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Unsolvability of the knot problem for surface complexes

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It is shown that the problem of deciding whether a polygonal curve c in a finite surface complex K is knotted in K is complete recursively enumerable, and hence unsolvable.

We refer to [6] for the definition of a finite surface complex, introductory remarks, and general references. In [6] it was shown that the problem of deciding whether an edge path c in a 2-dimensional simplicial complex K bounds a disc in K is NP-complete. Generalizing to an arbitrary polygonal path c in K gives an equivalent problem, since K may be simplicially subdivided to make c an edge path in polynomial time. Bounding a disc is equivalent to the existence of an isotopy which contracts c to a point without pulling it over any point twice.

In the present paper we discuss the equivalence of simple curves under more general isotopies in K, namely simplicial isotopies in an arbitrary simplicial decomposition of K. Curves c_1, c_2 are called *simplicially isotopic* with respect to a simplicial decomposition Σ of K, if there is a finite sequence of simple edge paths of Σ,

$$c_1 = c^{(1)}, c^{(2)}, \ldots, c^{(k)} = c_2 ,$$

such that $c^{(m+1)}$ is the result of pulling $c^{(m)}$ from one side to the other of a triangle in Σ. A curve c is called *simplicially unknotted* with respect to Σ if it is simplicially isotopic to a curve which bounds a disc, and *unknotted in* K if it is simplicially unknotted with respect

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to some simplicial decomposition Σ of K. By the Hauptvermutung for surface complexes, [7], c is unknotted if it is simplicially unknotted in a sufficiently fine simplicial decomposition, for example the nth barycentric subdivision for sufficiently large n.

The reason we do not use general isotopies in K to define knotting was pointed out by Alexander [1] in the case of classical knots in R^3. Alexander's example may be adapted to surface complexes using a "book with three leaves" K (see page 133).

The curve c is a trefoil knot when K is embedded in R^3; nevertheless the isotopy (1) → (2) → (3) reduces it to a curve bounding a disc.

It is clear that we can decide whether a curve is unknotted with respect to a given Σ by enumerating the finitely many possible simplicial isotopies. (In fact this can be done by a non-deterministic linear bounded Turing machine, or using Savitch's Theorem [4], by a deterministic Turing machine on quadratically bounded tape.) By applying this decision process in successive barycentric subdivisions of K we see that the set of pairs (K, c) for which c is an unknotted polygonal curve in K is recursively enumerable.

We now show that the set is *complete* recursively enumerable by reducing the word problem for finitely presented groups to it.

THEOREM 1. *Given a finite presentation G of a group, and a word w in G, we can effectively construct a finite surface complex $K(G)$ and a simple polygonal curve $c(w)$ such that*

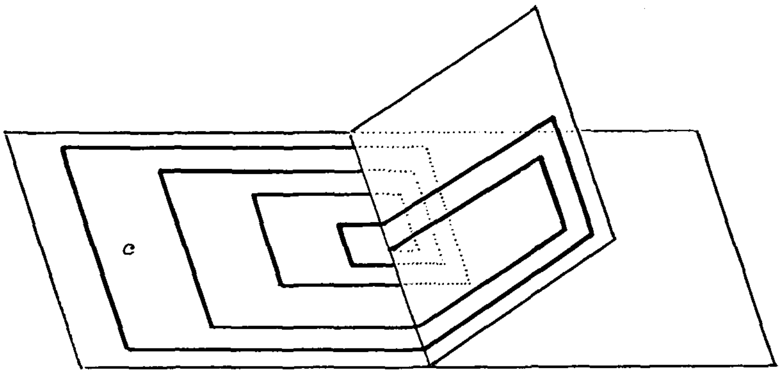
$$c(w) \text{ is unknotted in } K(G) \Leftrightarrow w = 1 \text{ in } G .$$

Proof. $K(G)$ is a slight modification of the complex used by Dehn [2] to realize an arbitrary finitely presented group

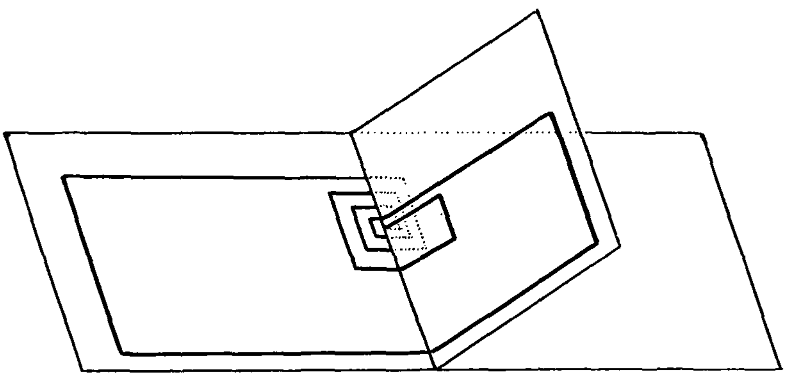
$$G = \langle a_1, \ldots, a_n ; r_1, \ldots, r_m \rangle .$$

Dehn takes a bouquet B of circles $a_1, \ldots, a_n$ to realize the generators, and realizes each relation $r_j = 1$ by attaching a disc D_j along its boundary to the path r_j (spelled as a product of a_i's) in B.

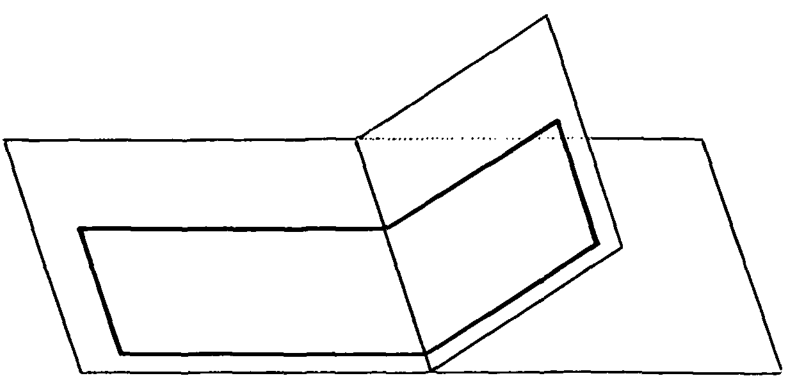
(1)



(2)



(3)



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We realize each a_i by an annulus A_i which has a_i as its centre-line, and let the different A_i meet along a common transverse segment $[0, 1]$ (and nowhere else). Given a word

$$w = a_{i_1}^{\varepsilon_1} a_{i_2}^{\varepsilon_2} \ldots a_{i_k}^{\varepsilon_k} , \quad \varepsilon_l = \pm 1 ,$$

we construct a simple arc $a(w)$ in $\bigcup_i A_i$ by taking points $0 < P_1 < P_2 < \ldots < P_{k+1} < 1$ on $[0, 1]$ and connecting each P_l to P_{l+1} by the "geodesic" (in a natural sense) in A_{i_l} with orientation implied by ε_l. For example if

$$w = a_1 a_2 a_2^{-1} a_1 a_2 ,$$

then $a(w)$ will resemble the curve in the figure on page 135. (It is not unique because of the arbitrariness in the choice of $P_1, \ldots, P_{k+1}$; however, different $a(w)$'s will be isotopic - a fact which is exploited below.)

It is clear that any word w is representable by a simple arc in this way, and hence if we attach $[0, 1]$ to the top side of a square S which is otherwise disjoint from $\bigcup_i A_i$ we can close $a(w)$ to a simple curve $c(w)$ by running round the other three sides of the square. Furthermore, the fundamental group of $A = \bigcup_i A_i \cup S$ is the free group generated by $a_1, \ldots, a_n$, since there is a deformation retraction of A onto the bouquet of circles $\bigcup_i a_i$, and $c(w)$ represents the element w.

We now attach a disc D_j which will allow us to insert or remove a subarc $a(r_j)$ of a $c(w)$ by an isotopy. Namely, take any points Q, R with $0 < Q < R < 1$ and let $\bar{a}(r_j)$ be any fixed $a(r_j)$ which runs from Q to R. Then $b(r_j) = \bar{a}(r_j) \cup RQ$ is taken as the boundary of D_j. Notice that the simple arc $\bar{a}(r_j)$ may be deformed isotopically into the

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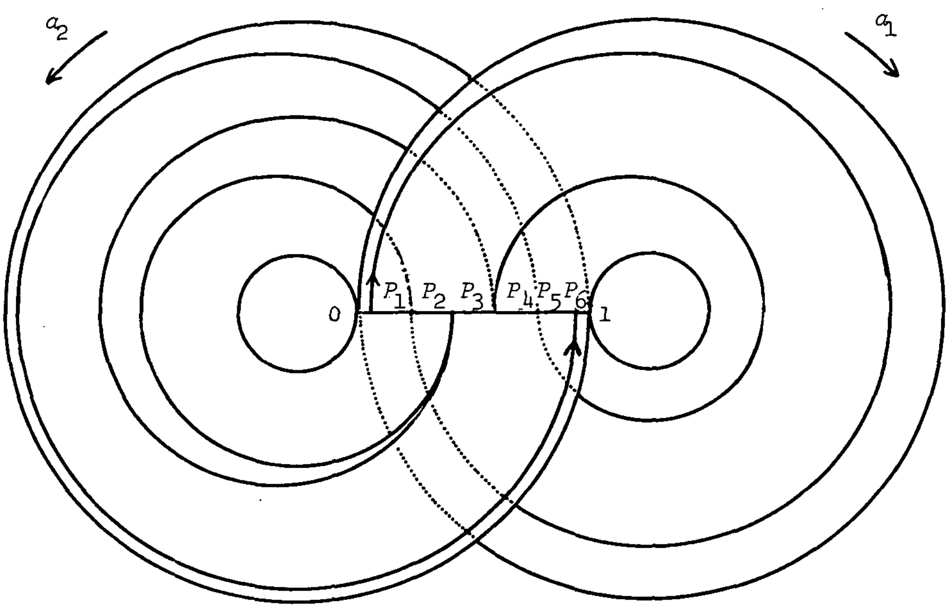

line segment QR by pulling it across D_j . We let $K(G) = A \cup \bigcup_j D_j$.

Then to remove a subarc $a(r_j)$ of $c(w)$ we first deform $c(w)$ isotopically so that $a(r_j)$ is carried onto $\bar{a}(r_j)$, then pull $\bar{a}(r_j)$ across D_j to the position QR . A further isotopy contracts QR to a point and gives a curve $c(w')$ where w' is the result of removing r_j from w . The reverse process simulates the insertion of r_j in w' to produce w . Insertion or removal of trivial relators $a_i a_i^{-1}$ or $a_i^{-1} a_i$ can obviously be accomplished by isotopies in A itself.

Since any word w which equals 1 in G can be converted to the

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empty word by a finite sequence of insertions or removals of relators, the corresponding curve $c(w)$ will be convertible to the boundary of the square S by a finite sequence of isotopies of the above type (which can be realized in a sufficiently fine simplicial decomposition of $K(G)$) and hence unknotted. On the other hand, it is clear from the Seifert-Van Kampen Theorem [5] that the fundamental group of $K(G)$ is precisely G ; hence when $w \neq 1$ in G the curve $c(w)$ will not even be homotopic, let alone isotopic, to the boundary of a disc. □

COROLLARY. *The set of pairs* (K, c) *for which* c *is a knotted polygonal curve in a finite 2-dimensional simplicial complex* K *is not recursively enumerable.*

Proof. If it were, the set $\{(K, c) \mid c \text{ is unknotted in } K\}$ would be recursive, and the construction of Theorem 1 would yield an algorithm for the word problem for groups. □

Another obvious corollary to this theorem is that the problem of deciding whether a polygonal curve in a surface complex is isotopic (in the general sense) to a point is unsolvable. Furthermore, we obtain unsolvability of both problems in a *fixed* $K(G)$ by choosing a G with unsolvable word problem. This shows that surface complexes constitute an exception to the remark of Haken [3] that isotopy problems are easier to solve than homotopy problems.

References

[1] J.W. Alexander, "Some problems in topology", *Verhandlungen des Internationalen Mathematiker-Kongresses*, Zürich, 1932. Band I. *Bericht und Allgemeine Vorträge*, 249-257 (Orell Füssli, Zürich, Leipzig, ND[1933]; Kraus Reprint Limited, Nendeln, 1967).

[2] M. Dehn, "Über die Topologie des dreidimensionalen Raumes", *Math. Ann.* **69** (1910), 137-168.

[3] Wolfgang Haken, "Connections between topological and group theoretical decision problems", *Word problems, decision problems and the Burnside problem in group theory*, 427-441 (Studies in Logic and the Foundations of Mathematics, . North-Holland, Amsterdam, London, 1973).

[4] Walter J. Savitch, "Relationship between nondeterministic and deterministic tape complexities", *J. Computer System Sci.* 4 (1970), 177-192.

[5] H. Seifert und W. Threlfall, *Lehrbuch der Topologie* (B.G. Teubner, Leipzig, Berlin, 1934. Reprinted Chelsea, New York, 1947).

[6] John C. Stillwell, "Isotopy in surface complexes from the computational viewpoint", *Bull. Austral. Math. Soc.* 20 (1979), 1-6.

[7] E.F. Whittlesey, "Classification of finite 2-complexes", *Proc. Amer. Math. Soc.* 9 (1958), 841-845.

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