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ON ASYMPTOTIC CENTERS AND FIXED POINTS OF NONEXPANSIVE MAPPINGS

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1. Introduction. Let X be a Banach space and B a bounded subset of X. For each $x \in X$, define $R(x) = \sup\{\|x - y\| : y \in B\}$. If C is a nonempty subset of X, we call the number $R = \inf\{R(x) : x \in C\}$ the *Chebyshev radius* of B in C and the set $\mathscr{C}(B, C) = \{x \in C : R(x) = R\}$ the *Chebyshev center* of B in C. It is well known that if C is weakly compact and convex, then $\mathscr{C}(B, C) \neq \emptyset$ and if, in addition, X is uniformly convex, then the Chebyshev center is unique; see e.g., [**9**].

Let $\{B_\alpha : \alpha \in \Lambda\}$ be a decreasing net of bounded subsets of X. For each $x \in X$ and each $\alpha \in \Lambda$, define

$$r_\alpha(x) = \sup\{\|x - y\| : y \in B_\alpha\} \quad \text{and}$$
$$r(x) = \lim_\alpha r_\alpha(x) = \inf_\alpha r_\alpha(x).$$

The number $ar(\{B_\alpha : \alpha \in \Lambda\}, C) = \inf\{r(x) : x \in C\} = r$ and the set $AC(\{B_\alpha : \alpha \in \Lambda\}, C) = \{x \in C : r(x) = r\}$ are called, respectively, the *asymptotic radius* and *asymptotic center* of $\{B_\alpha : \alpha \in \Lambda\}$ with respect to C. If $\{x_\beta : \beta \in A\}$ is a bounded net, the asymptotic center of $\{x_\beta : \beta \in A\}$ is defined to be the asymptotic center of the decreasing net of sets $\{B_\gamma : \gamma \in A\}$, where $B_\gamma = \{x_\beta : \beta \geqq \gamma, \beta \in A\}$ for each $\gamma \in A$.

The original notion of asymptotic center was introduced by M. Edelstein in [**7**] and had been used later in the study of fixed points of nonexpansive mappings; see e.g., [**7**], [**8**], [**16**], [**17**] and [**19**]. Our objective in this paper is twofold: (1) to present a set of elementary properties of asymptotic centers; (2) to apply these properties to obtain some results on fixed points of nonexpansive mappings.

The modulus of convexity [**4**] of X is the function

$$\delta(\epsilon) = \inf\{1 - \tfrac{1}{2}\|x + y\| : \|x\| \leqq 1, \|y\| \leqq 1, \|x - y\| \geqq \epsilon\}$$

defined for $0 \leqq \epsilon \leqq 2$. X is uniformly convex if and only if $\delta(\epsilon) > 0$ for $\epsilon > 0$.

LEMMA 1. ([**12**], [**3**]) *$\delta(\epsilon)$ is continuous on $[0, 2)$. For uniformly convex spaces, $\delta(\epsilon)$ is continuous on $[0, 2]$.*

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2. Basic properties of asymptotic centers. In this section, unless otherwise stated, X will denote a Banach space, $\{B_\alpha : \alpha \in \Lambda\}$ a decreasing net of bounded subsets of X, C a nonempty subset of X, and $r_\alpha(x)$, $r(x)$ and r will be defined as in Section 1.

The following lemma follows immediately from the definition.

LEMMA 2. *For $x, y \in X$ the inequalities $\|x\| \leqq r$, $\|y\| \leqq r$, and $\|x - y\| \geqq \epsilon$ imply $\|\frac{1}{2}(x + y)\| \leqq r(1 - \delta(\epsilon/r))$.*

LEMMA 3. (i) *If $r_\alpha(x) \leqq d$, $r_\alpha(y) \leqq d$, and $\|x - y\| \geqq \epsilon$ then*

$$r_\alpha((x + y)/2) \leqq d(1 - \delta(\epsilon/d)).$$

(ii) *If $r(x) \leqq d$, $r(y) \leqq d$ and $\|x - y\| \geqq \epsilon$, then*

$$r((x + y)/2) \leqq d(1 - \delta(\epsilon/d)).$$

Proof. (i). If $z \in B_\alpha$, then

$$\|x - z\| \leqq d, \|y - z\| \leqq d \text{ and } \|x - y\| \geqq \epsilon.$$

By Lemma 1, $\|(x + y)/2 - z\| \leqq d(1 - \delta(\epsilon/d))$. Hence

$$r_\alpha((x + y)/2) \leqq d(1 - \delta(\epsilon/d -)).$$

(ii). If $r_\alpha(x) \leqq d + \eta$, $r_\alpha(y) \leqq d + \eta$, then

$$r\left(\frac{x + y}{2}\right) \leqq r_\alpha\left(\frac{x + y}{2}\right) \leqq (d + \eta)\left(1 - \delta\left(\frac{\epsilon}{d + \eta}\right)\right).$$

Letting $\eta \to 0$, we obtain (ii).

LEMMA 4. (i) *$r(x)$ is convex and nonexpansive.*
(ii) $|r(x) - r(y)| \leqq \|x - y\| \leqq r(x) + r(y)$.

Proof. These follow from the corresponding properties for $r_\alpha(x)$.

PROPOSITION 1. *If C is a weakly compact convex subset of a Banach space or a closed convex subset of a reflexive Banach space, then the asymptotic center of $\{B_\alpha : \alpha \in \Lambda\}$ in C is nonempty, closed, convex and bounded. If, moreover, X is uniformly convex, the asymptotic center is unique.*

Proof. This follows from Lemma 4 and Lemma 3(ii); see also [**8**], [**16**].

THEOREM 1. *Let C be either* (i) *a closed convex nonempty subset of a uniformly convex Banach space X, or* (ii) *a compact convex nonempty subset of a Banach space X. Then the mapping $F(t) = \{x \in C : r(x) \leqq t\}$ for $t \geqq r$ is a continuous function, mapping $[r, \infty)$ into the family of nonempty closed convex subsets of C, equipped with the Hausdorff metric.*

Proof. By Theorem 1, $F(t)$ is nonempty for each $t \in [r, \infty)$. Let $\{t_i\}$ be a sequence in $[r, \infty)$ with limit t. Suppose that $F(t_i)$ does not converge to $F(t)$. It follows easily from the definition of the Hausdorff metric that there exists a monotonic subsequence of $\{t_i\}$ (which we still denote by $\{t_i\}$) such that either

(a) $\{t_i\}$ is decreasing and for each i, there exists $x_i \in F(t_i)$ with $\|x_i - y\| \geqq \epsilon$ for all $y \in F(t)$, or

(b) $\{t_i\}$ is increasing and for each i, there exists $x_i \in F(t)$ with $\|x_i - y\| \geqq \epsilon$ for all $y \in F(t_i)$.

In the former case, choose $y_i \in F(t)$ such that $\frac{1}{2}(x_i + y_i) \notin F(t)$ and in the latter case, choose $y_i \in F(t_i)$ such that $\frac{1}{2}(x_i + y_i) \notin F(t_i)$. Assume (i) of Theorem 1. Let $\lambda_i = \max(t_i, t)$. We have

$$r((x_i + y_i)/2) \leqq \lambda_i(1 - \delta(\epsilon/\lambda_i)).$$

Taking lim sup, we have

$$\limsup r((x_i + y_i)/2) \leqq t(1 - \delta(\epsilon/t)) < t.$$

Since $\lim t_i = t$,

$$r((x_i + y_i)/2) \leqq \min(t_i, t)$$

for some i. In either case, this a contradiction. Hence $F(t_i) \to F(t)$.

Now assume (ii) of Theorem 1. In case (a), taking a convergent subsequence of x_i yields a point $x \in F(t)$ such that $\|x - y\| \geqq \epsilon$ for all $y \in F(t)$, which is absurd. In case (b), we can extract a subsequence of $\{t_i\}$, which we again denote by $\{t_i\}$ such that $x_i \to x \in F(t)$ and $\|x - y\| \geqq \epsilon/2$ for all $y \in F(t_i)$. For each $z \in F(t) \cap B(x, \epsilon/4)$ we have $t_i < r(z) \leqq t$ and hence $r(z) = t$. Choose $w \in F(t_i)$. Since $t_i < t$, we have, for $0 < \lambda < 1$,

$$r(\lambda x + (1 - \lambda)w) \leqq \lambda r(x) + (1 - \lambda)r(w) \leqq \lambda t(1 - \lambda)t_i < t.$$

This contradicts the fact that the open segment $\{\lambda x + (1 - \lambda)w : 0 < \lambda < 1\}$ intersects $F(t) \cap B(x, \epsilon/4)$ in a nonempty set each of whose elements z satisfies $r(z) = t$.

COROLLARY 1. *Let C be a closed convex subset of a uniformly convex Banach space. Let $\{u_\beta : \beta \in \Lambda\}$ be a net in C. Then $r(u_\beta) \to r$ if and only if $u_\beta \to u$, where u is the unique asymptotic center of $\{B_\alpha : \alpha \in \Lambda\}$ in C.*

Proof. If $u_\beta \to u$, then $r(u_\beta) \to r(u) = r$ by the continuity of r. If $r(u_\beta) \to r$, then $u_\beta \to u$ by the continuity of F defined as in Theorem 1.

COROLLARY 2. *Let C be defined as in Corollary 1. For each $\alpha \in \Lambda$, let u_α be the unique Chebyshev center of B_α in C. Then $u_\alpha \to u$, where u is the asymptotic center of $\{B_\alpha : \alpha \in \Lambda\}$ in C.*

Proof. We have $r(u_\alpha) \leqq r_\alpha(u_\alpha) \leqq r_\alpha(u)$. Therefore, $\limsup r(u_\alpha) \leqq r(u) = r$. Since $r(u_\alpha) \geqq r$ for every $\alpha \in \Lambda$, we have $\liminf r(u_\alpha) \geqq r$. Hence $\lim r(u_\alpha) = r$.

We shall need the following result in the proof of Theorem 5. $B(x, r)$, ($B[x, r]$) will denote the open (closed) ball with center x and radius r; by S' we denote the complement of a subset S in X.

PROPOSITION 2. *Suppose that B_α is relatively compact for some $\alpha \in \Lambda$. Then the asymptotic center B_α (w.r.t. an arbitrary fixed nonempty subset C of X) is identical with the Chebyshev center of $\cap \{\bar{B}_\alpha : \alpha \in \Lambda\}$ (w.r.t. C).*

Proof. Write $B = \cap \{\bar{B}_\alpha : \alpha \in \Lambda\}$. It suffices to show that $R(x) = r(x)$ for every $x \in X$, where $r(x)$ and $R(x)$ are defined as in Section 1. Clearly $R(x) \leqq r_\alpha(x)$ for all $\alpha \in \Lambda$ and hence $R(x) \leqq r(x)$. Let $\epsilon > 0$. If for each $\alpha \in \Lambda$, $B[x, r_\alpha(x)]$ intersects $B[x, R(x) + \epsilon]'$, then $\bar{B}_\alpha$ intersects $B(x, R(x) + \epsilon/2)'$. Therefore, by compactness, B intersects $B(x, R(x) + \epsilon/2)'$, a contradiction. Hence $B[x, r_\alpha(x)] \subset B[x, R(x) + \epsilon]$ for some $\alpha \in \Lambda$; i.e.,

$$r(x) < r_\alpha(x) \leqq R(x) + \epsilon.$$

Consequently, $r(x) = R(x)$.

COROLLARY 3. *Let K be a compact subset of a Banach space X and $C \subseteq X$. Let $\{x_n\}$ be a dense countable subset of K. Then the asymptotic center of $\{x_n\}$ w.r.t. C is identical with the Chebyshev center of K w.r.t. C.*

Proof. The closure of each $B_n = \{x_i : i \geqq n\}$ is identical with K.

THEOREM 2 (Klee [**15**]). *For a Banach space X the following are equivalent.*

(i) *X is a Hilbert space or is two-dimensional;*

(ii) *for every bounded closed convex subset C of X, the Chebyshev center of C w.r.t. X intersects C;*

(iii) *for every compact convex subset C of X, the Chebyshev center of C w.r.t. X intersects C.*

We prove the following similar theorem for asymptotic centers.

THEOREM 3. *For a Banach space X, the following are equivalent.*

(i) *X is a Hilbert space or is two-dimensional;*

(ii) *for every decreasing net $\{B_\alpha : \alpha \in \Lambda\}$ of nonempty bounded subsets of X, the asymptotic center of $\{B_\alpha : \alpha \in \Lambda\}$ in X intersects*

$$\cap \overline{\mathrm{Co}}(B_\alpha : \alpha \in \Lambda);$$

(iii) *for every bounded sequence $\{u_n\}$ in X, the asymptotic center of $\{u_n\}$ in X intersects $\cap \overline{\mathrm{Co}}(u_n : n \geqq k)$;*

(iv) *for every bounded sequence $\{u_n\}$ in X, the asymptotic center of $\{u_n\}$ in X intersects $\overline{\mathrm{Co}}(u_n)_{n \geqq 1}$.*

Proof. (i) $\Rightarrow$ (ii). Suppose first that X is a Hilbert space. By Corollary 2, the asymptotic center c of $\{B_\alpha : \alpha \in \Lambda\}$ w.r.t. X is the limit of the net

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$\{u_\alpha : \alpha \in \Lambda\}$, where u_α is the unique Chebyshev center of B_α in X. Since $\overline{\text{Co}}(B_\alpha)$ and B_α have the same Chebyshev center in X, $u_\alpha \in \overline{\text{Co}}(B_\alpha)$ for each $\alpha \in \Lambda$, by Theorem 2. Since $u_\lambda \in \overline{\text{Co}}(B_\lambda) \subseteq \overline{\text{Co}}(B_\beta)$ if $\lambda \geqq \beta$, we must have $c \in \bigcap \{\overline{\text{Co}}(B_\alpha) : \alpha \in \Lambda\}$. Next, suppose that X is two-dimensional. By Proposition 2, $AC(\{B_\alpha : \alpha \in \Lambda\}, X) = \mathscr{C}(B, X)$, where $B = \bigcap \{\bar{B}_\alpha : \alpha \in \Lambda\}$. By Theorem 2, $\mathscr{C}(B, X) \cap \overline{\text{Co}}(B) \neq \emptyset$ and hence $\mathscr{C}(B, X) \cap \bigcap \overline{\text{Co}}(B_\alpha : \alpha \in \Lambda) \neq \emptyset$.

That (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is obvious.

To prove (iv) $\Rightarrow$ (i), let C be a compact convex subset of X. Since C is separable, there exists a sequence $\{u_n\}$ in C which is dense in C. By Corollary 3, $AC(\{u_n\}, X) = \mathscr{C}(C, X)$ and by hypothesis,

$$\mathscr{C}(C, X) \cap C = \mathscr{C}(C, X) \cap \overline{\text{Co}}(u_n)_{n \geqq 1} \neq \emptyset.$$

It follows from Theorem 2 that X is a Hilbert space or is two-dimensional.

The next theorem, whose counterpart for Chebyshev centers was proven by Garkavi [**10**], exhibits another similarity between Chebyshev centers and asymptotic centers. However, there are differences between them in some other aspects. For instance, while Chebyshev centers, being intersections of balls, are always nonvoid in conjugate Banach spaces, there exist such spaces in which asymptotic centers of some bounded sequences are void. See [**1**].

THEOREM 4. *If a Banach space X is uniformly convex in every direction, then the asymptotic center of $\{B_\alpha : \alpha \in \Lambda\}$ w.r.t. C consists of at most one point, where $\{B_\alpha : \alpha \in \Lambda\}$ is defined as in Section* 1 *and C is convex. On the other hand, if for every bounded sequence $\{x_n\}$ in X, the asymptotic center of $\{x_n\}$ w.r.t. X consists of at most one point, then X is uniformly convex in every direction.*

Proof. The proof is similar to that of Theorem VI in [**10**] and is thus omitted.

3. Nonexpansive self-mappings. For the definition of the terminology used in the next theorem, we refer to [**13**], [**16**].

THEOREM 5. *Let C be defined as in Corollary* 1 *and let S be a left reversible topological semigroup of nonexpansive actions on C such that $(s, x) \rightarrow s(x)$ is separately continuous. Let $x \in C$. For each $s \in S$, let u_s be the Chebyshev center of $sS(x)$ in C. Then u_s converges to a common fixed point of S, which is the asymptotic center of $\{\overline{sS(x)} : s \in S\}$. Here S is ordered by putting $a \geqq b$ if $aS \subseteq \overline{bS}$.*

Proof. This follows from Corollary 2 and the fact that the asymptotic center of $\{\overline{sS(x)} : s \in S\}$ is a common fixed point of S (see [**16**]).

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COROLLARY 4. *Let C be defined as in Corollary 1 and let $\mathcal{T}$ be a commutative semigroup of nonexpansive self-mappings of C. Let $x \in C$. For each $f \in \mathcal{T}$, let u_f be the unique Chebyshev center of $f\mathcal{T}(x)$ in C. Then u_f converges to a common fixed point of $\mathcal{T}$ which is the unique asymptotic center of $\{f\mathcal{T}(x) : f \in \mathcal{T}\}$ in C. Here $f \geqq g$ if and only if $f\mathcal{T} \subseteq g\mathcal{T}$.*

Proof. This follows from Corollary 3 by considering $\mathcal{T}$ as a topological semigroup with the discrete topology.

COROLLARY 5. (Edelstein [**7**]). *Let C be defined as in Corollary 1 and let T be a nonexpansive self-mapping of C. Let $x \in C$. For each n, let u_n be the Chebyshev center of $\{T^n x, T^{n+1}x, \ldots\}$ in C. Then u_n converges to a fixed point of T, which is the asymptotic center of u_n in C.*

Proof. Let $\mathcal{T} = \{T^n : n = 0, 1, 2, \ldots\}$ in Corollary 4.

LEMMA 5. *In Corollary 4, if $I_C \in \mathcal{T}$, then $f \geqq g$ if and only if $f \in g\mathcal{T}$ and the asymptotic center of $\{f\mathcal{T}(x) : f \in \mathcal{T}\}$ is the same as the asymptotic center of $\{f(x) : f \in \mathcal{T}\}$.*

THEOREM 6. *Let C be a closed convex subset of a uniformly convex Banach space X. Let $\mathcal{T}$ be a commutative semigroup of nonexpansive self-mappings of C. Assume that $I_C \in \mathcal{T}$. Order $\mathcal{T}$ by putting $f \geqq g$ if and only if $f \in g\mathcal{T}$. Suppose that the common fixed point set of $\mathcal{T}$ is nonempty. For each $x \in C$, let Px be the asymptotic center of $\{f(x) : f \in \mathcal{T}\}$. Then P is a retraction from C onto the common fixed point set $F(\mathcal{T})$ of $\mathcal{T}$. Moreover, $fP = Pf = P^2 = P$ and $Px = \lim_f Nfx$ for every $x \in C$, where $N : C \rightarrow F(\mathcal{T})$ assigns to each $x \in C$ the unique point Nx in $F(\mathcal{T})$ which is nearest to x. If X is a Hilbert space, P is nonexpansive.*

Proof. Since X is strictly convex and each $f \in \mathcal{T}$ is continuous, it is well known that $F(\mathcal{T})$ is closed and convex. Let $x \in C$. For each $f \in \mathcal{T}$, let $u_f = N(f(x))$. Let $c = Px$. We have

$$(1) \quad \begin{aligned} &r(u_f) \geqq r(c) = r \quad \text{and} \\ &\lim_f \inf r(u_f) \geqq r. \end{aligned}$$

If $g \geqq f_0$, then $g = f_0 f_1$ for some $f_1 \in \mathcal{T}$ and

$$\begin{aligned} \|u_{f_0} - g(x)\| = \|u_{f_0} - f_0 f_1(x)\| &= \|f_1(u_{f_0}) - f_1 f_0(x)\| \\ &\leqq \|u_{f_0} - f_0(x)\| \\ &\leqq \|c - f_0(x)\|. \end{aligned}$$

The last inequality follows from the definition of u_f and the fact that c is a common fixed point of $\mathcal{T}$. Hence for all $f_0 \in \mathcal{T}$,

$$r(u_{f_0}) = \lim_f \sup \|u_{f_0} - f(x)\| \leqq \|c - f_0(x)\|,$$

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and hence

$$\lim_f \sup r(u_f) \leqq \lim_f \sup \|c - f(x)\| = r.$$

Therefore, by (1), $\lim r(u_f) = r$. By Corollary 1, $u_f = N(f(x)) \to c = Px$. Obviously, $fP = P = P^2$. $Pfx = Px$ for every x since the asymptotic center of $\{g(x) : g \geqq f\}$ is the same as that of $\{f(x) : f \in \mathscr{T}\}$. If X is a Hilbert space, it is well known that N is nonexpansive. Therefore, being a pointwise limit of a net of nonexpansive mappings Nf, P is nonexpansive, where $f \in \mathscr{T}$.

COROLLARY 6. *Let C be defined as in Theorem 6 and $T : C \to C$ a nonexpansive self-mapping of C. If the fixed point set $F(T)$ of T is nonempty, then the mapping P which assigns to each $x \in C$ the unique asymptotic center of the sequence $\{x, Tx, T^2x, \ldots\}$ is a retraction from C onto $F(T)$. Moreover, $T^nP = PT^n = P^2 = P$ for all n and $Px = \lim_n NT^nx$ for every $x \in C$, where N is the nearest point mapping from C onto $F(T)$. If X is a Hilbert space, P is nonexpansive.*

4. Nonexpansive mappings satisfying inwardness conditions. Let C be a closed convex subset of a Banach space X and $x \in C$. Define the *inward set* of x relative to C by

$$I_C(x) = \{(1 - \alpha)x + \alpha y : y \in C, \alpha \geqq 0\}.$$

A mapping $T : C \to X$ is said to be *inward* (resp. *weakly inward*) if $Tx \in I_C(x)$ (resp. $Tx \in \overline{I_C(x)}$), for every $x \in C$. For a historical account of fixed point theorems for inward mappings see, e.g., [**6**].

PROPOSITION 3. *Let C be a closed convex subset of a uniformly convex Banach space X and $\{x_n\}$ a bounded sequence in C. If x is the asymptotic center of the sequence $\{x_n\}$ w.r.t. C, then it is also the asymptotic center w.r.t. $\overline{I_C(x)}$.*

Proof. Let y be the asymptotic center of x_n w.r.t. $\overline{I_C(x)}$. Assume that $y \neq x$. Since $\overline{I_C(x)} \supseteq C$, we have $y \in \overline{I_C(x)}\backslash C$ and $r(y) < r(x)$ by the uniqueness of asymptotic center. By the continuity of $r(\cdot)$, there exists $z \in I_C(x)\backslash C$ such that $r(z) < r(x)$. Thus $z = (1 - \alpha)(x + \alpha w)$ for some $w \in C$ and $\alpha > 1$ and

$$r(w) = r\left(\frac{1}{\alpha}z + \left(1 - \frac{1}{\alpha}\right)x\right) \leqq \frac{1}{\alpha}r(z) + \left(1 - \frac{1}{\alpha}\right)r(x) < r(x).$$

This contradicts the definition of x. Hence $y = x$.

THEOREM 7. *Let C and X be defined as in Proposition 3 with C bounded, and $T : C \to X$ be a nonexpansive weakly inward mapping. Let $x_0 \in C$. For each n, let x_n be the unique fixed point of the mapping T_n defined by*

$$T_nx = (1 - \alpha_n)x_0 + \alpha_nTx,$$

where $0 < \alpha_n < 1$ *and* $\lim \alpha_n = 1$. *Then the asymptotic center of* x_n *w.r.t.* C *is a fixed point of* T.

Proof. Each T_n is clearly a contraction with Lipschitz constant $\alpha_n < 1$. It is well known that each T_n has a unique fixed point; see, e.g., [**20**], [**5**]. Moreover, $\|x_n - Tx_n\| \to 0$. If x is an asymptotic center of x_n w.r.t. C, then

$$\begin{aligned} r(Tx) &= \limsup \|Tx - x_n\| \\ &= \limsup \|Tx - Tx_n\| \\ &\leqq \limsup \|x - x_n\| = r(x). \end{aligned}$$

Since $Tx \in I_C(x)$ and x is the asymptotic center w.r.t. $I_C(x)$ by Proposition 3, we must have $Tx = x$ by the uniqueness of asymptotic centers.

The following result improves a theorem of ([**6**], Theorem 4).

THEOREM 8. *Let* X *and* C *be defined as in Proposition* 3 *with* C *bounded and* $T : C \to K(X)$ *a nonexpansive mapping such that* $Tx \subseteq \overline{I_C(x)}$ *for every* $x \in C$, *where* $K(X)$ *denotes the family of nonempty compact subsets of* X, *equipped with the Hausdorff metric. Then there exists* $x \in C$ *such that* $x \in Tx$.

Proof. Let $x_0 \in C$. Define T_n as in the previous theorem except that in this case, T_n is multivalued. By a theorem of ([**6**], Corollary 2), T_n has a fixed point x_n; i.e., $x_n \in T_n(x_n)$. Arguing as in the proof of Theorem 1 in [**17**] (see also [**11**]), there exists a subsequence, which we again denote by $\{x_n\}$ of $\{x_n\}$ such that every subsequence of it has a common asymptotic radius equal to $\lim \|x - x_n\|$. Moreover, there exists $y \in Tx$ such that $\limsup \|y - x_{n_i}\| \leqq \lim \|x - x_n\|$ for some subsequence x_{n_i} of $\{x_n\}$. Since x is also the asymptotic center of x_n w.r.t. $\overline{I_C(x)}$ by Proposition 3 and $y \in \overline{I_C(x)}$, we have $y = x$ by the uniqueness of the asymptotic center. Hence $x \in Tx$.

5. Nonexpansive mappings with weakly compact attractor. Let H be a closed convex subset of a Banach space and let K be a weakly compact subset of H. Let $T : H \to H$ be a nonexpansive mapping. We call K an *attractor* of T if for every $x \in H$, $\overline{\text{Co}}\{x, Tx, \ldots, T^n x, \ldots\} \cap K \neq \emptyset$. This condition is implied by the following:

$$\{x, Tx, \ldots, T^n x, \ldots\}^- \cap K \neq \emptyset,$$

a condition considered in [**2**].

In [**14**], Kirk proved the following.

THEOREM 9. *Let* H *be a closed convex bounded subset of a Banach space* X *and let* $T : H \to H$ *be a nonexpansive mapping. If* T *has a weakly compact attractor, then* T *has a fixed point.*

We take this opportunity to prove the following more general result.

THEOREM 10. *Let H be a closed convex bounded subset of a Banach space and let K be a weakly compact subset of H. Let S be a left reversible topological semigroup of nonexpansive self-mappings of H satisfying the following conditions;*

(i) *for each $a \in H$, the mapping from S into H defined by $s \to s(a)$ is continuous;*

(ii) *for each $x \in H$ and $s \in S$, $[\overline{\mathrm{Co}}\, sS(x)] \cap K \neq \emptyset$. Then S has a common fixed point.*

Proof. Let M be a subset of H minimal with respect to being nonempty, closed, convex, mapped into itself by each member of X, and having a nonempty intersection with K. M exists by the weakly compactness of K and by a standard argument using Zorn's lemma applying the fact that closed convex sets are weakly closed. Let $x_0 \in M$. Direct S by an ordering $\geqq$ and define

$$\mathscr{W}(x_0) = \{W_s = \overline{sS(x_0)} : s \in S\}$$

as in Theorem 5. Suppose that M consists of more than one point. Let $r = ar(\mathscr{W}(x_0), C)$. Since H possesses normal structure, by Theorem 1 in [**16**], there exists $x \in M$ such that $r(x) = r_1 > r$. Here $r(\cdot)$ is understood to be relative to the decreasing net $\mathscr{W}(x_0)$. If r_2 is such that $r_1 > r_2 > r$, then the set $C = \{x \in M : r(x) \leqq r_2\}$ is a nonempty proper closed convex subset of M. Let $s_1 \in S$, $x \in C$ and $\epsilon > 0$. There exists $t \in S$ such that $tS(x) \subseteq W_t \subseteq B_{r_2+\epsilon}(x)$. Since s_1 is nonexpansive, we have $s_1tS(x) \subseteq B_{r_2+\epsilon}(s_1(x))$ so that

$$W_{s_1 t} \subseteq B_{r_2+\epsilon}(s_1(x)).$$

Since $\{W_s : s \in S\}$ is decreasing,

$$W_s \subseteq B_{r_2+\epsilon}(s_1(x)) \quad \text{for all} \quad s \geqq s_1t.$$

Hence $r(s_1(x)) \leqq r_2 + \epsilon$ and $s_1(x) \in C$ since $\epsilon > 0$ is arbitrary. Therefore, C is invariant under S. Let $x \in C$. Since C is closed convex and S-invariant, we have $\overline{\mathrm{Co}}\, sS(x) \subseteq C$. Therefore $C \cap K \neq \emptyset$ by condition (ii). This contradicts the minimality of M.

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