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EMBEDDING NEAR-RINGS INTO POLYNOMIAL NEAR-RINGS

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In this paper we will show that large classes of near-rings are embeddable into (or even isomorphic to) near-rings $G^{\mathscr{V}}[x]$ of polynomials over G in a suitably chosen variety $\mathscr{V}$ of Ω-groups.

For near-rings see [**2**], for polynomials [**1**]; for near-rings of polynomials over Ω-groups see [**3**], [**4**], [**5**].

We start with the surprising result that every near-ring is a polynomial near-ring:

Theorem 1. *For every near-ring N there is a variety $\mathscr{V}$ of Ω-groups and a $G \in \mathscr{V}$ such that*

$$N \hookrightarrow G[x].$$

Proof. By 1.102 of [**2**], N can be embedded into a near-ring $\bar{N}$ with identity 1. Now we take $\mathscr{V}$ as the subvariety of the variety of all near-rings with identity which is generated by $\bar{N}$ (this last restriction is not necessary for the sequel, but seems to be natural when looking at examples). Now we can consider the map

$$\phi: \bar{N} \to \bar{N}^{\mathscr{V}}[x]: n \to nx$$

ϕ easily turns out to be a near-ring homomorphism if $\bar{N}^{\mathscr{V}}[x]$ is considered as a near-ring w.r.t. addition and composition of polynomials. ϕ is even a monomorphism (hence an embedding map): take $n \in \operatorname{Ker} \phi$. Then $nx = 0$. Suppose that $n \neq 0$. Then the polynomial function $f: \bar{N} \to \bar{N}$ induced by nx would fulfill $f(1) = n1 = n \neq 0$. Hence $f \neq 0$ (zero function), whence $nx \neq 0$, a contradiction. This proves the theorem.

Now we fix the variety $\mathscr{V}$ and take for $\mathscr{V}$ some important varieties.

We start with the variety $\mathscr{G}$ of groups. Quite unexpectedly, we need information about the idempotents in $G^{\mathscr{G}}[x]$. $G[x]$ as an additive group is the free sum of the group G and an infinite cyclic group generated by x. The product is composition of functions:

$$(g_0 + z_1 x + \ldots + z_n x + g_n) \cdot h = g_0 + z_1 h + \ldots + z_n h + g_n$$

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where $g_i \in G$, $0 \leqq i \leqq n$, $z_i \in \mathbb{Z}$, $1 \leqq i \leqq n$ and $h \in G[x]$. Obviously, constant elements $g \in G$ and x are idempotent. But that's all:

Theorem 2. *The idempotent elements in $G[x]$ (G a group) are x and the constants.*

Proof. Let $e = g_0 + z_1 x + \dots + z_n x + g_n$ be in reduced (normal) form. $\deg(e) = n$ is the "*degree*" of e. If $n \leqq 1$ then it is fairly easy to see that $e = x$ or $e = c \in G$. So suppose that $n \geqq 2$.

We first remark that if a is an element of a free product of groups written in reduced normal form, then there is a unique element b of maximal length such that $a = -b + c + b$ and this expression on the right is already in reduced form. Then the reduced form of $2c$ is just $c + c$. Note that c is the cyclically reduced form of a. The reduced form of za is then $-b + zc + b$. In this expression for a, b may be 0, but c never is.

From the expression for e and the assumptions, we deduce that

$$e \circ e = g_0 + z_1 e + \dots + z_n e + g_n,$$

where g_0 or g_n or both or neither may be zero, but no other terms are zero. Let $e = -a + b + a$, with b the cyclically reduced form of e.

If $a \neq 0$, then $z_i e = -a + z_i b + a$ and

$$e \circ e = g_0 - a + z_1 b + a + \dots + g_{i-1} - a + z_i b + a + g_i - a + \dots.$$

The only place where cancellations could start is at g_i for some i. Also a ends in either g_n or $z_n x$, so the only cancellation possible is the replacement of g_i by $g_n + g_i - g_n$ which is non-zero. There is no cancellation at the ends of the expression for $e \circ e$, except if g_0 or g_n is of order 2. In any case the degrees considered are not affected. Hence

$$\deg(e \circ e) = \sum_{i=1}^{n} \deg(z_i e) \geqq 2 \deg e > \deg e$$

as $n \geqq 2$, $\deg e = n \geqq 2$ and $\deg(z_i e) \geqq \deg e$ since $|z_i| \geqq 1$. Thus in this case $e \circ e \neq e$.

Now suppose that $a = 0$. Then the sum $e + e$ has no cancellations. Consider $e \circ e$ at g_i:

$$\dots + z_i e + g_i + z_{i+1} e + \dots$$

If $\operatorname{sgn} z_i \neq \operatorname{sgn} z_{i+1}$, again we have

$$\pm e + g_i \mp e$$

and, as before, the only possible change is in replacing g_i by $g_n + g_i - g_n$ or $-g_0 + g_i + g_0$. So assume that $\operatorname{sgn} z_i = \operatorname{sgn} z_{i+1}$. Then, at g_i we have

$$\dots + z_n x + g_n + g_i + g_0 + z_1 x + \dots,$$

or

$$\dots - z_1 x - g_0 + g_i - g_n - z_n x + \dots.$$

If $g_n + g_i + g_0$ or $-g_0 + g_i - g_n$ is not zero then there is no more cancellation. If it is zero, then we have

$$2(z_1 x + \dots + z_n x),$$

or

$$-2(z_1 x + \dots + z_n x)$$

and the cancellation can not proceed more than halfway along the expression $\pm(z_1x+\ldots+z_nx)$. This means that each g_i can be considered separately. There is no cancellation at either end since $g_0 \neq -g_n$ apart from replacing g_0 by $2g_0$ or g_0-g_n, and g_n by $2g_n$ or $-g_0+g_n$. If g_0 or g_n is of order 2 then $2g_0$ or $2g_n$ may be 0. In any case the degrees considered are not affected. Also $\deg(\pm 2(z_1x+\ldots+z_nx)) \geqq \deg(z_1x+\ldots+z_nx)$.

So unless $\operatorname{sgn} z_i$ is constant for all i, $1 \leqq i \leqq n$, $|z_i|=1$ for $1 \leqq i \leqq n$ and $g_n+g_i+g_0=0$ ($\operatorname{sgn} z_i=+1$) or $-g_0+g_i-g_n=0$ ($\operatorname{sgn} z_i=-1$) for $1 \leqq i \leqq n-1$, we have

$$\deg(e \circ e) \geqq 2 \deg e = 2n > \deg e = n,$$

and hence $e \circ e \neq e$. Otherwise we have

$$e \circ e = g_0 + g_0 + n(z_1x+\ldots+z_nx) + g_n + g_n \quad (\operatorname{sgn} z_i = +1)$$

or

$$g_0 - g_n - n(z_1x+\ldots+z_nx) - g_0 + g_n \quad (\operatorname{sgn} z_i = -1)$$

If $e \circ e = e$, then $g_0 = g_n = 0$ and so $\operatorname{sgn} z_1 = \operatorname{sgn} z_n$ forces $\deg n(z_1x+\ldots+z_nx) = n \deg(z_1x+\ldots+z_nx) = n^2-(n-1) > n$ as $n \geqq 2$. Thus in all cases it is impossible for $e \circ e = e$ to hold if $n \geqq 2$. This finishes the proof.

Corollary 1. (a) *If e is idempotent in $G_0[x]$ then $e=0$ or $e=x$.*
(*b*) *$G[x]$ has exactly $|G|+1$ idempotents.*

Corollary 2. *Let N be a non-zero subnear-ring of $G[x]$ with identity e. Then $e=x$.*

This holds since e is zerosymmetric.

Corollary 3. *Let N be a near-ring with identity $e \neq 0$ and let $\phi: N \to G_0[x]$ be an embedding map. Then $\phi(e)=x$.*

Corollary 4. *Let ϕ be as above. If $d \in N$ ($d \neq 0$) is distributive then there is some $g \in G$ with $\phi(d)=g+x-g$ or $N=\mathbb{Z}$ and $\phi(d)=zx$, $z \in \mathbb{Z}$.*

Proof. d is zero-symmetric, hence $\phi(d)$ has the form

$$\phi(d) = \sum_{i=1}^{n} (g_i + z_ix - g_i) \quad (g_i \in G,\ z_i \in \mathbb{Z},\ g_i \neq g_{i+1}).$$

Now
$\phi(d)\circ(x+x) = \phi(d)\circ\phi(e+e) = \phi(d\circ(e+e)) = \phi(d\circ e + d\circ e) = \phi(d+d) = \phi(d)+\phi(d)$.
Hence

$$\sum_{i=1}^{n} (g_i + 2z_ix - g_i) = \sum_{i=1}^{n} (g_i + z_ix - g_i) + \sum_{i=1}^{n} (g_i + z_ix - g_i).$$

However, this relation can only hold if the right hand side collapses, i.e. if $g_n = g_1$, $z_n = -z_1$, $g_{n-1} = g_2$, and so on. So the right side is either zero, whence $g_1 + 2z_1 - g_1 + \ldots + g_n + 2z_nx - g_n = 0$, a contradiction, or $n=1$ in which case $\phi(d) = g_1 + z_1x - g_1$.

Computing $\phi(d\circ(e+d))$ in two ways and comparing the results we obtain $z_1=1$ or $g_1=0$. If $z_1=1$, we have $\phi(d)=g+x-g$. If $g_1=0$, then $\phi(d)=z_1x$. Suppose $z_1 \neq 1$ and

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$\phi(N) \supset \{zx; z \in \mathbb{Z}\}$. Then $z_1x \circ (a+b) = z_1(a+b)$, $z_1x \circ a + z_1x \circ b = z_1a + z_1b$. So $z_1(a+b) = z_1a + z_1b$. Take one of the pair a, b to be x, the other to lie in $\phi(N)$, but not of the form zx, $z \in \mathbb{Z}$. By examining the various cases that can arise it is easy to see that we get a contradiction. Thus $N = \mathbb{Z}$ and $\phi(N) = \{zx; z \in \mathbb{Z}\}$.

Now we prove one of the few existing results that certain embeddings can not take place.

Theorem 3. *There exist distributively generated near-rings which cannot be embedded into some* $G[x]$.

Proof. Let N be a distributively generated near-ring such that $(N, +)$ is not a free group (for instance, take N finite, but $\neq \{0\}$). Let ϕ be a monomorphism of N into some $G[x]$. Then $\phi(N) \subseteq G_0[x]$. But $(G_0[x], +)$ is a free group and hence so is $(\phi(N), +)$. As $(\phi(N), +) \cong (N, +)$ we have a contradiction.

Consequently, not every generalized distributively generated (g.d.g.) near-ring can be embedded into some $G[x]$. Here, a near-ring N is g.d.g. if N is (additively) generated by distributive and constant elements (see [**5**]). $G[x]$ itself is g.d.g.

On the other hand we have shown in [**5**] that every finite near-ring can be embedded into the g.d.g. near-ring $P(G)$ of all polynomial functions on a suitable finite, simple, non-abelian group G.

We now consider the embedding of $G[x]$ in $P(H)$ for a suitable group H. To do this we need some information on the structure of $G[x]$. From [**1**], each element of $G[x]$ can be written uniquely in the form

$$g_1 + z_1x - g_1 + \ldots + g_n + z_nx - g_n + h = \sum_{i=1}^{n} (g_i + x - g_i) + h,$$

where each $g_i + x - g_i$ is distributive and h is a constant element. It is immediate that

$$\mathrm{Gp}\langle g + x - g; g \in G\rangle = G_0[x],$$

is the free d.g. near-ring on the set,

$$\{g + x - g; g \in G\}$$

of distributive elements. Also

$$(g_1 + x - g_1) \circ (g_2 + x - g_2) = g_1 + g_2 + x - g_2 - g_1.$$

So the multiplicative set of distributive elements $\{g + x - g; g \in G\}$ is a group which is isomorphic as an abstract group to the additive group $(G, +)$ under the isomorphism

$$g + x - g \to g.$$

As an additive group $(G[x], +)$ has $(G_0[x], +)$ as a normal subgroup and $G[x] = G_0[x] + G_c[x]$ where $G_c[x]$ is the constant subnear-ring isomorphic as an additive group to G. Also, from the normal form given above, if $h \in G_c[x]$,

$$h + (g + x - g) - h = (h + g) + x - (h + g).$$

So the automorphism induced in $(G_0[x],+)$ by conjugation by an element $h\in G_c[x]$ maps

$$g+x-g\to(h+g)+x-(h+g).$$

The automorphisms induced in $(G_0[x],+)$ by the elements of $G_c[x]\cong G$ is that induced in the free group $(G_0[x],+)$ by the permutation of the free generators of $G_0[x]$ by the left regular representation.

This determines the additive structure of $G[x]$ completely. Now we consider the multiplicative structure. Since the right distributive law is satisfied, we only need to consider products of the form xy where y is a general element and x is either an element of the form $g+x-g$ or h. But $h\circ y=h$ for all $y\in G[x]$. So consider

$$\begin{aligned}(g+x-g)\circ\left(\sum_{i=1}^{n} g_i+x-g_i+h\right)&=g+\sum_{i=1}^{n} g_i+x-g_i+h-g\\&=\sum_{i=1}^{n}(g+g_i+x-g_i-g)+g+h-g.\end{aligned}$$

We now turn to the construction of the group H which we will need. Let $G^*=\{g^*;\, g\in G\}$ be a set in one-one correspondence with the elements of the group G and define K to be the free group on the set G^*. Let $\tau\colon G\to \operatorname{Aut} K$ be defined by $\tau(g)\,(g_1^*)=(g+g_1)^*$ for all $g_1\in K$. This is enough to define $\tau(g)\in\operatorname{Aut} K$ since K is a free group on G^*. Define H as the semidirect product of K by G using τ. Consider $P(H)$ and note that $P_0(H)\cong I(H)$, the near-ring generated by $\operatorname{Inn}(H)$, the inner automorphisms of H and $P_c(H)\cong H$. Let the constant map which sends H to $h\in H$ be denoted by $\theta(h)$.

Without loss of generality, assume that $G\subseteq H$. Consider the subnear-ring N of $P(H)$ generated by $\tau(G)$ and $\theta(G)$, where $\tau(G)$ are the inner automorphisms of H determined by G. We first show that $\operatorname{Gp}\langle\tau(G)\rangle$ is a free group on $\tau(G)$.

Consider $\sum_{i=1}^{n}\varepsilon_i\tau(g_i)$ where $\varepsilon_i=\pm1$, $1\leqq i\leqq n$. Then

$$\begin{aligned}\left(\sum_{i=1}^{n}\varepsilon_i\tau(g_i)\right)(0^*)&=\sum_{i=1}^{n}\varepsilon_i\tau(g_i)(0^*)\\&=\sum_{i=1}^{n}\varepsilon_i(g_i+0)^*\\&=\sum_{i=1}^{n}\varepsilon_i g_i^*.\end{aligned}$$

and this is not 0 unless $\sum_{i=1}^{n}\varepsilon_i\tau(g_i)$ is the trivial word since K is free on G^*. Hence $\sum_{i=1}^{n}\varepsilon_i\tau(g_i)\neq0$ in $P(H)$. It is trivial to check that $\tau(g_1)\circ\tau(g_2)=\tau(g_1+g_2)$. Also $\tau(g)0^*=g^*\neq g_1^*=\tau(g_1)0^*$ if $g\neq g_1$.

Thus $\tau(G)$ is a multiplicative group isomorphic to the additive group G and $\operatorname{Gp}\langle\tau(G)\rangle$ is the free d.g. near-ring on $\tau(G)$. Call this near-ring N_0. Then $N_0\cong G_0[x]$, with $\tau(g)\to g+x-g$. Again it is trivial that $\theta(g)+\tau(g_i)-\theta(g)=\tau(g+g_i)$. Hence

$(N,+)\cong(G[x],+)$ and $N_c\cong G_c[x]$, $N_0\cong G_0[x]$. Finally consider $\tau(g)\circ\left(\sum_{i=1}^{n}\tau(g_i)+\theta(k)\right)$ for $k\in G$. Let $h\in H$.

$$\tau(g)\circ\left(\sum_{i=1}^{n}\tau(g_i)+\theta(k)\right)(h)=\tau(g)\left(\sum_{i=1}^{n}\tau(g_i)(h)+k\right)$$
$$=\sum_{i=1}^{n}\tau(g+g_i)(h)+g+k-g=\left(\sum_{i=1}^{n}\tau(g+g_i)+\theta(g+k-g)\right)(h).$$

This is enough to complete the proof of the following theorem.

Theorem 4. *Let G be a group. Then there exists a group H such that $G[x]$ can be embedded in $P(H)$.*

Corollary 5. *Let G be a group. Then there exists a group H such that $G_0[x]$ can be embedded in $P_0(H)$.*

Note that $P_0(H)\cong I(H)$ and that the group H is isomorphic as a group to $(G[x],+)$ in the construction given above.

Now we change the variety of groups to $\mathscr{A}$, the variety of abelian groups. If $A\in\mathscr{A}$ then $A]x[:=A^{\mathscr{A}}[x]=\{a+zx\mid a\in A,\ z\in\mathbb{Z}\}$ is an abstract affine near-ring.

If we look at the variety $\mathscr{M}_R$ of R-modules over a ring R and $M\in\mathscr{M}_R$ then $M_R[x]:=M^{\mathscr{M}_R}[x]=\{m+rx\mid m\in M,\ r\in R\}$ is abstract affine, too.

Hence it is natural to ask which abstract affine near-rings can be embedded into $A]x[$ or $M_R[x]$. $A]x[$ does not suffice, but $M_R[x]$ does an excellent job:

Theorem 5. (i) *Not every abstract affine near-ring N can be embedded into some $A]x[$(A an abelian group).*

(ii) *Every abstract affine near-ring is isomorphic to a suitably chosen $M_R[x]$ (M_R an R-module).*

Proof. (i) We take $N:=\mathbb{R}\times\mathbb{R}$ with $+$ componentwise and $(r,s)\cdot(r',s'):=(r+sr',ss')$. Then N is an abstract affine near-ring. Suppose that A is an abelian group with $N\hookrightarrow A]x[$ by some monomorphism ϕ. $N_0=\{0\}\times\mathbb{R}$ will then be embedded into $(A]x[)_0\cong\mathbb{Z}$, a contradiction.

(ii) Let N be an abstract affine near-ring. Then N_0 is a ring and the constants N_c are an N_0-module. We consider the map $\psi\colon N\to(N_c)_{(N_0)}[x]$: $n_c+n_0\to n_c+n_0x$. ψ is obviously a group isomorphism. Also, for all $n,n'\in N$, $n=n_c+n_0$, $n'=n_c'+n_0'$, we get

$$\psi(nn')=\psi((n_c+n_0)(n_c'+n_0'))=\psi(n_c+n_0n_c'+n_0n_0').$$

Since $n_c+n_0n_c'\in N_c$ and $n_0n_0'\in N_0$, this expression equals

$$(n_c+n_0n_c')+(n_0n_0')x=(n_c+n_0x)\circ(n_c'+n_0'x)=\psi(n)\circ\psi(n').$$

Hence we get $N\cong(N_c)_{(N_0)}[x]$.

The authors hope that these results will help to solve the long-standing problem whether or not every zerosymmetric near-ring can be embedded into a d.g. near-ring (see [**2**], p. 178).

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