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A smooth, non-reflexive second conjugate space

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It is shown that a separable, quasi-reflexive Banach space of deficiency one admits an equivalent norm such that its second conjugate space is smooth; this answers a question raised by Ivan Singer [*Bull. Austral. Math. Soc.* 12 (1975), 407-416].

A Banach space B with conjugate space B^* is said to be *smooth* if for every $x \in S(B) \equiv \{x \in B : \|x\| = 1\}$, there exists a unique $x^* \in S(B^*)$ such that $x^*(x) = 1$.

The purpose of this note is to show the existence of a non-reflexive Banach space with smooth second conjugate space, a matter of interest raised by Rainwater [4]. Specifically, we show that the quasi-reflexive space J of James [3] admits an equivalent norm for which the second conjugate space is smooth; this answers a question raised by Singer [5].

A Banach space B is said to be *very smooth* (Sullivan [6]) if for every $x \in S(B)$ there exists a unique $x^{***} \in S(B^{***})$ such that $x^{***}\big(Q(x)\big) = 1$ where $Q : B \to B^{**}$ is the canonical embedding.

A Banach space B is said to be *extremely smooth* (Sullivan [6]) if whenever $x^{***}(x^{**}) = y^{***}(x^{**}) = 1$ where $x^{**} \in S(B^{**})$ and $x^{***}, y^{***} \in S(B^{***})$ then $x^{***} - y^{***} \in Q(B)^{\perp}$.

THEOREM. *If B is a separable, quasi-reflexive Banach space of deficiency one, then B admits an equivalent norm such that B^{**} is smooth.*

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Proof. Since both B and B^* are separable, B admits an equivalent norm that is both Fréchet and uniformly Gâteaux differentiable (see Day [2]). Thus we can and do assume that B has such a norm. It follows from results of Sullivan [6] that B is both very smooth and extremely smooth, since B is Fréchet and uniformly Gâteaux differentiable respectively. Write $B^{**} = Q(B) \oplus [b^{**}]$ where $[b^{**}]$ denotes the linear span of $\{b^{**}\}$.

We claim that B^{**} is smooth. To obtain a contradiction, suppose there exist $x^{**} \in S(B^{**})$ and $x^{***}, y^{***} \in S(B^{***})$ such that $x^{***} \neq y^{***}$ and $x^{***}(x^{**}) = y^{***}(x^{**}) = 1$. Write $x^{**} = Q(b) + \beta b^{**}$ where $b \in B$ and β is a scalar. Since B is very smooth, we have that $\beta \neq 0$. Also since B is extremely smooth, we have that $x^{***} - y^{***} \in Q(B)^{\perp}$. It follows that

$$0 = (x^{***}-y^{***})(x^{**}) = (x^{***}-y^{***})(\beta b^{**}) .$$

Since $\beta \neq 0$, it follows that $(x^{***}-y^{***})(b^{**}) = 0$, and hence $x^{***} = y^{***}$. This contradiction completes the proof.

REMARK. From the proof of the theorem, a quasi-reflexive Banach space of deficiency one that is both very smooth and extremely smooth has smooth second conjugate space. However, this implication is not true in general since c_0 admits an equivalent norm that is both Fréchet and uniformly Gâteaux differentiable, and Day [1] has shown that l^∞ admits no equivalent smooth norm. It is trivial that a Banach space is very smooth and extremely smooth if its second conjugate space is smooth.

References

[1] Mahlon M. Day, "Strict convexity and smoothness of normed spaces", *Trans. Amer. Math. Soc.* 78 (1955), 516-528.

[2] Mahlon M. Day, *Normed linear spaces*, 3rd ed. (Ergebnisse der Mathematik und ihrer Grenzgebiete, 21. Springer-Verlag, Berlin, Heidelberg, New York, 1973).

[3] Robert C. James, "A non-reflexive Banach space isometric with its second conjugate space", *Proc. Nat. Acad. Sci. USA* 37 (1951), 174-177.

[4] John Rainwater, "A non-reflexive Banach space has non-smooth third conjugate space", preprint.

[5] Ivan Singer, "On the problem of non-smoothness of non-reflexive second conjugate spaces", *Bull. Austral. Math. Soc.* 12 (1975), 407-416.

[6] Francis Sullivan, Geometrical properties determined by the higher duals of a Banach space", preprint.

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