numbers and the Fibonacci numbers is: $L_n = F_n + 2F_{n-1}$, $n \geqslant 1$ (see [1, p. 117, Ex. 5.44]). Combining this relation with the result given in (1) leads to a relatively simple integral representation of the Lucas numbers. It is

$$L_n = \frac{n}{2^n} \int_{-1}^{1} \left(5 + x\sqrt{5} - \frac{4}{n}\right)\left(1 + x\sqrt{5}\right)^{n-2} dx,$$

and is valid for $n \geqslant 1$. Many other integral representations of the Lucas numbers can be found by employing other known relations between the two numbers L_n and F_n which the reader may care to find.

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107.02 Collatz conjecture: coalescing orbits and conditions on a minimum counterexample

Introduction

Originally proposed by Lothar Collatz in the 1930s, the Collatz Conjecture, also known as the Collatz Problem, Syracuse Problem, and $3n + 1$ Conjecture, has become a notoriously difficult unsolved problem in mathematics. Much of its appeal is in the simplicity of the problem statement. The conjecture states that for every positive integer n, iterating over the function below will eventually yield the number 1:

$$C(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ 3n + 1 & \text{if } n \text{ is odd.} \end{cases}$$

The sequence of numbers generated by iterating over the Collatz Function is known as a Collatz Orbit. For example, the numbers 7 and 15 generate the following Collatz Orbits, respectively:

7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, …

15, 46, 23, 70, 35, 106, 53, 160, 80, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, …

The length of Collatz Orbits can be very erratic. For example, the number 27 goes through 111 iterations before returning to 1 while no number prior to 27 takes more than 23 iterations to return to 1. At time of writing, the Collatz Conjecture has been verified for all starting numbers up to 2^{68}. Numerous attempts have been made to tackle this problem as can be seen in Jeffrey Lagarias' surveys over the years. These attempts include heuristic arguments as well as efforts to bound a number within a Collatz Orbit below the starting number, suggesting that only a low percentage of numbers could be minimum counterexamples [1, 2, 3, 4]. In many cases, it is not obvious why different starting numbers (like 7 and 15 shown above) have orbits that eventually reach the same number, at which point the ensuing sub-sequences will obviously be identical. In this Note, we show why certain orbits coalesce in this way, a phenomenon which may help us more directly understand the dynamics of this problem. We also use these orbital relationships to establish specific conditions on a minimum counterexample.

Accelerated Collatz Function

Note that every positive odd integer can be written in the form $2^a 3^b k - 1$, where $a \in \mathbb{N}_1 = \{0, 1, 2, \dots\}$, $b \in \mathbb{N}_0 = \{1, 2, 3, \dots\}$, and k is a positive odd integer not divisible by 3. Consider the orbit generated by iterating over an arbitrary positive odd integer written in this form:

$$C(n) = 3\left(2^a 3^b k - 1\right) + 1 = 3\left(2^a\right)\left(3^b\right)k - 3 + 1 = \mathbf{2^a 3^{b+1} k - 2}$$

$$C^2(n) = \frac{2^a 3^{b+1} k - 2}{2} = \mathbf{2^{a-1} 3^{b+1} k - 1}$$

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$$C^{2a}(n) = \frac{2\left(3^{a+b}\right) - 2}{2} = \mathbf{3^{a+b} k - 1}.$$

In fact, every positive odd integer, written in the form $2^a 3^b k - 1$, will go to $3^{a+b} k - 1$ after $2a$ iterations. Additionally, $3^{a+b} k - 1$ is always even.

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As such, we can define the following Accelerated Collatz Function:

$$A(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{3^{a+b}k-1}{2} & \text{if } n \text{ is odd and written in the form } 2^a 3^b k - 1 \text{ given above.} \end{cases}$$

Define each iteration of this function as an accelerated iteration, and define the corresponding orbit as the accelerated orbit. Then, the numbers 7 and 15, which can be written as $2^3 3^0 - 1$ and $2^4 3^0 - 1$, respectively, generate the accelerated orbits below. These are notably more condensed than the standard Collatz Orbits shown above:

$$7, 13, 10, 5, 4, 2, 1, 1, 1, \ldots$$
$$15, 40, 20, 10, 5, 4, 2, 1, 1, 1, \ldots.$$

Coalescing Orbits

When using the Accelerated Collatz Function defined above, it becomes apparent why seemingly unrelated numbers may have orbits that coalesce. The lemmas and theorem below summarise these relationships. If you apply the lemmas and theorem below to our example numbers of 7 and 15, we can see, prior to iterating, why the orbits of these two numbers coalesce and why 15 cannot be a minimum counterexample.

Lemma 1: Let n be a positive odd integer written in the form $2^{a_n} 3^{b_n} k - 1$, and let m be a positive odd integer written in the form $2^{a_m} 3^{b_m} k - 1$ such that $a_n, a_m \in \mathbb{N}_1$, $b_n, b_m \in \mathbb{N}_0$, and k is a positive odd integer not divisible by 3. The orbit beginning with n will coalesce with that of m if $a_n + b_n = a_m + b_m$.

Proof: The proof is trivial when considering the Accelerated Collatz Function above. After one accelerated iteration, n and m will go to

$$\frac{3^{a_n + b_n} k - 1}{2} \quad \text{and} \quad \frac{3^{a_m + b_m} k - 1}{2},$$

respectively. Since we assumed that $a_n + b_n = a_m + b_m$, we can see that these two numbers are equal.

Lemma 2: For $a \in \mathbb{N}_1$, if $\frac{1}{2}\left(3^{a-1}k - 1\right)$ is odd, then the orbits of $n = 2^a k - 1$ and $m = 2^{a-1} k - 1$ will coalesce.

Proof: After one accelerated iteration, m will go to $\frac{1}{2}\left(3^{a-1}k - 1\right)$. If this is odd, another regular iteration takes it to $\frac{1}{2}\left[3\left(3^{a-1}k - 1\right) + 2\right] = \frac{1}{2}\left(3^a k - 1\right)$, which coincides with the first accelerated iteration of n.

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Conditions on a Minimum Counterexample

Theorem: Any minimum counterexample n to the Collatz Conjecture must be of the form $2^a k - 1$ where a is a positive integer strictly greater than 1, k is a positive odd integer not divisible by 3, $k \equiv 1 \bmod 4$ if a is odd, and $k \equiv -1 \bmod 4$ if a is even.

Proof: Trivially, n must be odd, so it can be written in the form $n = 2^a 3^b k - 1$ described further above. If $a = 1$, then after one accelerated iteration, n will go to a number smaller than n, so a must be strictly greater than 1. From Lemma 1, we can see that if $b > 0$, there is a smaller number whose orbit will coalesce with that of n after one accelerated iteration, so we must have $b = 0$. From Lemma 2, we can see that if $\frac{1}{2}\left(3^{a-1}k - 1\right)$ is odd, then $2^{a-1}k - 1$, which is smaller than n, will have an orbit that coalesces with that of n. It follows that a minimum counterexample must have $\frac{1}{2}\left(3^{a-1}k - 1\right)$ as even, which implies that $3^{a-1}k \equiv 1 \bmod 4$. Since $3^{a-1} \equiv 1 \bmod 4$ if a is odd and $3^{a-1} \equiv -1 \bmod 4$ if a is even, it follows that $k \equiv 1 \bmod 4$ if a is odd and $k \equiv -1 \bmod 4$ if a is even.

The above theorem allows us to immediately eliminate many non-trivial minimum counterexample candidates. The reader may care to check that after applying the theorem, the following would be the only remaining counterexample candidates below 100 (supposing that they had not already been verified as non-counterexamples):

$$7, 27, 31, 39, 43, 75, 91.$$

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