Bull. Aust. Math. Soc. **99** (2019), 302–310
doi:10.1017/S000497271800103X

ON GENERALISED METRISABILITY AND CARDINAL INVARIANTS IN QUASITOPOLOGICAL GROUPS

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(Received 25 July 2018; accepted 27 August 2018; first published online 17 October 2018)

Abstract

We consider generalised metrisability and cardinal invariants in quasitopological groups. We construct examples to show that some equalities of cardinal invariants in topological groups cannot be extended to quasitopological groups.

2010 *Mathematics subject classification*: primary 54A25; secondary 22A05, 54E25, 54G20, 54H11.

Keywords and phrases: cardinal invariant, semimetrisable, topological group, quasitopological group.

1. Introduction

All spaces are Hausdorff unless stated otherwise.

A *paratopological group* G is a group endowed with a topology such that the multiplication of G is jointly continuous. A *semitopological group* G is a group endowed with a topology such that the multiplication of G is separately continuous. A *topological group* (respectively, *quasitopological group*) is a paratopological group (respectively, semitopological group) G such that the inversion of G is continuous.

It is well known that a topological group is metrisable if and only if it is first-countable [4, 8], and a paratopological group is quasimetrisable if and only if it is first-countable [10, 11]. Recently, Li and Mou [9] and Shen [13] proved that a quasitopological group is semimetrisable if and only if it is first-countable. The authors in [14] discussed generalised metrisability of a special class of semi- and quasitopological groups and three-space properties in quasitopological groups. In this paper, we continue the study of quasitopological groups.

In Section 2, we consider generalised metrisability of quasitopological groups. We prove: (a) a semimetrisable quasitopological group G admits an invariant semimetric generating its topology if and only if G is balanced; (b) if H is a closed subgroup of a first-countable quasitopological group G, then the quotient space G/H is semimetrisable; and (c) the following properties are equivalent: (1) G is a semimetrisable quasitopological group with respect to a continuous left-invariant

Supported by the NSFC (Grant Nos.11471153, 11571158, 11801254).

semimetric, (2) G is a quasimetrisable quasitopological group with respect to a left-invariant quasimetric, (3) G is a metrisable topological group.

In Section 3, we investigate cardinal invariants in quasitopological groups. It is well known that cardinal functions behave much better on topological groups than on completely regular spaces (see [3, Section 5.2]). We construct some examples to show that some equalities of cardinal invariants in topological groups cannot be extended to quasitopological groups.

We denote by $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ and ω the set of all positive integers, the set of all rational numbers, the set of all real numbers and the first infinite ordinal, respectively. The neutral element of a group is denoted by e. Readers may consult [3, 5, 6] for notation and terminology not given here.

2. Generalised metrisability in quasitopological groups

A function $d : X \times X \to [0, \infty)$ is called *symmetric* on the set X if, for each $x, y \in X$, (i) $d(x, y) = 0$ if and only if $x = y$ and (ii) $d(x, y) = d(y, x)$. Define the set $B(x, \epsilon)$ by $B(x, \epsilon) = \{y \in X : d(x, y) < \epsilon\}$ for each $x \in X$ and $\epsilon > 0$. A space X is called *symmetrisable* [1] if there is a symmetric d on X such that a subset $U \subseteq X$ is open if and only if, for each $x \in U$, there exists $\epsilon > 0$ with $B(x, \epsilon) \subseteq U$. A symmetrisable space (X, d) is called a *semimetrisable space* [6, page 482] and d is called a *semimetric* on X if, for each $x \in X$, the collection $\{B(x, \varepsilon) : \varepsilon > 0\}$ forms a neighbourhood base of x.

A quasimetric (respectively, semimetric) $d(x, y)$ is called *continuous* if $d(x, y)$ is continuous with respect to both x and y; $d(x, y)$ is *left-continuous* (respectively, *right-continuous*) if $d(x, \cdot)$ (respectively, $d(\cdot, y)$) is continuous.

Let G be a group. A function $d : G \times G \to [0, \infty)$ is called *left-invariant* (respectively, *right-invariant*) if $d(x, y) = d(ax, ay)$ (respectively, $d(x, y) = d(xa, ya)$), for any $a, x, y \in G$. A function $d : G \times G \to [0, \infty)$ is called *invariant* if d is simultaneously left-invariant and right-invariant.

Lemma 2.1 [9, 13]. *A quasitopological group is semimetrisable if and only if it is first-countable.*

By using the proof of [13, Theorem 3.1], we can complement Lemma 2.1 as follows.

Proposition 2.2. Every first-countable quasitopological group G admits a right-invariant semimetric ϱ and a left-invariant semimetric λ, both generating the original topology of G.

Let G be a group. A subset A of G is said to be *invariant* [3, page 69] if $xAx^{-1} = A$, for each $x \in G$. It is clear that all subsets of Abelian groups are invariant. A semitopological group G is called *balanced* if it has a local base at the neutral element e consisting of invariant sets. A balanced semitopological group is also called a group with invariant basis.

https://doi.org/10.1017/S000497271800103X Published online by Cambridge University Press

Theorem 2.3. *A semimetrisable quasitopological group G admits an invariant semimetric generating its topology if and only if G is balanced.*

Proof. Suppose that ϱ is an invariant semimetric on G that generates the topology of G. For every $n \in \mathbb{N}$, let $U_n = \{x \in G : \varrho(e, x) < 1/n\}$. Then $\{U_n : n \in \mathbb{N}\}$ is a neighbourhood base at e in G. Since ϱ is invariant, for any $x \in U_n$ and $y \in G$,

$$\varrho(e, yxy^{-1}) = \varrho(y, yx) = \varrho(e, x) < \frac{1}{n},$$

and hence $yxy^{-1} \in U_n$. It follows that $yU_ny^{-1} = U_n$ for all $y \in G$ and $n \in \mathbb{N}$. Thus the family $\{U_n : n \in \mathbb{N}\}$ is an invariant base for G at the neutral element e. Hence the group G is balanced.

Conversely, suppose that the group G is balanced. Since G is first-countable, there exists a family $\xi = \{U_n : n \in \mathbb{N}\}$ of open, symmetric, invariant neighbourhoods of e in G satisfying $U_n \subseteq U_{n+1}$ for each $n \in \mathbb{N}$, such that ξ forms a local base for G at e. Define a function $d : G \times G \to [0, \infty)$ by $d(x, y) = \inf\{1/n : x^{-1}y \in U_n, n \in \mathbb{N}\}$. It is easy to see that d is symmetric on G. Moreover, the original topology on G coincides with the topology generated by d. Finally, $U_n = yU_ny^{-1}$ for each $y \in G$, so $d(e, yxy^{-1}) = d(e, x)$ for each $x, y \in G$, that is, d is invariant. □

The next proposition is an easy modification of the proof of [13, Theorem 3.1].

Proposition 2.4. *Let G be a quasitopological group. If the neutral element e is a G_δ-point in G, then there is a weaker symmetrisable topology on G.*

Proof. Let $\{e\} = \bigcap_{n \in \mathbb{N}} U_n$, where U_n is an open neighbourhood of e for each $n \in \mathbb{N}$. Since G is a quasitopological group, we may assume that $U_n = U_n^{-1}$ and $U_n \subseteq U_{n+1}$ for each $n \in \mathbb{N}$. We will define a new topology by the function $d : G \times G \to [0, \infty)$, where $d(x, y) = \inf\{1/n : x^{-1}y \in U_n, n \in \mathbb{N}\}$. It is easy to see that d is symmetric on G. Moreover, $xU_{n+1} = B(x, 1/n)$ for each $x \in G$ and $n \in \mathbb{N}$. Obviously, the topology generated by d is weaker than the original topology on G. □

In general, semimetrisability of topological spaces is not preserved by open continuous mappings. By Lemma 2.1, we have the following corollary.

Corollary 2.5. *Suppose that f is an open continuous homomorphism of a semimetrisable quasitopological group G onto a quasitopological group H. Then H is also semimetrisable.*

The next theorem is a generalisation of Corollary 2.5 and Lemma 2.1 to quotient spaces of semimetrisable quasitopological groups.

Theorem 2.6. *Let G be a quasitopological group. Then G is first-countable if and only if the quotient space G/H is semimetrisable for every closed subgroup H of G.*

https://doi.org/10.1017/S000497271800103X Published online by Cambridge University Press

PROOF. Suppose the quotient space G/H is semimetrisable for every closed subgroup H of G. Put $H = \{e\}$. Then $G/H = G$ is semimetrisable and so G is first-countable.

Conversely, suppose G is first-countable. By Proposition 2.2, there exists a right-invariant semimetric d on G which generates the topology of G. For arbitrary points $x, y \in G$, define a function $\varrho : G/H \times G/H \to [0, \infty)$ by

$$\varrho(xH, yH) = \inf\{d(xh_1, yh_2) : h_1, h_2 \in H\}.$$

Since d is right-invariant and H is a subgroup of G, we have $\varrho(xH, yH) = d(x, yH) \geq 0$ for all $x, y \in G$. The function ϱ is symmetric: that is,

$$\begin{aligned}\varrho(xH, yH) = d(x, yH) &= \inf\{d(x, yh) : h \in H\} = \inf\{d(xh^{-1}, y) : h \in H\}\\ &= \inf\{d(y, xh^{-1}) : h \in H\} = d(y, xH) = \varrho(yH, xH).\end{aligned}$$

Since H is closed in G and d is a semimetric on G, we have $\varrho(xH, yH) = d(x, yH) = 0$ if and only if $x \in yH$, that is, $xH = yH$.

We will show that ϱ generates the topology of the quotient space G/H. For each $x \in G$ and $\varepsilon > 0$, let

$$O_\varepsilon(x) = \{y \in G : d(x, y) < \varepsilon\} \quad \text{and} \quad B_\varepsilon(xH) = \{yH : y \in G, \varrho(xH, yH) < \varepsilon\}.$$

Denote by π the quotient mapping of G onto G/H, so that $\pi(x) = xH$ for each $x \in G$. By the definition of ϱ, it is easy to check that $\pi(O_\varepsilon(x)) = B_\varepsilon(xH)$ for all $x \in G$ and $\varepsilon > 0$. Since the family $\{O_\varepsilon(x) : \varepsilon > 0\}$ forms a neighbourhood base at x in G and the mapping $\pi : G \to G/H$ is continuous and open, we conclude that the family $\{B_\varepsilon(xH) : \varepsilon > 0\}$ forms a neighbourhood base at xH in G/H for the original topology of the space G/H. This completes the proof. □

Let (X, τ) be a space. A function $g : \mathbb{N} \times X \to \tau$ is called a *g-function* on X if $x \in g(n + 1, x) \subseteq g(n, x)$ for every $x \in X$ and $n \in \mathbb{N}$. A space X is called a *β-space* [6, page 475] if there is a g-function on X such that $p \in g(n, x_n)$ for each $n \in \mathbb{N}$ implies that the sequence $\{x_n\}_{n\in\mathbb{N}}$ has an accumulation point. A space X is called a *γ-space* [6, page 491] if there is a g-function on X such that (i) $\{g(n, x) : n \in \mathbb{N}\}$ is a neighbourhood base of x and (ii) for every $n \in \mathbb{N}$ and $x \in X$ there exists $m \in \mathbb{N}$ such that $y \in g(m, x)$ implies that $g(m, x) \subseteq g(n, y)$. We know that every first-countable quasitopological group G is a β-space [9, Theorem 2.5] and that if X is a β-space and a γ-space, then X is developable [6, Theorem 10.7]. Therefore, the following corollary is immediate.

COROLLARY 2.7. *Let G be a quasitopological group G. If G is a γ-space, then G is developable.*

By Proposition 2.2, every first-countable quasitopological group G admits a left-invariant semimetric λ that generates the original topology of G. However, G need not be metrisable. In fact, let $G = \mathbb{R}^2$ be an additive group endowed with the bowtie topology $\mathcal{D}$ [6, Example 9.10] . For each $p = (x_1, x_2) \in \mathbb{R}^2$ and $\varepsilon > 0$, let

$$U(p, \varepsilon) = \{p\} \cup \{(x, y) : 0 < |x - x_1| < \varepsilon, |(y - y_1)/(x - x_1)| < \varepsilon\}.$$

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Then $\{U(p,\varepsilon) : \varepsilon > 0\}$ is a a local base at each point $p \in G$ and $(G, \mathcal{D})$ is a completely regular quasitopological group. Since G is not a σ-space, G is not metrisable. Hence, it is natural to consider what conditions need to be added to a first-countable quasitopological group so that the group is metrisable. We obtain the following result.

Lemma 2.8 [6, Lemma 9.3 and Lemma 10.2(i)].

(1) *Suppose that X is symmetrisable with respect to a symmetric d. Then $x_n \to x$ in X if and only if $d(x_n, x) \to 0$.*
(2) *Suppose that X is quasimetrisable with respect to the quasimetric d. Then $x_n \to x$ in X if and only if $d(x, x_n) \to 0$.*

Theorem 2.9. *The following statements are equivalent:*

(1) *G is a semimetrisable quasitopological group with respect to a continuous left-invariant semimetric;*
(2) *G is a quasimetrisable quasitopological group with respect to a left-invariant quasimetric; and*
(3) *G is a metrisable topological group.*

Proof. The implications $(3) \Rightarrow (1), (2)$ are trivial. For (1) or (2) $\Rightarrow$ (3), we only need to prove that the quasitopological group G in (1) or (2) is a topological group. It is sufficient to prove that the multiplication of G is jointly continuous. Obviously, G is first-countable, so $G \times G$ is first-countable and it is enough to show that the product map of $G \times G$ into G is sequentially continuous. This is equivalent to the condition that $a_n b_n \to e$ in G whenever $a_n \to e$ and $b_n \to e$ in G, where e is the neutral element of G.

$(1) \Rightarrow (3)$. Assume that G is a semimetrisable quasitopological group with respect to a continuous left-invariant semimetric ϱ. Then

$$\varrho(a_n b_n, e) = \varrho(a_n^{-1} a_n b_n, a_n^{-1}) = \varrho(b_n, a_n^{-1}).$$

Since G is a quasitopological group, $a_n \to e$ implies that $a_n^{-1} \to e$. By the continuity of ϱ and $b_n \to e$, we have $\varrho(a_n b_n, e) = \varrho(b_n, a_n^{-1}) \to 0$. Thus $a_n b_n \to e$ by Lemma 2.8(1).

$(2) \Rightarrow (3)$. Let G be a quasimetrisable quasitopological group with respect to a left-invariant quasimetric d. Fix $n \in \mathbb{N}$. Since d a left-invariant quasimetric, $d(e, a_n b_n) \le d(e, a_n) + d(a_n, a_n b_n) = d(e, a_n) + d(e, b_n)$. Since $a_n \to e$ and $b_n \to e$, we have $d(e, a_n) \to 0$ and $d(e, b_n) \to 0$. By Lemma 2.8(2), $d(e, a_n b_n) \to 0$, that is, $a_n b_n \to e$. □

Remark 2.10. The conditions 'continuous' in Theorem 2.9(1) and 'left-invariant' in Theorem 2.9(2) are essential. By Proposition 2.2, every first-countable quasitopological group G admits a left-invariant semimetric. The additive group $G = \mathbb{R}^2$ endowed with the bowtie topology is a first-countable quasitopological group which is not a metrisable space. Further, $G_1 = \mathbb{Q}^2 \subseteq G$ endowed with the subspace topology is a metrisable quasitopological group, but G_1 is not a topological group.

The following question remains open.

Question 2.11. If G is a quasimetrisable quasitopological group, is G metrisable?

https://doi.org/10.1017/S000497271800103X Published online by Cambridge University Press

3. Cardinal invariants in quasitopological groups

Given a space X, we denote by $w(X), nw(X), d(X), c(X), l(X), \chi(X), \pi w(X)$ and $\pi\chi(X)$ the weight, network weight, density, cellularity, Lindelöf number, character, π-weight and π-character of X, respectively (see [7]). Let τ be an infinite cardinal and let G be a semitopological group. We call G *τ-narrow* [15, Section 6] if, for every neighbourhood U of the identity in G, there is a subset $K \subseteq G$ with $|K| \leq \tau$ such that $KU = UK = G$. The *index of narrowness* $\mathrm{In}(G)$ of G is defined in the following way: $\mathrm{In}(G) = \min\{\tau \geq \omega : G \text{ is } \tau\text{-narrow}\}$. By [3, Propositions 5.2.3, 5.2.6 and Theorem 5.2.5], every topological group G satisfies the following equalities:

$$\pi\chi(G) = \chi(G), \tag{3.1}$$
$$w(G) = \pi w(G), \tag{3.2}$$
$$w(G) = d(G) \cdot \chi(G), \tag{3.3}$$
$$w(G) = l(G) \cdot \chi(G), \tag{3.4}$$
$$w(G) = c(G) \cdot \chi(G), \tag{3.5}$$
$$w(G) = \mathrm{In}(G) \cdot \chi(G), \tag{3.6}$$
$$w(G) = nw(G) \cdot \chi(G). \tag{3.7}$$

However, one cannot extend these equalities to quasitopological groups. In this section, we construct examples to show that the above equalities do not hold in quasitopological groups.

Example 3.1. There exists a completely regular quasitopological group G such that $\chi(X) > \pi\chi(G) = \omega$.

Proof. Consider the additive group $G = \mathbb{R}^2$ endowed with the topology $\mathcal{T}^*$ [2, page 112]. By [2, Property 3.2], G is a completely regular quasitopological group and $\chi(X) > \omega$. Also, $\pi w(G) = \omega$ from [2, Property 3.4], so $\pi\chi(G) = \omega$. □

Example 3.2. There exists a completely regular quasitopological group G such that $w(G) > \pi w(G) = \omega$.

Proof. Consider the additive group $G = \mathbb{R}^2$ endowed with the bowtie topology. Then G is a completely regular quasitopological group and $\pi w(G) = \omega$. However, $w(G) > \omega$ since G contains an uncountable closed discrete subspace $\{0\} \times \mathbb{R}$. □

Example 3.3. There exists a completely regular quasitopological group G such that $w(G) > nw(G) = \chi(G) = \omega$.

Proof. Consider the additive group $G_1 = \mathbb{R}^2$ endowed with the bowtie topology $\mathcal{D}$. $(G_1, \mathcal{D})$ is a completely regular quasitopological group. Let $G = \mathbb{R} \times \mathbb{Q}$ endowed with the subspace topology of G_1. Clearly, G_1 is a completely regular, first-countable quasitopological group. Since the subspace $\mathbb{R} \times \{0\}$ of G is homeomorphic to $\mathbb{R}$ with the usual topology, $\mathbb{R} \times \{0\}$ has a countable network. Thus $G = \bigcup_{y \in \mathbb{Q}} \mathbb{R} \times \{y\}$ has a

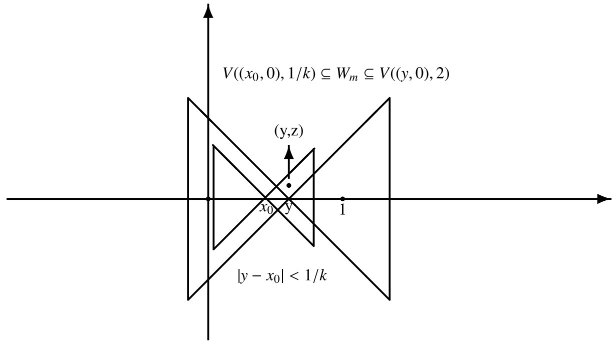

FIGURE 1. Construction for Example 3.3.

countable network. That is $nw(G) = \omega$. We will show that $w(G) > \omega$. Let $I = [0, 1]$ be a subspace of $\mathbb{R}$ with the usual topology. Then $A = I \times \{0\}$ is compact in G.

Claim. The compact set A does not have a countable neighbourhood base in G.

For any $a \in G$ and $\varepsilon > 0$, let $V(a, \varepsilon) = U(a, \varepsilon) \cap G$. Then $V(a, \varepsilon)$ is a neighbourhood of x in G. Suppose that A has a countable neighbourhood base $\{W_n : n \in \mathbb{N}\}$ in G. For any $x \in I$, since $V((x, 0), 2)$ is a neighbourhood of A in G, there is $n(x) \in \mathbb{N}$ such that $W_{n(x)} \subseteq V((x, 0), 2)$. Clearly, I is an uncountable set. Hence there is $m \in \mathbb{N}$ such that $B = \{x \in I : n(x) = m\}$ is uncountable. Let x_0 be an accumulation point of the set B in I. There exists $k \in \mathbb{N}$ such that $V((x_0, 0), 1/k) \subseteq W_m$. Take $y \in B$ with $y \neq x_0$ and $|y - x_0| < 1/k$. Then $V((x_0, 0), 1/k) \subseteq W_m \subseteq V((y, 0), 2)$. However, we can pick a rational number z such that $0 < z < |y - x_0|/k$. Thus $(y, z) \in V((y, 0), 1/k)$ and $(y, z) \notin V((y, 0), 2)$, which is a contradiction, as illustrated in Figure 1. Therefore the compact set A does not have a countable neighbourhood base in G.

If $w(G) = \omega$, then G is second-countable. Since G is completely regular, G is metrisable by [6, Theorem 1.1]. But this means that the compact set A has a countable neighbourhood base in G, which is a contradiction. So $w(G) > \omega$. □

REMARK 3.4. Examples 3.1 and 3.2 show that neither equation (3.1) nor equation (3.2) is valid for quasitopological groups. It is well known that every topological space X satisfies $c(X) \leq d(X) \leq nw(X)$ and $l(X) \leq nw(X)$. By [12, Proposition 2.6], $\mathrm{In}(G) \leq d(G)$ for every semitopological group G. Therefore Example 3.3 shows that equations (3.3)–(3.7) cannot be extended to quasitopological groups.

It is known that every subgroup of a τ-narrow topological group is τ-narrow. However, a (closed) subgroup of a τ-narrow quasitopological group can fail to be τ-narrow. In fact, the subgroup $H = \{0\} \times \mathbb{R}$ of $G = \mathbb{R}^2$ endowed with the bowtie topology is closed and discrete. Clearly, $\mathrm{In}(H) = 2^{\omega}$. But $d(G) = \omega$, so $\mathrm{In}(G) = \omega$.

A subset B of a semitopological group G is called *τ-narrow* if, for every neighbourhood U of the identity e in G, there is a subset $K \subseteq G$ with $|K| \leq \tau$ such that $B \subseteq KU \cap UK$. Clearly, G is τ-narrow if and only if G is τ-narrow in itself, and every subset of a τ-narrow semitopological group is τ-narrow in this group. This leads to the following theorem.

THEOREM 3.5. *Let G be a semitopological group. If $l(B) \leq \tau$, then B is τ-narrow.*

PROOF. If U is an open neighbourhood of the identity in G, then $\{xU : x \in G\}$ and $\{Ux : x \in G\}$ are two open coverings of G. Since $l(B) \leq \tau$, there are two subsets C_1, C_2 of G such that $|C_i| \leq \tau$ $(i = 1, 2)$ and the families $\{xU : x \in C_1\}$ and $\{Ux : x \in C_2\}$ cover B, that is, $B \subseteq C_1 U \cap UC_2$. Hence B is τ-narrow. □

Let G be a topological group. If $c(G) = \omega$, then G is ω-narrow [3, Theorem 3.4.7]. Therefore it is natural to ask the following question.

QUESTION 3.6. If G is a quasitopological group with $c(G) = \omega$, is G ω-narrow?

Acknowledgements

The authors would like to thank Professor F. Lin for his valuable suggestions, which led to a much better presentation of this paper. We also wish to thank the reviewer for the detailed list of corrections and suggestions which have improved the paper.

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https://doi.org/10.1017/S000497271800103X Published online by Cambridge University Press