MAPS OF CERTAIN ALGEBRAIC CURVES INVARIANT UNDER CYCLIC INVOLUTIONS OF PERIODS THREE, FIVE, AND SEVEN

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1. Introduction. In earlier papers **(4; 5; 6)**, certain space curves, invariant under cyclic involutions of periods three, five, and seven, have been mapped. Lucien Godeaux **(2; 3)** in 1916 mapped plane cubic curves, invariant under an involution of period three, onto a cubic surface in ordinary three-space. Mlle. J. Dessart **(1)** in 1931 mapped plane quintic curves, invariant under an involution of order five, onto a fifth order surface in a space of four dimensions.

This paper concerns itself with the mapping of plane septimic curves, invariant under the cyclic involution

$$T \qquad x_1' : x_2' : x_3' = x_1 : Ex_2 : E^2x_3 \text{ where } E^7 = 1,$$

onto a linear space (S_5) of five dimensions.

The general system of plane curves of order seven is, in general, non-invariant under the transformation T. It can be split up, however, into seven invariant curves:

$$\sum_{i=0}^{6} \lambda_i C_i = 0, \tag{1}$$

where

$$C_0 = v_0x_1^7 + v_1x_1^3x_2x_3^3 + v_2x_1^2x_2^3x_3^2 + v_3x_1x_2^5x_3 + v_4x_2^7 + v_5x_3^7,$$

$$C_1 = u_0x_1^4x_3^3 + u_1x_1^3x_2^2x_3^2 + u_2x_1^2x_2^4x_3 + u_3x_1x_2^6 + u_4x_2x_3^6,$$

$$C_2 = u_0x_1^4x_2x_3^2 + u_1x_1^3x_2^3x_3 + u_2x_1^2x_2^5 + u_3x_1x_3^6 + u_4x_2^2x_3^5,$$

$$C_3 = u_0x_1^5x_3^2 + u_1x_1^4x_2^2x_3 + u_2x_1^3x_2^4 + u_3x_1x_2x_3^5 + u_4x_2^3x_3^4,$$

$$C_4 = u_0x_1^5x_2x_3 + u_1x_1^4x_2^3 + u_2x_1^2x_3^5 + u_3x_1x_2^2x_3^4 + u_4x_2^4x_3^3,$$

$$C_5 = u_0x_1^6x_3 + u_1x_1^5x_2^2 + u_2x_1^2x_2x_3^4 + u_3x_1x_2^3x_3^3 + u_4x_2^5x_3^2,$$

$$C_6 = u_0x_1^6x_2 + u_1x_1^3x_3^4 + u_2x_1^2x_2^2x_3^3 + u_3x_1x_2^4x_3^2 + u_4x_2^6x_3.$$

The (1,1) correspondence between the ∞^5 curves of C_0 and the hyperplanes of S_5 defines the transformation

$$\frac{X_0}{x_1^7} = \frac{X_1}{x_1^3x_2x_3^3} = \frac{X_2}{x_1^2x_2^3x_3^2} = \frac{X_3}{x_1x_2^5x_3} = \frac{X_4}{x_2^7} = \frac{X_5}{x_3^7}. \tag{2}$$

By eliminating the x_i's from these equations, one gets for the new surface F the equations

$$\left\| \begin{matrix} X_1 & X_2 & X_3 & X_0X_5 \\ X_2 & X_3 & X_4 & X_1^2 \end{matrix} \right\| = 0. \tag{3}$$

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This surface F is the branch point surface in S_5 of the transformation.

2. Harmonic homology. A careful study of the invariant curves C_i, shows that the homography

$$\Omega \qquad y_1 : y_2 : y_3 = x_3 : x_2 : x_1$$

in the plane containing the involution I_7, is a harmonic homology of centre $A(x_2 = 0, x_1 + x_3 = 0)$ and with axis a $(x_1 - x_3 = 0)$. This homology transforms O_1 (1,0,0) into O_3 (0,0,1), O_3 into O_1, and O_2 (0,1,0) into itself. Furthermore, the homology also transforms the totality of curves C_0 into C_0, C_1 into C_6, C_2 into C_5, and C_3 into C_4.

This harmonic homology corresponds in S_5 to the harmonic homology

$$\Omega' \qquad \frac{X_0}{Y_5} = \frac{X_1}{Y_1} = \frac{X_2}{Y_2} = \frac{X_3}{Y_3} = \frac{X_4}{Y_4} = \frac{X_5}{Y_0}$$

with centre at A' $(X_1 = X_2 = X_3 = X_4 = 0, X_0 + X_5 = 0)$ and axis the hyperplane a' $(X_0 - X_5 = 0)$. This homology transforms the surface F into itself.

3. Image curves. We will designate by Γ_0 the hyperplane sections of F, which correspond to the curves C_0. Likewise, Γ_1, Γ_2, Γ_3, Γ_4, Γ_5, and Γ_6 are the curves on F, which correspond, respectively, to the curves C_1, C_2, C_3, C_4, C_5, and C_6.

By the indicated homology, then, these curves are transformed as follows: Γ_0 goes into itself, Γ_1 into Γ_6, Γ_2 into Γ_5, and Γ_3 into Γ_4.

It is observed that any curve from the system C_i $(i = 1, 2, \ldots, 6)$ intersects a curve from C_0 in forty-nine points, forming seven groups of the involution I_7. It follows, therefore, that Γ_i $(i = 1, 2, \ldots 6)$ will intersect Γ_0 in seven points, making the curves Γ_i of order seven.

The equations of Γ_1 are

$$\left\| \begin{matrix} X_1 & X_2 & X_3 & X_0X_5 & (-u_4X_5) \\ X_2 & X_3 & X_4 & X_1^2 & (u_0X_1 + u_1X_2 + u_2X_3 + u_3X_4) \end{matrix} \right\| = 0.$$

In the equations of the remaining Γ_i, the matrices differ only in the last column. For $i = 2, \ldots, 6$ the last columns are, respectively:

$$(-u_4X_5),\ (u_0X_1 + u_1X_2 + u_2X_3 + u_3X_5);\ (-u_1X_0X_1 - u_2X_0X_2),$$

$$(u_3X_1^2 + u_4X_1X_2 + u_0X_0X_1);\ (-u_3X_1X_5 - u_4X_2X_5), (u_0X_1^2 + u_1X_1X_2 + u_2X_1X_5);$$

$$(u_1X_0), (u_0X_0 + u_2X_1 + u_3X_2 + u_4X_3);\ (-u_0X_0), (u_1X_1 + u_2X_2 + u_3X_3 + u_4X_4).$$

4. Branch point O_0'. To point O_1 on the plane corresponds on F the point $O_0'(1,0,0,0,0,0)$. The singularities of this point will now be studied. Consider the family (∞^4) of curves from the system C_0, passing through the invariant point O_1. The equation for this family of curves is

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$$v_1x_1^3x_2x_3^3 + v_2x_1^2x_2^3x_3^2 + v_3x_1x_2^5x_3 + v_4x_2^7 + v_5x_3^7 = 0. \tag{4}$$

Applying to equations (4) the quadratic transformation

$$U \qquad x_1' : x_2' : x_3' = z_1^2 : z_1z_2 : z_2z_3$$

and simplifying, we get

$$z_1^7(v_1z_3^3 + v_2z_2z_3^2 + v_3z_2^2z_3 + v_4z_2^3) + v_5z_2^3z_3^7 = 0. \tag{5}$$

This shows that to the point O_{12} (the first order neighbourhood of O_1 on $x_3 = 0$), corresponds the triple point ($z_2 = z_3 = 0$) for the curves (5).

In order to obtain the points, infinitely near $O_0'(1,0,0,0,0,0)$ on F, which correspond to the points infinitely near the point ($z_2 = z_3 = 0$), it is necessary, first, to project the surface F from the point O_0' onto the hyperplane $X_0 = 0$. This gives the independent equations

$$F_1 \qquad X_1X_3 = X_2^2, \quad X_2X_4 = X_3^2, \quad X_0 = 0,$$

and the dependent equation

$$X_1X_4 = X_2X_3.$$

It is noted that the plane $X_0 = X_2 = X_3 = 0$ satisfies the independent equations, but not the dependent equation. Thus F_1 must be a cubic surface **(1)**.

Second, apply the transformation U to the transformation (2) in which $X_0 = 0$ and obtain the simplified expression

$$\frac{X_1}{z_1^7z_3^3} = \frac{X_2}{z_1^7z_2z_3^2} = \frac{X_3}{z_1^7z_2^2z_3} = \frac{X_4}{z_1^7z_2^3} = \frac{X_5}{z_2^3z_3^7}.$$

Since one is interested in approaching the point ($z_2 = z_3 = 0$) from all directions, let $z_3 = kz_2$ and substitute in the last equations. Let z_2 approach zero, which implies that $X_5 = 0$. Eliminating k from the resulting equations, one obtains the cubic cone

$$X_1X_3 = X_2^2, \quad X_2X_4 = X_3^2, \quad X_5 = 0. \tag{6}$$

This cone intersects F_1 in a twisted cubic curve

$$(\gamma_1) \qquad X_1X_3 = X_2^2, \quad X_2X_4 = X_3^2, \quad X_0 = X_5 = 0.$$

This shows that the points of the first order neighbourhood of the point ($z_2 = z_3 = 0$), as well as the points in the domain of the first order neighbourhood of O_{12}, correspond projectively to points of this cubic curve. The points of this curve are projections of the points infinitely near O_0' and on the cubic tangent cone to F at the point O_0'.

Applying now the quadratic transformation

$$V \qquad x_1' : x_2' : x_3' = z_1^2 : z_2z_3 : z_1z_3$$

three times, successively, to the curves (4), one gets

$$z_1^{21}(v_1z_2 + v_5z_3) + v_2z_1^{14}z_2^3z_3^5 + v_3z_1^7z_2^5z_3^{10} + v_4z_2^7z_3^{15} = 0. \tag{7}$$

This shows that to O_{1333} in the third order neighbourhood of O_1 on $x_2 = 0$

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corresponds simply the point ($z_2 = z_3 = 0$). Thus, the curves (4) pass simply through O_{13}, O_{133}, and O_{1333} on the line $x_2 = 0$.

Transforming the curves (7) of the z-plane to S_5 by (2) with $X_0 = 0$, we obtain

$$\frac{X_1}{z_1^{21} z_2} = \frac{X_2}{z_1^{14} z_2^{3} z_3^{5}} = \frac{X_3}{z_1^{7} z_2^{5} z_3^{10}} = \frac{X_4}{z_2^{7} z_3^{15}} = \frac{X_5}{z_1^{21} z_3}.$$

Again substitute $z_3 = kz_2$ and allow z_2 to approach zero, obtaining

$$(8) \qquad X_5 = kX_1, \quad X_2 = X_3 = X_4 = 0.$$

It follows, therefore, that to the points infinitely near the point O_{1333} correspond projectively on the surface F_1 the points on the straight line

$$(a_1) \qquad X_0 = X_2 = X_3 = X_4 = 0.$$

This also means that the points infinitely near O_{1333} correspond to the points infinitely near the point O_0' on F and lying in the plane

$$(9) \qquad X_2 = X_3 = X_4 = 0.$$

This plane is tangent to F at the point O_0'.

We have now established the fact that to the invariant point O_1 of the involution I_7, corresponds on the surface F a branch point O_0', which is a quadruple point. The quartic tangent cone at this point has degenerated into a cubic tangent cone (6) and the tangent plane (9).

Moreover, the cubic curve γ, and the straight line a_1 have in common only the point

$$X_0 = X_2 = X_3 = X_4 = X_5 = 0,$$

which is designated by $O_1'(0,1,0,0,0,0)$.

The cone (6) and the plane (9) have in common only the straight line

$$(10) \qquad X_2 = X_3 = X_4 = X_5 = 0.$$

Hence, the tangent plane is also tangent to the cubic cone along this triple line.

5. Images of curves C_1 at O_0'. The curves of the system C_1 have triple points at the invariant point O_1. Each branch is tangent to the invariant direction $x_3 = 0$.

Applying the transformation U to the curves C_1 gives

$$(11) \qquad z_1^7(u_0 z_3^3 + u_1 z_2 z_3^2 + u_2 z_2^2 z_3 + u_3 z_2^3) + u_4 z_2^4 z_3^6 = 0.$$

These curves have a triple point at $z_2 = z_3 = 0$, and the tangents to the curves at this point have the equations

$$u_0 z_3^3 + u_1 z_2 z_3^2 + u_2 z_2^2 z_3 + u_3 z_2^3 = 0.$$

Thus, the point O_{12}, in the first order neighbourhood of O_1, is triple. Hence, there are three simple variable points in the first order neighbourhood of O_{12}.

To approach the point ($z_2 = z_3 = 0$) along the curves (11), one substitutes $z_3 = kz_2$ into their equation and allows z_2 to approach zero. It follows that

$$z_1^7(u_0k^3z_2^3 + u_1k^2z_2^3 + u_2kz_2^3 + u_3z_2^3) + u_4k^6z_2^{10} = 0,$$

or

$$z_1^7(u_0k^3 + u_1k^2 + u_2k + u_3) + u_4k^6z_2^7 = 0,$$

and hence

(12) $$u_0k^3 + u_1k^2 + u_2k + u_3 = 0.$$

One has learned earlier that the points in the first order neighbourhood of O_{12} project into the points of the twisted cubic curve γ_1. Any one member of the system C_1 has three points in the first order neighbourhood of O_{12}, and their projections on F_1 are therefore on the twisted cubic curve γ_1. The three values for k, found by solving equation (12), will locate the three points on γ_1.

Assume that the roots of (12) are k', k'', and k'''. The three points then have the coordinates:

(Q_1) $(0, k'^3, k'^2, k', 1, 0)$

(Q_2) $(0, k''^3, k''^2, k''\ 1, 0)$

(Q_3) $(0, k'''^3, k'''^2, k''' ,1, 0)$.

So the equation of the plane, $Q_1Q_2Q_3$, can now be written as

$$u_0X_1 + u_1X_2 + u_2X_3 + u_3X_4 = 0, \quad X_0 = X_5 = 0.$$

These results show that the images, designated by Γ_1, of the curves C_1 mapped upon the surface F, have a triple point at O_0', and the tangents to the curves at this point are the intersections of the cubic tangent cone (6) with the hyperplane

$$u_0X_1 + u_1X_2 + u_2X_3 + u_3X_4 = 0.$$

When the curves Γ_1 are projected from O_0' upon the cubic surface F_1, the equations for the curves, now designated by Γ_1', become

$$\left\| \begin{matrix} X_1 & X_2 & X_3 & (-u_4X_5) \\ X_2 & X_3 & X_4 & (u_0X_1 + u_1X_2 + u_2X_3 + u_3X_4) \end{matrix} \right\| = 0,$$
$$X_0 = 0,$$

and are of order four.

The same general procedure is then applied to the remaining systems of curves C_2, C_3, C_4, C_5, and C_6 to complete the study of the behavior of these curves at the branch point O_0'.

The existing harmonic homology permits one then to deduce the behavior of the same curves at the branch point $O_5'(0,0,0,0,0,1)$.

To analyse the systems of curves C_i ($i = 0, 1, 2, \ldots, 6$) at the branch point $O_4'(0,0,0,0,1,0)$, it becomes necessary to project the surface F onto the hyper-

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plane $X_4 = 0$ from the point O_4'. New quadratic transformations must be used, and they are

W, $$x_1 : x_2 : x_3 = z_1z_3 : z_2^2 : z_2z_3$$

and

R, $$x_1 : x_2 : x_3 = z_1z_2 : z_2^2 : z_1z_3.$$

6. Summary. The results of this paper show that the image of a plane cyclic involution of period seven may be taken as a surface of order seven in a linear space of five dimensions. The surface has two quadruple branch points, whose tangent cones are formed by a cubic cone and a plane. The surface has also a third branch point, which is a binode infinitely near to two binodes not on the surface.

There exist on the surface six linear systems (∞^4) of twisted septimic curves. One system, Γ_1, passes triply through one of the quadruple points, with the three tangent lines lying on the cubic cone; Γ_1 also passes simply through the other quadruple point, with the tangent lying in the tangent plane at that point. Finally, Γ_1 passes simply through the binode, with its tangent line in one of the two tangent planes at this point.

A second system, Γ_6, has the same characteristics as Γ_1, only the roles of the two quadruple points and the two tangent planes at the binode are reversed.

A third system, Γ_2, passes triply through one of the two quadruple points, with two tangent lines on the cubic cone and one on the tangent plane. It passes simply through the other quadruple point, with its tangent line on the cubic cone at that point. Finally, Γ_2 passes simply through the binode, with its tangent line the line of intersection of the two tangent planes at that point.

A fourth system, Γ_5, has results analogous to Γ_2, except that the roles of the two quadruple points are reversed.

A fifth system, Γ_3, passes doubly through one of the two quadruple points, with the two tangent lines lying on the cubic cone. It passes doubly through the other quadruple point, with one tangent line on the cubic cone and the other on the tangent plane. Finally, Γ_3 passes simply through the binode. It has for its tangent line the line of intersection of the two tangent planes at this point.

A sixth and final system Γ_4 has the same properties as Γ_3, with the roles of the two quadruple points reversed.

When the systems of curves, Γ_1 and Γ_6, are projected from one of the two quadruple points onto a hyperplane, two new systems result, which are twisted quartic curves.

When the systems Γ_2 and Γ_5 are projected from the binode, and when the systems Γ_3 and Γ_4 are projected from the line connecting this point with its adjacent binode, O_3', both onto respective hyperplanes, twisted quintic curves result.

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