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WEIERSTRASS DIVISION IN QUASIANALYTIC LOCAL RINGS

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1. Introduction. In this paper we consider the problem of extending the Weierstrass division theorem to quasianalytic local rings of germs of functions of k real variables which properly contain the local ring of germs of analytic functions. After some background material (§ 2) and some technical preliminaries (§ 3), we show by examples that the so-called generic division theorem fails in such rings if $k \geqq 1$ and that the Weierstrass division theorem fails in such rings if $k \geqq 2$ (§ 4). Guided by these examples, we give a necessary condition for an element of such rings to have the Weierstrass division property (Theorem 5.1). Using the fact that the quasianalytic local rings are "analytically uniform" in the sense of Ehrenpreis [**4**], and hence that the elements of these rings have Fourier integral representations, we give an estimate which is both necessary and sufficient for an element of these rings to have the Weierstrass division property (Theorem 6.2). This estimate, however, is often difficult to verify in practice. We conclude the paper with a brief discussion of whether or not the quasianalytic local rings are Noetherian unique factorizations domains and the relation of the answer to this question to the division theorem (§ 7).

2. Background. Let k be a positive integer, $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a k-tuple of nonnegative integers, $x = (x_1, \ldots, x_k)$ be the canonical coordinates on $\mathbf{R}^k$, and $r > 0$. We use the standard notations $|\alpha| = \alpha_1 + \ldots + \alpha_k$, $D^\alpha = D_x^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_k^{\alpha_k}$, and we use the notation $\Delta_k(r) = \{x : x \in \mathbf{R}^k, |x_j| < r$ for $1 \leqq j \leqq k\}$.

If $0 < r_2 < r_1$, there is a natural restriction map from $C^\infty(\Delta_k(r_1))$ into $C^\infty(\Delta_k(r_2))$. Recall that the ring $\mathscr{C}_k^\infty = {}_0\mathscr{C}_k^\infty$ of germs of complex-valued C^∞ functions at $0 \in \mathbf{R}^k$ is the direct limit of $C^\infty(\Delta_k(r))$ as r decreases to 0. (In general, we will not distinguish notationally between the germ of a function and a representative of the germ.)

Fix a sequence $\{M_n\}_{n=0}^{+\infty}$ with $M_n \geqq n!$ for all n. We assume

(2.1) $\quad M_n = \exp(g(n)) \quad$ for all n,

where g is a nonnegative, increasing, convex function defined on $\{t : t \geqq 0\}$, $g(0) = 0$, and $t^{-1}g(t) \to +\infty$ as $t \to +\infty$. (These assumptions on the sequence $\{M_n\}$ are made without loss of generality; cf. [**7**].)

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Definition 2.1. Let k, ν, and N be positive integers. We define

$$\rho_{\nu,N}(f) = \rho_{k,\nu,N}(f) = \rho_{k,\nu,N,\{M_n\}}(f)$$

(2.2) $$= \sup_{n} \sup_{\substack{|\alpha| \leqq n, \\ x \in \Delta_k(1/\nu)}} |D^\alpha f(x)|/(N^n M_n) \quad \text{for } f \in C^\infty(\Delta_k(1/\nu)),$$

and

$$E_{\nu,N} = E_{k,\nu,N} = E_{k,\nu,N}(\{M_n\})$$

(2.3) $$= \{f : f \in C^\infty(\Delta_k(1/\nu)), \rho_{\nu,N}(f) < +\infty\}.$$

Finally, we define

$$\xi_k = {}_0\xi_k = {}_0\xi_k(\{M_n\})$$

(2.4) $$= \{f : f \in {}_0\mathscr{C}_k^\infty, f \text{ is represented by an element of } E_{\nu,N} \text{ for all } \nu \text{ and } N \text{ sufficiently large}\}.$$

Since $M_0 = 1$ and $\{M_n\}$ is logarithmically convex by (2.1), it follows that (cf. [**8**])

$$M_j M_{n-j} \leqq M_n \quad \text{for } 0 \leqq j \leqq n.$$

Using this inequality, it is easy to show that ξ_k is a ring. ξ_k is clearly a local ring, i.e., ξ_k has a unique maximal ideal $\{f : f \in \xi_k, f(0) = 0\}$.

There are several elementary structural properties of the local rings ξ_k which we will not need but which are of such sufficient interest in themselves that we mention them and indicate briefly how proofs of the more difficult results proceed. In each case, further restrictions on $\{M_n\}$ are required.

If $f_1, \ldots, f_j \in \xi_k$ are nonunits and real-valued, and if $g \in \xi_j$, then $g \circ f \in \mathscr{C}_k^\infty$, where $f = (f_1, \ldots, f_j)$. If we assume

(2.5) the sequence $\{M_n/n!\}$ is logarithmically convex,

then we can prove $g \circ f \in \xi_k$. The proof uses the following formula for the derivatives of a composite function (cf. [**1**]):

$$\frac{(u \circ v)^{(n)}(x)}{n!} = \sum_{p=1}^{n} \frac{u^{(p)}[v(x)]}{p!} \sum_{\substack{k=(k_1,\ldots,k_n), \\ |k|=p, \text{ and} \\ k_1+2k_2+\ldots+nk_n=n}} \binom{n-1}{p-1} \times \left[\frac{v'(x)}{1!}\right]^{k_1} \cdots \left[\frac{v^{(n)}(x)}{n!}\right]^{k_n}.$$

In addition, the following inequality, which follows from an inductive argument using (2.5), is required:

$$\frac{M_p}{p!}\left(\frac{M_1}{1!}\right)^{k_1} \cdots \left(\frac{M_n}{n!}\right)^{k_n} \leqq \left(\frac{M_1}{1!}\right)^p \frac{M_n}{n!},$$

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where $k = (k_1, \ldots, k_n)$ is any n-tuple of nonnegative integers such that $|k| = p$ and $k_1 + 2k_2 + \ldots + nk_n = n$.

If $f \in \xi_k$, then $\partial f/\partial x_j \in \mathscr{C}_k^{\infty}$ for $1 \leqq j \leqq k$. If we assume

(2.6) the sequence $\{M_n\}$ has the property that there exists A such that for all nonnegative integers p, $\sup_n M_{n+p}/(A^n M_n) < +\infty$,

then we can easily prove that $\partial f/\partial x_j \in \xi_k$ for $1 \leqq j \leqq k$. With this same additional assumption on $\{M_n\}$, we can also easily prove that if $x^\alpha = x_1^{\alpha_1} \ldots x_k^{\alpha_k}$ divides $f \in \xi_k$ in the ring of formal power series at $0 \in \mathbf{R}^k$, then x^α divides f in ξ_k.

PROPOSITION 2.2. *Suppose there exists a positive integer k with the following property*:

(2.7) $f \in \xi_k$ *and* $D^\alpha f(0) = 0$ *for all* α *imply* $f = 0 \in \xi_k$.

Then (2.7) *holds for all positive integers k.*

Proof. If (2.7) holds for some positive integer $k = k_0$, then it holds for $k = 1$; for we can imbed ξ_1 in ξ_{k_0} by $\xi_1 \ni f \rightarrow g \in \xi_{k_0}$ where $g(x_1, \ldots, x_{k_0}) = f(x_1)$.

Conversely, if (2.7) holds for $k = 1$, then it holds for an arbitrary positive integer $k = k_0$. For if $f \in \xi_{k_0}$ and $D^\alpha f(0) = 0$ for all α, we define $g(y) = f(yx)$. It is easy to verify that this defines a germ $g \in \xi_1$ with $d^n g(0)/dy^n = 0$ for all n. Thus by assumption, $g = 0 \in \xi_1$, and so $f(x) = g(1) = 0$. Since x was arbitrary, $f = 0 \in \xi_{k_0}$.

We can now define the local rings with which we are concerned in this paper.

Definition 2.3. The local rings $\xi_k = \xi_k(\{M_n\})$, $k = 1, 2, \ldots$, are called *quasi-analytic* if (2.7) holds.

The question of which sequences $\{M_n\}$ yield quasianalytic classes was answered by Denjoy and Carleman; see [**8**].

Let

(2.8) $$\Lambda(x) = \sum_{n=0}^{+\infty} |x|^n/M_n$$

and

(2.9) $$\lambda(x) = \sup_n |x|^n/M_n.$$

THEOREM 2.4. (Denjoy-Carleman) *The following conditions are equivalent*:

(i) $\xi_k(\{M_n\})$, $k = 1, 2, \ldots$, *are quasianalytic.*

(ii) $\int_0^{+\infty} \log (\Lambda(x))/(1 + x^2)\, dx = +\infty$.

(iii) $\int_0^{+\infty} \log (\lambda(x))/(1 + x^2)\, dx = +\infty$.

(iv) $\sum_{n=0}^{+\infty} M_n/M_{n+1} = +\infty$.

(v) $\sum_{n=0}^{+\infty} (1/M_n)^{1/n} = +\infty$.

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We use the standard notation $\mathcal{O} = {}_0\mathcal{O}_k = {}_0\xi_k(\{n!\})$ for the ring of germs of analytic functions at $0 \in \mathbf{R}^k$. Also, we now let t and $x = (x_1, \ldots, x_{k-1})$ be the canonical coordinates on $\mathbf{R}$ and $\mathbf{R}^{k-1}$, respectively, so that (t, x) are the canonical coordinates on $\mathbf{R}^k$.

Definition 2.5. *If* $\lambda = (\lambda_1, \ldots, \lambda_p) \in \mathbf{C}^p$, then

$$P(t, \lambda) = t^p + \sum_{j=1}^{p} \lambda_j t^{p-j} \tag{2.10}$$

is called a *generic monic polynomial in t of degree p*.

Definition 2.6. An element $f = f(t, x) \in \mathcal{C}_k^\infty$ is said to be *regular in t of order p* if

$$f(0, 0) = \partial f(0, 0)/\partial t = \ldots = \partial^{p-1} f(0, 0)/\partial t^{p-1} = 0, \quad \text{while} \quad \partial^p f(0, 0)/\partial t^p \neq 0. \tag{2.11}$$

THEOREM 2.7. (generic division theorem in $\mathcal{O}_k$) *Let* $P = P(t, \lambda)$ *be a generic monic polynomial in t of degree p. If* $g = g(t, x) \in \mathcal{O}_k$, *then there exist unique elements* $q = q(t, x) \in \mathcal{O}_k$ *and* $r_j = r_j(x) \in \mathcal{O}_{k-1}$, $1 \leqq j \leqq p$, *such that*

$$g = Pq + r, \quad \text{where} \quad r = \sum_{j=1}^{p} r_j t^{p-j}. \tag{2.12}$$

Proof. See [**9**].

THEOREM 2.8. (Weierstrass division theorem in $\mathcal{O}_k$) *Let* $f = f(t, x) \in \mathcal{O}_k$ *be regular in t of order p. If* $g = g(t, x) \in \mathcal{O}_k$, *then there exist unique elements* $q = q(t, x) \in \mathcal{O}_k$ *and* $r_j = r_j(x) \in \mathcal{O}_{k-1}$, $1 \leqq j \leqq p$, *such that*

$$g = fq + r, \quad \text{where} \quad r = \sum_{j=1}^{p} r_j t^{p-j}. \tag{2.13}$$

Proof. See [**6**] or [**9**].

In this paper we consider the questions of when the generic division theorem and the Weierstrass division theorem continue to hold with $\mathcal{O}_k$ replaced by a quasianalytic ring ξ_k throughout. In the case of the Weierstrass division theorem, we find it convenient to consider when the theorem so extends for a specific $f \in \xi_k$.

Definition 2.9. Let ξ_k be quasianalytic and $f = f(t, x) \in \xi_k$ be regular in t of order p. Then f is said to have the *Weierstrass division property* if given $g = g(t, x) \in \xi_k$, there exist unique elements $q = q(t, x) \in \xi_k$ and $r_j = r_j(x) \in \xi_{k-1}$, $1 \leqq j \leqq p$, such that equations (2.13) hold.

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We remark that the uniqueness assumption in Definition 2.9 is superfluous. This is because equations (2.13) uniquely determine the formal power series at the origin of q and r_j, $1 \leqq j \leqq p$ (cf. [**2**]); and so $q \in \xi_k$ and $r_j \in \xi_{k-1}$, $1 \leqq j \leqq p$, are uniquely determined since ξ_{k-1} and ξ_k are assumed to be quasianalytic.

3. Preliminaries. In this section we establish two estimates which we will need. Recall the function λ defined in equation (2.9).

LEMMA 3.1. *If there exist* $\epsilon > 0$, $A > 0$, *and* $C > 0$ *such that*

$$\exp(\epsilon a) \leqq C\lambda(a) \quad \textit{for } a > A, \tag{3.1}$$

then there exist $\alpha > 0$ *and* $\beta > 0$ *such that*

$$M_n \leqq \alpha\beta^n n! \quad \textit{for all } n. \tag{3.2}$$

Proof. Let n_0 be a nonnegative integer. Since by (3.1), $\exp(\epsilon a) \leqq C\lambda(a)$ for $a > A$,

$$\begin{aligned} a^{n_0}/(\epsilon^{-n_0}n_0!) &\leqq \textstyle\sum_{n=0}^{+\infty} a^n/(\epsilon^{-n}n!) = \exp(\epsilon a) \\ &\leqq C\lambda(a) = C \sup_n a^n/M_n \quad \text{for } a > A. \end{aligned} \tag{3.3}$$

The idea of the proof is to show there exists a positive integer N_0 such that if $n_0 > N_0$, then there exists $a_0 = a_0(n_0)$ with $a_0 > A$ and $\sup_n a_0^{\,n}/M_n = a_0^{\,n_0}/M_{n_0}$. It will then follow from (3.3) that $M_{n_0} \leqq C(\epsilon^{-1})^{n_0}n_0!$ for $n_0 > N_0$. If we choose $\alpha = \max(C, \max_{0 \leqq n \leqq N_0} \epsilon^n M_n/n!)$ and $\beta = \epsilon^{-1}$, then (3.2) will be true, and so the lemma will be established.

In equation (2.1) we made the assumption that there exists a nonnegative, increasing, convex function $g = g(x)$, defined for $x \geqq 0$, such that $M_n = \exp(g(n))$ for all n, $g(0) = 0$, and $x^{-1}g(x) \to +\infty$ as $x \to +\infty$. Since $g(x)$ is convex for $x \geqq 0$ and $g(0) = 0$, there exists a monotone increasing function $m = m(t)$, defined for $t \geqq 0$, such that

$$g(x) = g(x) - g(0) = \int_0^x m(t)dt \quad \text{for } x \geqq 0.$$

Since $x^{-1}g(x) \to +\infty$ as $x \to +\infty$, $m(t) \to +\infty$ as $t \to +\infty$. Thus there exists a positive integer N_0 such that if $n_0 > N_0$, then $\exp(m(n_0)) > A$. We will now show that if we choose $a_0 = a_0(n_0) = \exp(m(n_0))$, then $\sup_n a_0^{\,n}/M_n = a_0^{\,n_0}/M_{n_0}$.

Since $M_n = \exp(g(n))$ for all n and $\log a_0 = m(n_0)$, we must show $\sup_n(nm(n_0) - g(n)) = n_0m(n_0) - g(n_0)$. Since $m = m(t)$ is monotone increasing for $t \geqq 0$, if $n \leqq n_0$, then $m(t) \leqq m(n_0)$ for $n \leqq t \leqq n_0$, and so

$$\begin{aligned} nm(n_0) - g(n) &= \int_0^n (m(n_0) - m(t))dt \leqq \int_0^{n_0} (m(n_0) - m(t))dt \\ &= n_0m(n_0) - g(n_0); \end{aligned}$$

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while if $n \geqq n_0$, then $m(t) \geqq m(n_0)$ for $n \geqq t \geqq n_0$, and so

$$nm(n_0) - g(n) = \int_0^n (m(n_0) - m(t))dt \leqq \int_0^{n_0} (m(n_0) - m(t))dt$$
$$= n_0 m(n_0) - g(n_0).$$

Thus $\sup_n(nm(n_0) - g(n)) = n_0 m(n_0) - g(n_0)$. This completes the proof of the lemma.

Let E be a Banach space and $F = \cup_{n=1}^{+\infty} F_n$ be an inductive limit of Banach spaces. As a special case of a more general proposition of Grothendieck [**5**], if $T : E \to F$ is a continuous linear map, then there exists a positive integer n_0 such that $T(E) \subseteq F_{n_0}$. We apply this fact to a special continuous linear map T, which we now define, and obtain the second estimate which we will need.

Fix a quasianalytic local ring $\xi_k = {}_0\xi_k(\{M_n\})$ and $f = f(t, x) \in \xi_k$ which is regular in t of order p, and suppose that f has the Weierstrass division property. Let ν_0 be the smallest positive integer such that f is represented by an element of $E_{\nu,N}$ for some N and all $\nu \geqq \nu_0$. Define a map

$$(q, (r_1, \ldots, r_p)) \to g = fq + r,$$
$$\bigcup_{\substack{\nu=\nu_0,\\ N=1}}^{+\infty} \left(E_{k,\nu,N} \oplus \left(\bigoplus_1^p E_{k-1,\nu,N}\right)\right) \to \bigcup_{\substack{\nu=\nu_0,\\ N=1}}^{+\infty} E_{k,\nu,N}, \quad \text{where}$$

$$q = q(t,x) \in \bigcup_{\substack{\nu=\nu_0,\\ N=1}}^{+\infty} E_{k,\nu,N} \quad \text{and} \tag{3.4}$$
$$r = r(t,x) = \sum_{j=1}^{p} r_j(x) t^{p-j} \quad \text{with}$$
$$r_j = r_j(x) \in \bigcup_{\substack{\nu=\nu_0,\\ N=1}}^{+\infty} E_{k-1,\nu,N} \quad \text{for } 1 \leqq j \leqq p.$$

Note that for all positive integers ν and N, $E_{\nu,N}$ is a Banach space. The map is clearly linear. It is continuous because its restriction to $E_{k,\nu,N} \oplus (\oplus_1^p E_{k-1,\nu,N})$ is continuous for all $\nu \geqq \nu_0$ and all N. The assumption that $\xi_k(\{M_n\})$ is quasianalytic implies that the map is injective. The assumption that f has the Weierstrass division property means precisely that the map is surjective. If we denote by T the inverse of this map, then the closed graph theorem implies that T is continuous. Applying the fact quoted above to this continuous linear map T, we obtain

LEMMA 3.2. *Fix $\nu \geqq \nu_0$ and N. Then there exist positive integers $\nu' \geqq \nu_0$ and N' and a constant $A > 0$ such that for each $g = g(t, x) \in \xi_k$ which is represented by*

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an element of $E_{k,\nu,N}$ *there exist unique elements* $q = q(t, x) \in E_{k,\nu',N'}$ *and* $r_j = r_j(x) \in E_{k-1,\nu',N'}$ *for* $1 \leqq j \leqq p$ *with*

$$g = fq + r, \quad \text{where} \quad r = \sum_{j=1}^{p} r_j t^{p-j}, \tag{3.5}$$

and q *and the* r_j *satisfy the estimates*

$$\rho_{k,\nu',N'}(q) \leqq A\rho_{k,\nu,N}(g) \quad \text{and}$$
$$\rho_{k-1,\nu',N'}(r_j) \leqq A\rho_{k,\nu,N}(g) \quad \text{for } 1 \leqq j \leqq p.$$

4. Examples. We give first an example which shows that Theorem 2.7, the generic division theorem in $\mathcal{O}_k$, does not generalize to any quasianalytic local ring $\xi_k(\{M_n\}) \supsetneq \mathcal{O}_k$ for $k \geqq 1$.

Suppose a generic division theorem held in a quasianalytic local ring $\xi_k = {}_0\xi_k(\{M_n\}) \supseteq \mathcal{O}_k$ for some $k \geqq 1$. Let $\lambda = (0, 1) \in \mathbf{C}^2$, and set $P = P(t, \lambda) = t^2 + 1$. For each $a \in \mathbf{R}$, $g = g(t, x) = g(t, x, a) = e^{iat}$ defines an element of ξ_k. Thus, for each $a \in \mathbf{R}$, we may write

$$e^{iat} = (t^2 + 1)q(t, x, a) + r_1(x, a)t + r_2(x, a), \tag{4.1}$$

where $q = q(t, x, a) \in \xi_k$ and $r_1 = r_1(x, a)$, $r_2 = r_2(x, a) \in \xi_{k-1}$. ($\xi_0(\{M_n\}) = \mathbf{C}$.) Since

$$\begin{aligned} \rho_{k,1,1}(g) &= \sup_n \sup_{\substack{|\alpha| \leqq n, \\ (t,x) \in \Delta_k(1)}} |D_{(t,x)}{}^\alpha e^{iat}|/(1^n M_n) \\ &\leqq \sup_n \sup_{\substack{|\alpha| \leqq n, \\ t \in \Delta_1(1)}} |(ia)^{|\alpha|} e^{iat}|/M_n \\ &= \sup_n \sup_{|\alpha| \leqq n} |a|^{|\alpha|}/M_n \\ &= \sup_n \sup_{|\alpha| \leqq n} (|a|^{|\alpha|}/M_{|\alpha|})(M_{|\alpha|}/M_n) \\ &\leqq \sup_n \lambda(a)(M_n/M_n) \\ &= \lambda(a) \quad < +\infty, \end{aligned} \tag{4.2}$$

it follows that $g = g(t, x, a) \in E_{k,1,1}$ for all $a \in \mathbf{R}$. Applying Lemma 3.2, we obtain that there exist positive integers ν' and N' and a constant $A > 0$, all independent of $a \in \mathbf{R}$, such that the germs in equation (4.1) are represented by elements of $E_{\nu',N'}$ and $r_1 = r_1(x, a)$ in particular satisfies

$$\begin{aligned} |r_1(x, a)| &\leqq \rho_{k-1,\nu',N'}(r_1) \\ &\leqq A\rho_{k,1,1}(g) \\ &\leqq A\lambda(a) \quad \text{for } x \in \Delta_{k-1}(1/\nu') \text{ and } a \in \mathbf{R}, \end{aligned} \tag{4.3}$$

where the last inequality follows from inequality (4.2).

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Now the roots of $t^2 + 1 = 0$ are $t = \pm i$, and so equation (4.1) yields the system of equations

$$e^{-a} = ir_1(x, a),$$
$$e^{a} = -ir_1(x, a).$$

Thus $r_1(x, a) = i(e^a - e^{-a})/2$. Since $|r_1(x, a)|$ is thus asymptotic to $e^a/2$ as $a \to +\infty$, we obtain from inequality (4.3) that there exist constants $C, K > 0$, both independent of $a \in \mathbf{R}$, such that $e^a \leqq K\lambda(a)$ for $a > C$. Applying Lemma 3.1, it follows that there exist constants $\alpha, \beta > 0$ such that $M_n \leqq \alpha\beta^n n!$ for all n. Thus $\xi_k(\{M_n\}) = \mathcal{O}_k$.

We give now an example which shows that Theorem 2.8, the Weierstrass division theorem in ξ_k, does not generalize to any quasianalytic local ring $\xi_k(\{M_n\}) \supsetneq \mathcal{O}_k$ for $k \geqq 2$.

Suppose a Weierstrass division theorem held in a quasianalytic local ring $\xi_k = {}_0\xi_k(\{M_n\}) \supseteq \mathcal{O}_k$ for some $k \geqq 2$. Let $f = f(t, x) = t^2 + x_1$. Certainly f is regular in t of order two. For each $a \in \mathbf{R}$, $g = g(t, x) = g(t, x, a) = e^{iat}$ defines an element of ξ_k. Thus, for each $a \in \mathbf{R}$, we may write

$$(4.4) \qquad e^{iat} = (t^2 + x_1)q(t, x, a) + r_1(x, a)t + r_2(x, a),$$

where $q = q(t, x, a) \in \xi_k$ and $r_1 = r_1(x, a)$, $r_2 = r_2(x, a) \in \xi_{k-1}$. Now repeating exactly the same argument used above, we again obtain inequality (4.3). Since the roots of $t^2 + x_1 = 0$ are $t = \pm i\sqrt{x_1}$, equation (4.4) yields the system of equations

$$e^{-a\sqrt{x_1}} = i\sqrt{x_1}r_1(x, a),$$
$$e^{a\sqrt{x_1}} = -i\sqrt{x_1}r_1(x, a).$$

Thus $r_1(x, a) = i(e^{a\sqrt{x_1}} - e^{-a\sqrt{x_1}})/2\sqrt{x_1}$. Choose $\epsilon > 0$ such that $\epsilon^2 < 1/\nu'$, and set $x = (\epsilon^2, 0, \ldots, 0)$. We then obtain from inequality (4.3) that

$$|(e^{\epsilon a} - e^{-\epsilon a})/2\epsilon| \leqq A\lambda(a)$$

for $a \in \mathbf{R}$. Since $(e^{\epsilon a} - e^{-\epsilon a})/2\epsilon$ is asymptotic to $e^{\epsilon a}/2\epsilon$ as $a \to +\infty$, we obtain that there exist constants $C, K > 0$, both independent of $a \in \mathbf{R}$, such that $e^{\epsilon a} \leqq K\lambda(a)$ for $a > C$. Applying Lemma 3.1, we can again deduce that $\xi_k(\{M_n\}) = \mathcal{O}_k$.

5. A necessary condition. Let $\xi_k(\{M_n\}) \supsetneq \mathcal{O}_k$, where $k \geqq 2$, be quasianalytic. Abstracting from the examples in the previous section, we give a condition which an element $f = f(t, x) \in \xi_k$ with the Weierstrass division property must satisfy.

THEOREM 5.1. *Fix a quasianalytic local ring $\xi_k = \xi_k(\{M_n\}) \supsetneq \mathcal{O}_k$, where $k \geqq 2$, and $f = f(t, x) \in \xi_k$ which is regular in t of order p. Suppose f has the Weierstrass division property. Then there is a unique polynomial $P = P(t, x) \in \xi_{k-1}[t]$,*

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which is monic in t of degree p, such that f/P is a unit in ξ_k. Further, there is a neighborhood U of 0 in $\mathbf{R}^{k-1}$ such that the germ P is defined on $\mathbf{C} \times U$, and for all $x \in U$, the roots of $P(t, x) = 0$ are real.

Proof. Since f has the Weierstrass division property, we may perform the division

(5.1) $\quad t^p = f(t, x)q(t, x) + \sum_{j=1}^{p} r_j(x)t^{p-j},$

where $q = q(t, x) \in \xi_k$ and $r_j = r_j(x) \in \xi_{k-1}$ for $1 \leqq j \leqq p$. Let $P = P(t, x) = t^p + \sum_{j=1}^{p} a_j(x)t^{p-j}$, where $a_j = -r_j$ for $1 \leqq j \leqq p$. Equation (5.1) becomes

(5.2) $\quad P(t, x) = f(t, x)q(t, x).$

Since f is regular in t of order p,

$$\begin{aligned} p! &= \partial^p P(0, 0)/\partial t^p \\ &= \sum_{j=1}^{p} \binom{p}{j} \partial^j f(0, 0)/\partial t^j \partial^{p-j} q(0, 0)/\partial t^{p-j} \\ &= \partial^p f(0, 0)/\partial t^p q(0, 0). \end{aligned}$$

Thus $q(0, 0) \neq 0$, and so q is a unit in ξ_k. Let u be the unit $1/q$ in ξ_k. Then equation (5.2) becomes

$$f(t, x)/P(t, x) = u(t, x).$$

The uniqueness of P follows by reversing the above process to write

$$t^p = f(t, x)(1/u(t, x)) - \sum_{j=1}^{p} a_j(x)t^{p-j},$$

and then applying the uniqueness of Weierstrass division.

We now turn to establishing the assertion concerning the zeros of P. Since $g = g(t, x) = g(t, x, a) = e^{iat}$ defines an element of ξ_k for each $a \in \mathbf{R}$, and since f has the Weierstrass division property implies P has the Weierstrass division property, for each $a \in \mathbf{R}$ we may write

(5.3) $\quad e^{iat} = P(t, x)q(t, x, a) + \sum_{j=1}^{p} r_j(x, a)t^{p-j},$

where $q = q(t, x, a) \in \xi_k$ and $r_j = r_j(x, a) \in \xi_{k-1}$ for $1 \leqq j \leqq p$. Choose $\epsilon > 0$ such that all the germs in equation (5.3) are defined for $(t, x) \in \Delta_k(\epsilon)$. We can use exactly the same argument employed to obtain inequality (4.3) to show that we can decrease ϵ, if necessary, and find a constant $A > 0$, with both ϵ and A independent of $a \in \mathbf{R}$, such that

(5.4) $\quad |r_j(x, a)| \leqq A\lambda(a) \quad$ for $x \in \Delta_{k-1}(\epsilon)$, $a \in \mathbf{R}$, and $1 \leqq j \leqq p$.

Fix $x \in \Delta_{k-1}(\epsilon)$. Suppose, in order to obtain a contradiction, that some of the roots $t_1 = t_1(x), \ldots, t_p = t_p(x)$ of $P(t, x) = 0$ have nonzero imaginary parts. In particular, the t_j's are not all zero, and so some $a_j = a_j(x)$ is not zero, where $P(t, x) = t^p + \sum_{j=1}^{p} a_j(x)t^{p-j}$. Suppose $a_p = \ldots = a_{\alpha+1} = 0$, but $a_\alpha \neq 0$. Let

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the distinct roots be $t_{j_1}, \ldots, t_{j_\beta}$, with respective multiplicities $m_1, \ldots, m_\beta$, where $\beta \geqq 2$ and $t_{j_\beta} = 0$. (If none of the t_j's is zero, then $\alpha = p$ and $m_\beta = 0$.) We first derive an expression for r_α.

Let $r = r(t, x, a) = \sum_{j=1}^{p} r_j(x, a) t^{p-j}$. Suppose for the moment that there are p distinct t_j's. Since substitution in equation (5.3) yields the system of equations

$$e^{iat_1} = r(t_1, x, a),$$
$$\ldots$$
$$e^{iat_p} = r(t_p, x, a),$$

it follows that r is the unique polynomial in t of degree $\leqq p - 1$ which interpolates e^{iat} at $t = t_1, \ldots, t = t_p$. Let Γ be a positively oriented circle about the origin in $\mathbf{C}$ which contains all the t_j's. From the theory of interpolating polynomials (c.f. [**3**]), we obtain

$$\begin{aligned} r(z, x, a) &= \frac{1}{2\pi i}\int_\Gamma \frac{P(t, x) - P(z, x)}{P(t, x)(t - z)} e^{iat}\, dt \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{e^{iat}}{t - z}\, dt - \frac{1}{2\pi i}\int_\Gamma \frac{P(z, x)}{t - z}\,\frac{e^{iat}}{P(t, x)}\, dt. \end{aligned} \tag{5.5}$$

Moreover, this formula continues to hold when some of the t_j's coalesce. Thus we drop the momentary assumption that there are p distinct t_j's. A simple computation shows that

$$\begin{aligned} \frac{1}{(p - \alpha)!}\left(\frac{\partial}{\partial z}\right)^{p-\alpha}\bigg|_{z=0} \frac{P(z, x)}{t - z} &= \sum_{\nu=0}^{p-\alpha} \frac{a_{p-\nu}}{t^{p-\alpha-\nu+1}} \\ &= \frac{a_\alpha}{t}, \end{aligned}$$

where the second equality follows from the fact that $a_p = \ldots = a_{\alpha+1} = 0$. Thus, differentiating under the signs of integration in equation (5.5), we obtain

$$\begin{aligned} r_\alpha(x, a) &= \frac{1}{(p - \alpha)!}\left(\frac{\partial}{\partial z}\right)^{p-\alpha}\bigg|_{z=0} r(z, x, a) \\ &= \frac{1}{2\pi i}\int_\Gamma \frac{e^{iat}}{t^{p-\alpha+1}}\, dt - \frac{a_\alpha}{2\pi i}\int_\Gamma \frac{e^{iat}}{tP(t, x)}\, dt. \end{aligned} \tag{5.6}$$

We apply the residue theorem to evaluate the integrals in equation (5.6).

$$\frac{1}{2\pi i}\int_\Gamma \frac{e^{iat}}{t^{p-\alpha+1}}\, dt = \frac{(ia)^{p-\alpha}}{(p - \alpha)!}, \quad \text{and}$$

$$\begin{aligned} \frac{a_\alpha}{2\pi i}\int_\Gamma \frac{e^{iat}}{tP(t, x)}\, dt = a_\alpha \operatorname*{Res}_{t=t_{j1}} \frac{e^{iat}}{tP(t, x)} + \ldots + a_\alpha \operatorname*{Res}_{t=t_{j(\beta-1)}} \frac{e^{iat}}{tP(t, x)} \\ + a_\alpha \operatorname*{Res}_{t=t_{j\beta}=0} \frac{e^{iat}}{tP(t, x)}. \end{aligned} \tag{5.7}$$

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For $1 \leqq \nu \leqq \beta - 1$,

$$\operatorname*{Res}_{t=t_{j_\nu}} \frac{e^{iat}}{tP(t,x)} = \operatorname*{Res}_{t=t_{j_\nu}} \frac{e^{iat}Q_\nu(t)}{(t-t_j)^{m_\nu}},$$

where $Q_\nu(t)$ is analytic in a neighborhood of $t = t_{j_\nu}$ and $Q_\nu(t_{j_\nu}) \neq 0$. Thus, for $1 \leqq \nu \leqq \beta - 1$,

$$(5.8)\quad \begin{aligned} \operatorname*{Res}_{t=t_{j_\nu}} \frac{e^{iat}}{tP(t,x)} &= \frac{1}{(m_\nu - 1)!}\left(\frac{d}{dt}\right)^{m_\nu - 1}\Bigg|_{t=t_j} (Q_\nu(t)e^{iat}) \\ &= e^{iat_{j_\nu}}\left[\frac{1}{(m_\nu-1)!}\sum_{\mu=0}^{m_\nu-1}\binom{m_\nu - 1}{\mu} Q_\nu^{(\mu)}(t_{j_\nu})(ia)^{m_\nu-1-\mu}\right]. \end{aligned}$$

Note that since $Q_\nu(t_{j_\nu}) \neq 0$, the bracketed expression in equation (5.8) is a polynomial in a of degree exactly $m_\nu - 1$. For $\nu = \beta$,

$$\operatorname*{Res}_{t=t_{j_\beta}=0} \frac{e^{iat}}{tP(t,x)} = \operatorname*{Res}_{t=0} \frac{e^{iat}Q(t)}{t^{m_\beta+1}},$$

where $Q(t)$ is analytic in a neighborhood of $t = 0$ and $Q(0) \neq 0$. Thus

$$(5.9)\quad \begin{aligned} \operatorname*{Res}_{t=t_{j_\beta}=0} \frac{e^{iat}}{tP(t,x)} &= \frac{1}{m_\beta!}\left(\frac{d}{dt}\right)^{m_\beta}\Bigg|_{t=0} (Q(t)\, e^{iat}) \\ &= \frac{1}{m_\beta!}\sum_{\mu=0}^{m_\beta}\binom{m_\beta}{\mu} Q^{(\mu)}(0)\,(ia)^{m_\beta-\mu}, \end{aligned}$$

which is a polynomial in a of degree exactly m_β since $Q(0) \neq 0$. To summarize, since $a_\alpha \neq 0$, equations (5.6) – (5.9) show that

$$(5.10)\quad r_\alpha(x,a) = \sum_{\nu=1}^{\beta} e^{iat_{j_\nu}}\gamma_\nu(x,a),$$

where the γ_ν's are nonzero polynomials in a.

Returning to the proof of the theorem, we now assume one of the t_j's has a positive imaginary part. (A proof similar to the one which follows works when one of the t_j's has a negative imaginary part.) List those roots satisfying $t_{j_\mu} = \max_{1\leqq\nu\leqq\beta} \operatorname{Im} t_{j_\nu} \equiv y > 0$; say $t_{j_{\nu_1}}, \ldots, t_{j_{\nu_\phi}}$ are those roots. Write $t_{j_{\nu_\mu}} = x_\mu + iy$ for $1 \leqq \mu \leqq \phi$. Then equation (5.10) becomes

$$(5.11)\quad \begin{aligned} r_\alpha(x,a) &= \sum_{\mu=1}^{\phi} e^{iat_{j_{\nu_\mu}}}\gamma_{\nu_\mu} + \sum_{\substack{1\leqq\nu\leqq\beta,\\ \nu\notin\{\nu_1,\ldots,\nu_\phi\}}} e^{iat_{j_\nu}}\gamma_\nu \\ &= e^{-ay}\left[\sum_{\mu=1}^{\phi} e^{iax_\mu}\gamma_{\nu_\mu} + \sum_{\substack{1\leqq\nu\leqq\beta,\\ \nu\notin\{\nu_1,\ldots,\nu_\phi\}}} e^{ia(t_{j_\nu}-iy)}\gamma_\nu\right]. \end{aligned}$$

Let $F = F(a) = \sum_{\mu=1}^{\phi} e^{iax_\mu}\gamma_{\nu_\mu}$. We will show that there exist constants B_1, C_1, $L > 0$ such that each interval of length L contained in $(-\infty, -B_1)$ contains a number $\hat{a}$ with $|F(\hat{a})| \geqq C_1$. Assuming this for the moment, we show how to use it to obtain the contradiction we seek. Observe that the second sum in the bracketed expression on the right hand side of equation (5.11) $\to 0$ as $\mathbf{R} \ni a \to$

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$-\infty$. Using equations (5.4) and (5.11), we therefore obtain that there exist constants $B, K > 0$ such that each interval of length L contained in $(B, +\infty)$ contains a number $\hat{a}$ with $e^{\hat{a}y} \leqq K\lambda(\hat{a})$. In the proof of Lemma 3.1, we showed that given n sufficiently large, there exists $a_n > 2B$ with $a_n{}^n/M_n = \lambda(a_n)$. We also saw that $a_n \to +\infty$ as $n \to +\infty$. Thus we may assume $a_n \geqq 2L$. Hence we may choose $\hat{a}_n$ with $\frac{1}{2}\, a_n \leqq \hat{a}_n \leqq a_n$ so that $e^{\hat{a}_n y} \leqq K\lambda(\hat{a}_n)$. We therefore have

$$\begin{aligned}(\hat{a}_n)^n/y^{-n}n! &\leqq e^{\hat{a}_n y}\\ &\leqq K\lambda(\hat{a}_n) = K\sup_\sigma(a_n{}^\sigma/M_\sigma)(\hat{a}_n/a_n)^\sigma\\ &\leqq K\sup_\sigma a_n{}^\sigma/M_\sigma = K\lambda(a_n)\\ &= \mathrm{K}a_n{}^n/M_n\\ &= K[(\hat{a}_n)^n/M_n][a_n/\hat{a}_n]^n\\ &\leqq K2^n(\hat{a}_n)^n/M_n.\end{aligned}$$

Thus $M_n \leqq K(2/y)^n n!$ for all sufficiently large n. Applying Lemma 3.1, we obtain $\xi_k(\{M_n\}) = \mathscr{O}_k$, which contradicts our original assumption that $\xi_k(\{M_n\}) \supsetneqq \mathscr{O}_k$.

Thus, to complete the proof of the theorem, there remains only to verify the assertion concerning $F(a) = \sum_{\mu=1}^{\phi} e^{iax_\mu}\gamma_{\nu_\mu}$. Let d be the maximum of the degrees of the γ_{ν_μ}'s as polynomials in a. Then $a^{-d}F(a) = \sum_{\mu=1}^{\phi} e^{iax_\mu}(a^{-d}\gamma_{\nu_\mu})$ is asymptotic to $G = G(a) = \sum_{\mu=1}^{\phi} e^{iax_\mu}c_\mu$ as $a \to -\infty$, where c_μ is the coefficient of a^d in γ_{ν_μ}, so that not all the c_μ's are zero. Since

$$t_{j_{\nu_1}} = x_1 + iy, \ldots, t_{j_{\nu_\phi}} = x_\phi + iy$$

are all distinct, the x_μ's are all distinct. Thus $G \not\equiv 0$. Recall from the theory of almost periodic functions that if $H : \mathbf{R} \to \mathbf{C}$ and $\delta > 0$, then $\tau = \tau(\delta) \in \mathbf{R}$ is called a *translation number of H corresponding to δ* whenever $|H(x+\tau) - H(x)| < \delta$ holds for all $x \in \mathbf{R}$. H is called *almost periodic* if H is continuous and if for each $\delta > 0$, there exists a length $L = L(\delta)$ such that each interval of length L contains at least one translation number $\tau = \tau(\delta)$ of H. Now it is an elementary proposition in the theory of almost periodic functions that functions of the form $G(a) = \sum_{\mu=1}^{\phi} e^{iax_\mu}c_\mu$ are almost periodic. Since $G \not\equiv 0$, we can choose $a \in \mathbf{R}$ such that $|G(a)| = c > 0$. Let $L = L(c/2)$ be a positive number such that each interval of length L contains a translation number $\tau = \tau(c/2)$ of G. If I is such an interval, choose $\tau \in I - a$. Then $a + \tau \in I$ and $|G(a+\tau) - G(a)| < c/2$, so that $|G(a+\tau)| > c/2$. Thus each interval of length L contains a point $\hat{a}$ such that $|G(\hat{a})| > c/2 > 0$. Since $a^{-d}F(a)$ is asymptotic to $G(a)$ as $a \to -\infty$, there thus exist constants $B_1, C_1 > 0$ such that each interval of length L contained in $(-\infty, -B_1)$ contains a point $\hat{a}$ with $|F(\hat{a})| \geqq C_1$. This completes the proof of the theorem.

6. A necessary and sufficient condition. Let $\xi_k = \xi_k(\{M_n\})$, $k \geq 1$, be quasianalytic. In [**4**] Ehrenpreis has shown that ξ_k is "analytically uniform"

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and, therefore, that the elements of ξ_k have Fourier integral representations. After stating Ehrenpreis' result more precisely, we use it together with Lemma 3.2 to give a condition which is both necessary and sufficient for an element $f = f(t, x) \in \xi_k$ to have the Weierstrass division property. We remark, however, that this condition is often difficult to apply in practice.

Recall the definition of the function λ in equation (2.9). We will also use the notation

$$\operatorname{Im} a = (\operatorname{Im} a_1, \ldots, \operatorname{Im} a_k)$$

and

$$(t, x)\cdot a = ta_1 + x_1 a_2 + \ldots + x_{k-1} a_k$$

for $(t, x) = (t, x_1, \ldots, x_{k-1}) \in \mathbf{R}^k$ and $a = (a_1, \ldots, a_k) \in \mathbf{C}^k$.

PROPOSITION 6.1. (Ehrenpreis) *Let $\xi_k = \xi_k(\{M_n\})$, $k \geqq 1$, be quasianalytic and $f = f(t, x) \in \xi_k$. There exist constants $\epsilon > 0$, $B_f > 0$, and a complex Borel measure μ on $\mathbf{C}^k$ such that*

$$(6.1) \qquad f(t, x) = \int_{\mathbf{C}^k} \frac{e^{i(t,x)\cdot a} d\mu(a)}{\lambda(a/B_f) e^{\epsilon|\operatorname{Im} a|}}, \quad (t, x) \in \Delta_k(\epsilon).$$

Proof. See [**4**].

THEOREM 6.2. *Let $\xi_k = \xi_k(\{M_n\})$, $k \geqq 1$, be quasianalytic. $f = f(t, x) \in \xi_k$ has the Weierstrass division property if and only if we can perform the division*

$$(6.2) \qquad e^{i(t,x)\cdot a} = f(t, x) q(t, x, a) + \textstyle\sum_{j=1}^{p} r_j(x, a) t^{p-j},$$

where $q = q(t, x, a) \in \xi_k$ and $r_j = r_j(x, a) \in \xi_{k-1}$ for $1 \leqq j \leqq p$ and all $a \in \mathbf{C}^k$, and where for each $\epsilon > 0$, there exist $A > 0$ and positive integers ν and N, all independent of $a \in \mathbf{C}^k$, such that

$$(6.3) \qquad \begin{aligned} &\rho_{k-1,\nu,N}(r_j(\cdot, a)) \leqq A\lambda(a) e^{\epsilon|\operatorname{Im} a|}, \quad 1 \leqq j \leqq p, \quad \textit{and} \\ &\rho_{k,\nu,N}(q(\cdot, a)) \leqq A\lambda(a) e^{\epsilon|\operatorname{Im} a|}. \end{aligned}$$

Proof. We first assume that f has the Weierstrass division property. Then we can perform division (6.2), and we must obtain estimate (6.3). Let $\epsilon > 0$ be given. Choose a positive integer ν' such that $1/\nu' < \epsilon$. Then

$$(6.4) \qquad \begin{aligned} \rho_{k,\nu',1}(e^{i(t,x)\cdot a}) &= \sup_n \sup_{\substack{|\beta| \leqq n, \\ (t,x) \in \Delta_k(1/\nu')}} |D_{(t,x)}{}^{\beta} e^{i(t,x)\cdot a}|^n M_n \\ &\leqq |a|^{|\beta|} e^{\epsilon|\operatorname{Im} a|}/M_n \\ &= (|a|^{|\beta|}/M_{|\beta|})(M_{|\beta|}/M_n)\, e^{\epsilon|\operatorname{Im} a|} \\ &\leqq \lambda(a)\, e^{\epsilon|\operatorname{Im} a|}. \end{aligned}$$

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It follows that $e^{i(t,x)\cdot a} \in E_{k,\nu'1}$ for all $a \in \mathbf{C}^k$. Thus, according to Lemma 3.2, there exist $A > 0$ and positive integers ν and N, all independent of $a \in \mathbf{C}^k$, such that

$$(6.5) \quad \rho_{k-1,\nu,N}(r_j(\cdot, a)) \leqq A\rho_{k,\nu',1}(e^{i(t,x)\cdot a}), \quad 1 \leqq j \leqq p, \quad \text{and}$$
$$\rho_{k,\nu,N}(q\cdot, a)) \leqq A\rho_{k,\nu',1}(e^{i(t,x)\cdot a}).$$

Combining inequalities (6.4) and (6.5), we obtain inequalities (6.3).

We now assume that we can perform the division (6.2) with the estimates (6.3) holding, and we show that f has the Weierstrass division property. Let $\epsilon > 0$ and $g = g(t, x) \in \xi_k$ be defined on $\Delta_k(\epsilon)$. By Proposition 6.1, there exist δ, with $0 < \delta < \epsilon$, $B_g > 0$, and a complex Borel measure μ on $\mathbf{C}^k$ such that g has the Fourier integral representation

$$(6.6) \quad g(t, x) = \int_{\mathbf{C}^k} \frac{e^{i(t,x)\cdot a} d\mu(a)}{\lambda(a/B_g)e^{\epsilon|\mathrm{Im}a|}}, \quad (t, x) \in \Delta_k(\delta).$$

(We may replace the sequence $\{M_n\}$ by the sequence $\{B_g^n M_n\}$; thus we may assume $B_g = 1$.) Substituting for $e^{i(t,x)\cdot a}$ from equation (6.2) in equation (6.6), we get

$$g(t, x) = f(t, x) \int_{\mathbf{C}^k} \frac{q(t, x, a)d\mu(a)}{\lambda(a)e^{\epsilon|\mathrm{Im}a|}} + \sum_{j=1}^{p} t^{p-j} \int_{\mathbf{C}^k} \frac{r_j(x, a)d\mu(a)}{\lambda(a)e^{\epsilon|\mathrm{Im}a|}}.$$

We estimate

$$D_x^\alpha \int_{\mathbf{C}^k} \frac{r_j(x, a)d\mu(a)}{\lambda(a)e^{\epsilon|\mathrm{Im}a|}}$$

for $|\alpha| \leqq n$, $x \in \Delta_{k-1}(\delta)$, and $1 \leqq j \leqq p$.
Using estimate (6.3), we see that for such α and x,

$$|D_x^\alpha r_j(x, a)| \leqq A\lambda(a)e^{\epsilon|\mathrm{Im}a|}N^n M_n.$$

Thus we get

$$\left| D_x^\alpha \int_{\mathbf{C}^k} \frac{r_j(x, a)d\mu(a)}{\lambda(a)e^{\epsilon|\mathrm{Im}a|}} \right| = \left| \int_{\mathbf{C}^k} \frac{D_x^\alpha r_j(x, a)d\mu(a)}{\lambda(a)e^{\epsilon|\mathrm{Im}a|}} \right|$$
$$\leqq \int_{\mathbf{C}^k} \frac{|D_x^\alpha r_j(x, a)|d|\mu|(a)}{\lambda(a)e^{\epsilon|\mathrm{Im}a|}} \leqq \int_{\mathbf{C}^k} \frac{A\lambda(a)e^{\epsilon|\mathrm{Im}a|}N^n M_n d|\mu|(a)}{\lambda(a)e^{\epsilon|\mathrm{Im}a|}}$$
$$= (A||\mu||)N^n M_n.$$

A similar computation using estimate (6.3) shows that

$$D_{(t,x)}{}^\beta \int_{\mathbf{C}^k} \frac{q(t, x, a)d\mu(a)}{\lambda(a)e^{\epsilon|\mathrm{Im}a|}}$$

can be estimated by $(A||{-}||)N^n M_n$ for $|\beta| \leqq n$ and $(t, x) \in \Delta_k(\delta)$. It follows that

$$(6.7) \quad g(t, x) = f(t, x)Q(t, x) + \textstyle\sum_{j=1}^{p} R_j(x)t^{p-j},$$

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where

$$Q(t, x) = \int_{\mathbf{C}^k} \frac{q(t, x, a) d\mu(a)}{\lambda(a) e^{\epsilon|\mathrm{Im}\, a|}}$$

represents an element of ξ_k on $\Delta_k(\delta)$ and

$$R_j(x) = \int_{\mathbf{C}^k} \frac{r_j(x, a) d\mu(a)}{\lambda(a) e^{\epsilon|\mathrm{Im}\, a|}}$$

represents an element of ξ_{k-1} on $\Delta_{k-1}(\delta)$ for $1 \leqq j \leqq p$. Thus f has the Weierstrass division property.

7. An algebraic question. Let R_k, $k = 1, 2, \ldots$, be local rings of smooth functions in k variables which contain $\mathcal{O}_k$. Assume that R_{k+1} is the "natural extension" to $(k + 1)$ variables of R_k. If a Weierstrass division theorem with uniqueness of the quotient and the remainder were to hold in the R_k, then by repeating standard arguments, it would be possible to show that the R_k are Noetherian unique factorization domains; see [**6**]. It may be that, conversely, if the R_k are Noetherian unique factorization domains, then a Weierstrass division theorem with uniqueness of quotient and remainder would hold in the R_k. This conjecture is supported by the fact that the Weierstrass division theorem can be formulated algebraically as a finiteness condition: Let m_j be the maximal ideal in R_j, and if u: $R_j \rightarrow R_k$ is an algebra homomorphism, let $u(m_j)R_k$ be the ideal generated in R_k be the image $u(m_j)$ of m_j. u is said to be *quasi-finite* if the induced map $\bar{u} : \mathbf{C} \approx R_j/m_j \rightarrow R_k/u(m_j)R_k$ makes $R_k/u(m_j)R_k$ into a finite dimensional $\mathbf{C}$-vector space, and u is said to be *finite* if it makes R_k into a finitely generated R_j-module. If u is finite, it is always quasi-finite. The Weierstrass division theorem is equivalent to the assertion that the converse implication always holds, i.e., if u is quasi-finite, then it is finite. See Wall [**9**] for a complete discussion.

In closing we remark that, in light of our results in this paper, were this conjecture established, we would know that the $\xi_k(\{M_n\})$ are not Noetherian unique factorization domains in the quasianalytic case with $k \geqq 2$.

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