+j-1},a_{M-k+j}}/B+$$

$$+\sum_{j=0}^{k-1}(-1)^k\bar{L}_A^{a_0a_1\ldots a_{M-k+j}}\alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0a_1\ldots a_{M-k+j-1},a_{M-k+j}}/B+$$

$$+(-1)^k\bar{L}_A^{a_0a_1\ldots a_M}{}_{,a_M}\alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0a_1\ldots a_{M-1}}/B+(-1)^k\bar{L}_A^{a_0\ldots a_M}\alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0\ldots a_{M-1},a_M}/B+$$

$$+(-1)^{k+1}\sum_{j=0}^{k+1}\bar{L}_A^{a_0a_1\ldots a_{M-k+j-1}}\alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0a_1\ldots a_{M-k+j-1}}/B.$$

The fourth expression can be absorbed in the first by extending the summation in the latter from $j=k-1$ to $j=k$. In the second and third expressions we replace the summation over j by one over $j-1$ and include a $j=0$ term (which is zero by (2.3)). From the sixth expression we extract the term $j=k+1$ and combine it with the fifth expression. Finally we group the remaining terms in the sixth expression with those in the third to find that the left hand side of (2.9) reads

$$\sum_{j=0}^{k}(-1)^j\left\{\sum_{p=0}^{k-j}(-1)^p\bar{L}_A^{a_0\ldots a_{M-p}}{}_{,a_{M-p}\cdots a_{M-k+j}}+(-1)^{k-j+1}\bar{L}_A^{a_0\ldots a_{M-k+j-1}}\right\}\times$$

$$\times\ (\alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0a_1\ldots a_{M-k+j-1}}-\alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0\ldots a_{M-k+j-2},a_{M-k+j-1}})/B+$$

$$+(-1)^{k+1}\bar{L}_A^{a_0\ldots a_M}(\alpha^{Ai_0\ldots a_{M-k-1}}_{Ba_0\ldots a_M}-\alpha^{Ai_0i_1\ldots i_{M-k-1}}_{Ba_0a_1\ldots a_{M-1},a_M})/B.$$

By virtue of (2.3) this is the right hand side of (2.9), which establishes the lemma.

With the aid of this lemma we are now in a position to prove

THEOREM 1. *If under* (1.1) ρ^A *and* L *transform according to* (1.2) *and* (1.5) *respectively then*

$$BE_A(L)=C_A^B\bar{E}_B(\bar{L}). \tag{2.10}$$

PROOF. In (2.5) we set $k=M-1$ to find

$$\sum_{p=0}^{M-2}(-1)^pL_B^{i_0\ldots i_{M-p}}{}_{,i_{M-p}\ldots i_2}+(-1)^{M-1}L_B^{i_0i_1}\ =$$

$$=\sum_{j=0}^{M-2}(-1)^j\left\{\sum_{p=0}^{M-j-2}(-1)^p\bar{L}_A^{a_0\ldots a_{M-p}}{}_{,a_{M-p}\ldots a_{j+2}}+\right.$$

$$\left.+\ (-1)^{M-j-1}\bar{L}_A^{a_0\ldots a_{j+1}}\right\}\alpha^{Ai_0}_{Ba_0\ldots a_j}B^{i_1}_{a_{j+1}}/B\ +$$

$$+\ (-1)^{M-1}\bar{L}_A^{a_0\ldots a_M}\alpha^{Ai_0}_{Ba_0\ldots a_{M-1}}B^{i_1}_{a_M}/B.$$

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By differentiating the latter with respect to x^{i_1}, multiplying by $(-1)^{-M}$ and recalling (1.4) we see that

$$E_B(L) = \sum_{j=0}^{M-1}(-1)^j \sum_{p=0}^{M-j-1}(-1)^{p-M}\bar{L}_A^{a_0\dots a_{M-p}}{}_{,a_{M-p}\dots a_{j+1}}\alpha^{Ai_0}_{Ba_0\dots a_j}/B +$$
$$+\sum_{j=0}^{M-2}(-1)^j\left\{\sum_{p=0}^{M-k-2}(-1)^{p-M}\bar{L}_A^{a_0\dots a_{M-p}}{}_{,a_{M-p}\dots a_{j+2}}+(-1)^{-j-1}\bar{L}_A^{a_0\dots a_{j+1}}\right\}\times$$
$$\times\ \alpha^{Ai_0}_{Ba_0\dots a_j,a_{j+1}}/B-\bar{L}_A^{a_0\dots a_M}\ \alpha^{Ai_0}_{Ba_0\dots a_{M-1},a_M}/B+$$
$$+\ \sum_{j=0}^{M}\bar{L}_A^{a_0\dots a_j}\ \alpha^{Ai_0}_{Ba_0\dots a_j}/B.$$

By virtue of (2.3) only the $j = 0$ terms in the first and final expresssions survive on the right hand side—all others cancel. We thus find

$$E_B(L) = \left\{\sum_{p=0}^{M-1}(-1)^{p-M}\bar{L}_A^{a_0\dots a_{M-p}}{}_{,a_{M-}\ \dots a_1}+\bar{L}_A^{a_0}\right\}\alpha^{Ai_0}_{Ba_0}/B,$$

which, in view of (2.4), is (2.10).

From Theorem 1 we see immediately that if ρ^A are the components of a relative tensor of covariant valency r, contravariant valency s and weight w, then $E_A(L)$ are the components of a relative tensor of covariant valency s, contravariant valency r and weight $(1-w)$.

We remark that Theorem 1 can be extended to quantities ρ^A which transform according to

$$\bar{\rho}^A = C^A_B\rho^B + \theta^A \tag{2.11}$$

where the C^A_B and θ^A are functions of $\bar{x}^a$ and are completely determined by the transformation (1.1). [A typical example of this would be the symmetric affine connection $\Gamma^h_{i\ j}$ which transforms according to

$$\bar{\Gamma}_b{}^a{}_c = B^m_bB^r_cA^a_h\Gamma_m{}^h{}_r + A^a_hB_b{}^h{}_c.]$$

In fact we can prove the

THEOREM. *If under* (1.1) *L is a scalar density and* ρ^A *transform according to* (2.11) *then*

$$BE_A(L) = C^B_A\bar{E}_B(\bar{L}).$$

However, we will not consider this case in any further detail, but will return to (1.2). If we multiply (2.10) by ρ^A and note (1.2) we have

THEOREM 2. *If under* (1.1) ρ^A *and L transform according to* (1.2) *and* (1.5) *respectively then* ρ^AE_A *is a scalar density.*

We shall now give a direct proof of the frequently asserted

THEOREM 3. *A divergence satisfies the Euler-Lagrange equations identically*[4], *i.e. if a Lagrangian* $\underset{(D)}{L}$ *is of the form*

$$\underset{(D)}{L} = S^i{}_{,i}$$

where

$$S^i = S^i(\rho^A; \rho^A{}_{,i_1}; \ldots; \rho^A{}_{,i_1 \ldots i_{M-1}}; \lambda^\alpha; \lambda^\alpha{}_{,i_1}; \ldots; \lambda^\alpha{}_{,i_1 \ldots i_p}),$$

and p *is any positive integer then*

$$E_A(\underset{(D)}{L}) \equiv 0.$$

PROOF. It is clear that[5]

$$\underset{(D)}{L} = \sum_{r=0}^{M-1} S^{i;}{}_B{}^{j_0 j_1 \ldots j_r} \rho^B{}_{,j_0 j_1 \ldots j_r i} + \sum_{r=0}^{p} S^{i;}{}_\alpha{}^{j_0 j_1 \ldots j_r} \lambda^\alpha{}_{,j_0 j_1 \ldots j_r i} \tag{2.12}$$

where

$$S^{i;}{}_B{}^{j_0 j_1 \ldots j_r} = \partial S^i / \partial \rho^B{}_{,j_0 j_1 \ldots j_r}$$

and

$$S^{i;}{}_\alpha{}^{j_0 j_1 \ldots j_r} = \partial S^i / \partial \lambda^\alpha{}_{,j_0 j_1 \ldots j_r}.$$

For $1 \leqq k \leqq M-1$ it is easily seen, from (2.12), that

$$\begin{aligned} \underset{(D)}{L}{}_A{}^{i_0 i_1 \ldots i_k} = & \sum_{r=0}^{M-1} (\partial S^{i;}{}_B{}^{j_0 j_1 \ldots j_r} / \partial \rho^A{}_{,i_0 i_1 \ldots i_k}) \rho^B{}_{,j_0 j_1 \ldots j_r i} + \\ & + \sum_{r=0}^{p} (\partial S^{i;}{}_\alpha{}^{j_0 j_1 \ldots j_r} / \partial \rho^A{}_{,i_0 i_1 \ldots i_k}) \lambda^\alpha{}_{,j_0 j_1 \ldots j_r i} + \\ & + S^{[i_k;}{}_A{}^{\|i_0\| i_1 \ldots i_{k-1}]}, \end{aligned} \tag{2.13}$$

where the square bracket denotes complete symmetrisation over $i_1 \ldots i_k$ (since i_0 is excluded from this symmetrisation process we place it in braces). However, for $1 \leqq k \leqq M-1$, it is easily seen that

$$\begin{aligned} S_A^{i; i_0 i_1 \ldots i_k}{}_{,i} = & \sum_{r=0}^{M-1} (\partial S_A^{i; i_0 i_1 \ldots i_k} / \partial \rho^B{}_{,j_0 j_1 \ldots j_r}) \rho^B{}_{,j_0 j_1 \ldots j_r i} + \\ & + \sum_{r=0}^{p} (\partial S_A^{i; i_0 i_1 \ldots i_k} / \partial \lambda^\alpha{}_{,j_0 j_1 \ldots j_r}) \lambda^\alpha{}_{,j_0 j_1 \ldots j_r i}, \end{aligned}$$

[4] This result is not at variance with [1] p. 121.

[5] Summation over α from 1 to q.

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which, when taken together with (2.13), yields

$$\underset{(D)}{L}_A^{i_0 i_1 \dots i_k} = S_A^{i;i_0 i_1 \dots i_k}{}_{,i} + S_A^{[i_k;||i_0||i_1 \dots i_{k-1}]}.$$

From the latter we see that for $1 \leqq k \leqq M-1$

$$\underset{(D)}{L}_A^{i_0 i_1 \dots i_k}{}_{,i_0 i_1 \dots i_k} = S_A^{i_{k+1};i_0 i_1 \dots i_k}{}_{,i_0 i_1 \dots i_{k+1}} + S_A^{i_k;i_0 i_1 \dots i_{k-1}}{}_{,i_0 i_1 \dots i_k}$$

from which we conclude that

$$\sum_{k=1}^{M-1} (-1)^k \underset{(D)}{L}_A^{i_0 i_1 \dots i_k}{}_{,i_0 i_1 \dots i_k} + (-1)^M S_A^{i_M;i_0 i_1 \dots i_{M-1}}{}_{,i_0 i_1 \dots i_{M-1}} + S_A^{i_1;i_0}{}_{,i_1} = 0.$$

In view of (2.12) the last two terms on the left hand side are respectively the $k = M$ and $k = 0$ terms of the first expression, so that

$$\underset{(D)}{E_A(L)} \equiv 0.$$

3. Degenerate Lagrange densities

For the reasons indicated in section 1 we now wish to investigate the consequences of adopting the quantity

$$\mathscr{L} = \rho^A E_A(L) \tag{3.1}$$

as a Lagrangian. If it is assumed that the ρ^A transform according to (1.2) then Theorem 2 assures us that $\mathscr{L}$ is a scalar density—one of the requirements usually made of a Lagrangian. Furthermore, if L is of the type (1.3) then in general $E_A(L)$ will involve derivatives of ρ^A up to order $2M$ and derivatives of λ^α up to order $M+Q$, in which case

$$\mathscr{L} = \mathscr{L}(\rho^A; \rho^A{}_{,i_1}; \dots; \rho^A{}_{,i_1 i_2 \dots i_{2M}}; \lambda^\alpha; \lambda^\alpha{}_{,i_1}; \dots; \lambda^\alpha{}_{,i_1 i_2 \dots i_{M+Q}}).$$

This in turn suggests that the associated Euler-Lagrange expression $E_A(\mathscr{L})$ [*i.e.* (1.4) with L replaced by $\mathscr{L}$ and M replaced by $2M$] will be of order $4M$ in ρ^A. In fact this is not the case as is shown by

THEOREM 4. *If*

$$L = L(\rho^A; \rho^A{}_{,i_1}; \dots; \rho^A{}_{,i_1 i_2 \dots i_M}; \lambda^\alpha; \lambda^\alpha{}_{,i_1}; \dots; \lambda^\alpha{}_{,i_1 \dots i_Q})$$

and

$$\mathscr{L} = \rho^A E_A(L)$$

then $E_A(\mathscr{L})$ is at most of order $2M$ in ρ^A i.e. $\mathscr{L}$ is a degenerate Lagrange density.

PROOF. For any positive integer r, $2 \leqq r \leqq M$ consider the quantity

$$S^{i_{r-j}}{}_{,i_{r-j}}$$

where

$$S^{i_{r-j}} = (-1)^{r-j}\rho^{A}{}_{,i_r i_{r-1}\dots i_{r-j+1}} L_A^{i_0 i_1 \dots i_r}{}_{,i_0 i_1 \dots i_{r-j-1}}$$

and $1 \leqq j \leqq r-1$. Clearly we have

$$S^{i_{r-j}}{}_{,i_{r-j}} = (-1)^{r-j}\rho^{A}{}_{,i_r \dots i_{r-j}} L_A^{i_0 i_1 \dots i_r}{}_{,i_0 i_1 \dots i_{r-j-1}} +$$

$$- (-1)^{r-j+1}\rho^{A}{}_{,i_r \dots i_{r-j+1}} L_A^{i_0 i_1 \dots i_r}{}_{,i_0 \dots i_{r-j}},$$

from which we conclude that

$$\sum_{j=1}^{r-1} S^{i_{r-j}}{}_{,i_{r-j}} = -\rho^{A}{}_{,i_0 i_1 \dots i_r} L_A^{i_0 i_1 \dots i_r} - (-1)^r \rho^{A}{}_{,i_r} L_A^{i_0 \dots i_r}{}_{,i_0 \dots i_{r-1}}.$$

In view of the fact that the last term on the right hand side can be expressed in the form

$$(-1)^{r+1}\{(\rho^A L_A^{i_0 \dots i_r}{}_{,i_0 \dots i_{r-1}})_{,i_r} - \rho^A L_A^{i_0 \dots i_r}{}_{,i_0 \dots i_r}\},$$

we find that for $2 \leqq r \leqq M$

$$(-1)^r \rho^A L_A^{i_0 \dots i_r}{}_{,i_0 \dots i_r} = \rho^{A}{}_{,i_0 \dots i_r} L_A^{i_0 \dots i_r} + \sum_{j=1}^{r-1} S^{i_{r-j}}{}_{,i_{r-j}}$$

$$+ (-1)^r (\rho^A L_A^{i_0 \dots i_r}{}_{,i_0 \dots i_{r-1}})_{,i_r}.$$

Consequently we find

$$\sum_{r=0}^{M} (-1)^r \rho^A L_A^{i_0 \dots i_r}{}_{,i_0 \dots i_r} = \sum_{r=0}^{M} \rho^{A}{}_{,i_0 \dots i_r} L_A^{i_0 \dots i_r} + \sum_{r=2}^{M} \sum_{j=1}^{r-1} S^{i_{r-j}}{}_{,i_{r-j}} + \tag{3.2}$$

$$+ \sum_{r=1}^{M} (-1)^r (\rho^A L_A^{i_0 \dots i_r}{}_{,i_0 \dots i_{r-1}}\}_{,i_r}$$

which, in view of (3.1), is of the form

$$\mathscr{L} = \mathscr{L}_1 + T^{i}{}_{,i} \tag{3.3}$$

where

$$\mathscr{L}_1 = \sum_{r=0}^{M} \rho^{A}{}_{,i_0 \dots i_r} L_A^{i_0 \dots i_r} \tag{3.4}$$

and

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$$T^i = \sum_{r=2}^{M} \sum_{j=1}^{r-1} S^{i_r - j}\delta^i_{i_{r-j}} + \sum_{r=1}^{M} (-1)^r \rho^A L_A^{i_0 \dots i_r}{}_{,i_0 \dots i_{r-1}} \delta^i_{i_r}.$$

However, by virtue of the fact that $T^i{}_{,i}$ is a divergence, Theorem 3 assures us that it will not contribute to $E_A(\mathscr{L})$. Therefore we find from (3.3) that

$$E_A(\mathscr{L}) = E_A(\mathscr{L}_1).$$

Furthermore it is clear from (3.4) that

$$\mathscr{L}_1 = \mathscr{L}_1(\rho^A; \rho^A{}_{,i_1}; \dots; \rho^A{}_{,i_1 \dots i_M}; \lambda^\alpha; \lambda^\alpha{}_{,i_1}; \dots; \lambda^\alpha{}_{,i_1 \dots i_Q}\},$$

so that $E_A(\mathscr{L})$ is at most of order $2M$ in ρ^A. This establishes the theorem.

We have shown that if $\mathscr{L}$, given by (3.1), is used as a Lagrangian then the corresponding Euler-Lagrange equations are also obtained from $\mathscr{L}_1$, given by (3.4). We also know that $\mathscr{L}$ is a scalar density. It is obviously of interest to establish the tensorial character of $\mathscr{L}_1$. This can easily be accomplished as follows. Under (1.1) we have

$$\begin{aligned} \sum_{r=0}^{M} \rho^B{}_{,i_0 \dots i_r} L_B^{i_0 \dots i_r} &= \sum_{r=0}^{M} \rho^B{}_{,i_0 \dots i_r} \sum_{j=0}^{M} \bar{L}_A^{a_0 \dots a_j} \alpha^{A i_0 \dots i_r}_{B a_0 \dots a_j} / B \\ &= \sum_{j=0}^{M} \left\{ \sum_{r=0}^{M} \rho^B{}_{,i_0 \dots i_r} \alpha^{A i_0 \dots a_r}_{B a_0 \dots a_j} \right\} \bar{L}_A^{a_0 \dots a_j} / B. \end{aligned}$$

By (2.1) and 2.3) the quantity in brackets on the right hand side is $\bar{\rho}^A{}_{,a_0 \dots a_j}$, which establishes

THEOREM 5. *The Euler-Lagrange expressions obtained from*

$$\mathscr{L} = \rho^A E_A(L)$$

and

$$\mathscr{L}_1 = \sum_{r=0}^{M} \rho^A{}_{,i_0 \dots i_r} L_A^{i_0 \dots i_r}$$

are identical, and, furthermore, both $\mathscr{L}$ *and* $\mathscr{L}_1$ *are scalar densities.*

We are now in a position to give a partial answer to the third question posed in section 1. From Theorem 5 we see that if $\mathscr{L}_1$ is proportional to L *i.e.* if

$$\mathscr{L}_1 = kL$$

where k is a non-zero constant then

$$E_A(\mathscr{L}) = kE_A(L).$$

By virtue of the definition of $\mathscr{L}_1$ we thus have

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THEOREM 6. *If L is homogeneous of degree $k(\neq 0)$ in the variables $(\rho^A; \rho^A{}_{,i_1}; \ldots; \rho^A{}_{,i_1 \ldots i_M})$, i.e. if*

$$\sum_{r=0}^{M} \rho^A{}_{.i_0 i_1 \ldots i_r} L_A^{i_0 i_1 \ldots i_r} = kL,$$

then the Lagrange density $\rho^A E_A(L)$ will also have as its Euler-Lagrange equations precisely

$$E_A(L) = 0.$$

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