ON PLANAR CONTINUOUS FAMILIES OF CURVES

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1. Introduction. In a recent paper (**3**), Grünbaum has found a general and unifying setting for a number of properties of some special lines associated with a planar convex body. Besides various interesting results, two conjectures are stated and two kinds of convexity and polygonal connectedness are introduced.

In the present paper, we shall prove one of Grünbaum's conjectures (§ 3, Theorem 1); we consider the other in § 4 and establish some related results in §§ 5 and 6. Six-partite problems are studied in this general setting (§ 7) and a question raised by Ceder (**2**) is answered. We give a generalization of the notion of a continuous family of curves in § 8, and discuss some new kinds of connectedness in § 9.

2. Definitions and notation. Throughout the paper, C will be a simple closed curve (topological image of a circle) in the plane and D the bounded domain with boundary C. Following (**3**), a family $\mathfrak{L}$ of simple arcs (topological images of a segment) is called a continuous family of curves in D provided:

(i) Each curve of $\mathfrak{L}$ is contained in D, except for the extremities which belong to C;

(ii) Each point $p \in C$ is an endpoint of one and only one curve $L(p) \in \mathfrak{L}$ (throughout, $L(p)$ denotes the curve in $\mathfrak{L}$ with an endpoint at $p \in C$);

(iii) The curve $L(p)$ depends continuously on $p \in C$;

(iv) If $L_1, L_2 \in \mathfrak{L}$ and $L_1 \neq L_2$, then $L_1 \cap L_2$ is a single point.

Let us remark that from axioms (i)–(iii) it follows that for different $L_1, L_2 \in \mathfrak{L}$, the endpoints of L_1 separate on C those of L_2. Consequently, $L_1 \cap L_2 \neq \emptyset$, and axiom (iv) only points out that $\operatorname{card}(L_1 \cap L_2) = 1$. In §§ 8 and 9 we shall use the term "generalized continuous family of curves" for a continuous family of curves, in the preceding sense, satisfying the following weaker condition instead of axiom (iv):

(iv′) If $L_1, L_2 \in \mathfrak{L}$, then $L_1 \cap L_2$ is connected.

The set of all points in D that belong to at least q different curves in $\mathfrak{L}$ is denoted by $M_q(\mathfrak{L})$. The set of those points each of which belongs to all the curves in $\mathfrak{L}$ having endpoints in some non-degenerate arc of C is denoted by $M_\infty(\mathfrak{L})$. Obviously,

$$D = M_1(\mathfrak{L}) \supset M_2(\mathfrak{L}) \supset \ldots \supset M_q(\mathfrak{L}) \supset \ldots \supset M_\infty(\mathfrak{L}).$$

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An arc in D consisting of n arcs of curves in $\mathfrak{L}$ is called an n-curve. By a polygon we mean a connected open set in D bounded by finitely many arcs on C or on curves in $\mathfrak{L}$, its intersections with these curves being empty. An m-gon is a polygon bounded by m arcs of curves in $\mathfrak{L}$ or of C, but not also by some $m - 1$ arcs. For example, the bounded component of the complement of three non-concurrent curves in $\mathfrak{L}$ is called a triangle (3-gon).

In the following we shall also use the following notation:

(1) $-p$ for the other endpoint of $L(p)$;

(2) xy for the arc of a curve in $\mathfrak{L}$, joining x and y, or, if $x, y \in C$, for one of the arcs of C with endpoints x and y;

(3) $(a_0, \ldots, a_n)$ for the n-curve consisting of arcs $a_0a_1, \ldots, a_{n-1}a_n$ of curves in $\mathfrak{L}$;

(4) $[b_0, \ldots, b_m]$ for the $(m + 1)$-polygon bounded by arcs $b_0b_1, \ldots, b_{m-1}b_m, b_mb_0$ of curves in $\mathfrak{L}$ or of C;

(5) $\overline{xy}$ for the line through x and y.

3. A conjecture of Grünbaum.

The following theorem improves a result and proves a conjecture in (**3**). It involves the notion of an $L_2(\mathfrak{L})$-set: P is an $L_2(\mathfrak{L})$-set provided that for each pair $x, y \in P$ there is a 2-curve connecting x and y, and included in P (see also § 9).

THEOREM 1. *Let $\mathfrak{L}$ be a continuous family of curves. Then $M_3(\mathfrak{L})$ is an $L_2(\mathfrak{L})$-set.*

Proof. Let us consider the point $x \in M_3(\mathfrak{L})$ and the curves $L(p_1)$, $L(p_2)$, and $L(p_3)$ which pass through x. We may suppose, without loss of generality, that

(1) the order of the endpoints of $L(p_1)$, $L(p_2)$, $L(p_3)$ on C is the following: $p_1, p_2, p_3, -p_1, -p_2, -p_3$, and

(2) another arbitrary point $y \in M_3(\mathfrak{L})$ lies in the triangle $[p_1, x, p_2]$ (see the notation (4) in § 2) or on its sides p_1x or p_2x.

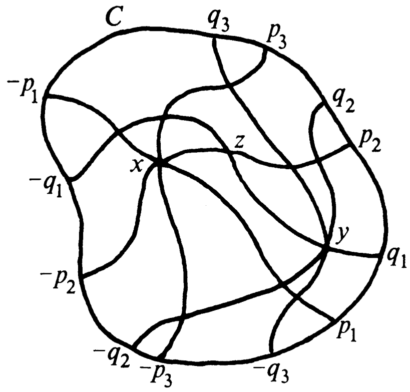
Let $L(q_1)$, $L(q_2)$, and $L(q_3)$ be three (other) curves in $\mathfrak{L}$ containing $y \in M_3(\mathfrak{L})$. Now one has to distinguish between three essentially different cases.

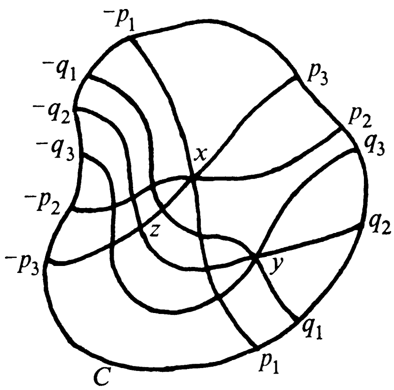
The order of endpoints of the six curves considered in $\mathfrak{L}$ on the arc $-p_1p_1$ of C containing p_2 is the following:

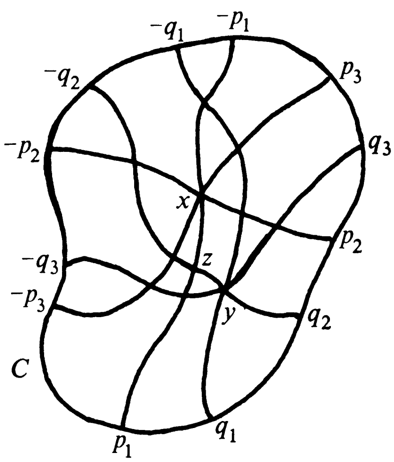
(a) $p_1, q_1, p_2, q_2, p_3, q_3$. Using the Jordan curve theorem, $L(q_1)$ must intersect either p_1x or p_2x at a point z situated between x and either $L(p_1) \cap L(q_2)$ or $L(p_2) \cap L(q_3)$. The arcs yz (if not degenerate) of $L(q_1)$ and zx (if not degenerate) of $L(p_1)$ or $L(p_2)$ are included in $\{x, y\}$ added to the triangle whose boundary consists of arcs of $L(q_2)$, $L(p_3)$, $L(q_3)$. Hence, the union T of all triangles with arcs of curves in $\mathfrak{L}$ as sides contains $(xz \cup zy) - \{x, y\}$.

(b) $p_1, q_1, q_2, q_3, p_2, p_3$. At least one of the arcs p_3x and $-p_3x$ of $L(p_3)$ contains at least two points of $L(q_1) \cup L(q_2) \cup L(q_3)$. We may suppose, without loss of generality, that

$$L(p_3) \cap (L(q_2) \cup L(q_3)) \subset -p_3x.$$







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Setting $\{z\} = L(q_2) \cap L(p_3)$, the triangle formed by $L(q_1)$, $L(p_2)$, $L(q_3)$ contains $yz - \{y\}$ and the triangle formed by $L(p_1)$, $L(p_2)$, $L(q_3)$ contains $zx - \{x\}$ (if $x \neq z$). It follows that

$$xz \cup zy \subset T \cup \{x, y\}.$$

(c) $p_1, q_1, q_2, p_2, q_3, p_3$. In this case, let $\{z\} = L(q_2) \cap (p_1x \cup p_2x)$. Equalities $x = z$ or $y = z$ are possible. The arcs yz of $L(q_2)$ and zx of $L(p_1)$ or $L(p_2)$ are contained (except for x and y) in the triangle formed by $L(q_1)$, $L(p_3)$, $L(q_3)$. Therefore,

$$xz \cup zy \subset T \cup \{x, y\}.$$

Now, by (**3**, Lemma 2), $T \subset M_3(\mathfrak{L})$; it follows that in every case $xz \cup zy$ is a 2-curve completely included in $M_3(\mathfrak{L})$. Hence, $M_3(\mathfrak{L})$ is an $L_2(\mathfrak{L})$-set.

4. Topological properties of $M_n(\mathfrak{L})$ for even and odd n. Let $\mathfrak{L}$ be a continuous family of curves.

THEOREM 2. *Let n be an even number.*
(1) $\operatorname{int}(M_n(\mathfrak{L}) - M_{n+1}(\mathfrak{L})) = \emptyset$,
(2) *if* $M_n(\mathfrak{L}) - M_\infty(\mathfrak{L}) \neq \emptyset$, *then* $\operatorname{int} M_{n+1}(\mathfrak{L}) \neq \emptyset$.

Proof. Let

$$x \in M_n(\mathfrak{L}) - M_\infty(\mathfrak{L}).$$

There exist at least n curves $L(p_0), \ldots, L(p_{n-1}) \in \mathfrak{L}$ that pass through x. If $p_n = -p_0$, then we may consider, without loss of generality, that $p_0, \ldots, p_n$ lie in this order on C. Let us consider the points q_i in the arcs p_ip_{i+1} with disjoint interiors of C ($i = 0, 2, 3, \ldots, n-2$), such that the triangles T_i and T_i' formed by $L(q_i)$, $L(p_1)$, $L(p_2)$, and $L(p_0)$, $L(q_i)$, $L(p_{n-1})$, respectively, do not degenerate; this is possible since $x \notin M_\infty(\mathfrak{L})$. It is easy to verify that either $T_0 \cap T_i$ or $T_0' \cap T_i'$ is not void; denote it by U_i ($i = 2, \ldots, n-2$). Then x is a boundary point of each U_i. There must exist $\frac{1}{2}n - 1$ polygons U_i with non-void intersection since

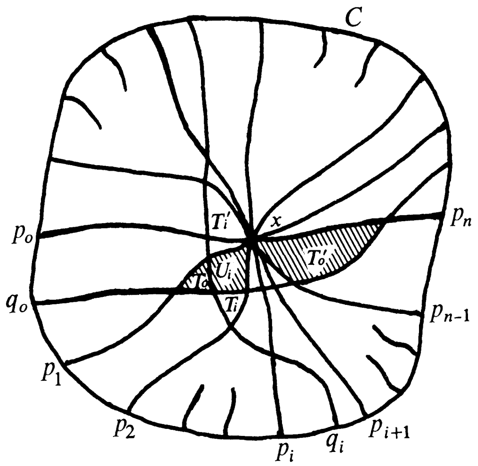
$$\{U_i\colon U_i \subset T_0\} \neq \emptyset \text{ implies } \{\cap U_i\colon U_i \subset T_0\} \neq \emptyset,$$

$$\{U_i\colon U_i \subset T_0'\} \neq \emptyset \text{ implies } \{\cap U_i\colon U_i \subset T_0'\} \neq \emptyset,$$

and if $\{U_i\colon U_i \subset T_0\}$ and $\{U_i\colon U_i \subset T_0'\}$ have cardinality at most $\frac{1}{2}n - 2$, then $\operatorname{card}\{U_i\} \leqq n - 4$, which is absurd. Let U_{ij} be these polygons with non-void intersection ($j = 1, \ldots, \frac{1}{2}n - 1$). Their union is evidently either included in T_0 or in T_0', and x is a boundary point of

$$V = \bigcap_{j=1}^{\frac{1}{2}n-1} U_{ij}.$$

Through each point of $T_i \cup T_i'$ pass two curves of $\mathfrak{L}$, one with an endpoint in the arc p_iq_i and the other with an endpoint in the arc q_ip_{i+1}. It follows



that each point of U_i ($i = 2, \ldots, n-2$) belongs to four curves of $\mathfrak{K}$ with endpoints on the arcs p_0q_0, q_0p_1, p_iq_i, q_ip_{i+1}. Finally, each point of V lies on p_0q_0, q_0p_1, $p_{i_1}q_{i_1}$, $q_{i_1}p_{i_1+1}, \ldots, p_{i_{\frac{1}{2}n-1}}q_{i_{\frac{1}{2}n-1}}$, $q_{i_{\frac{1}{2}n-1}}p_{i_{\frac{1}{2}n-1}+1}$, and on p_1p_2 if $V \subset T_0$ or on $p_{n-1}p_n$ if $V \subset T_0'$. Thus, $V \subset M_{n+1}(\mathfrak{K})$. Since V is open, (2) is proved.

For each point $x \in M_n(\mathfrak{K}) - M_\infty(\mathfrak{K})$ (n even) we found an open set $M_{n+1}(x)$ included in $M_{n+1}(\mathfrak{K})$, such that $x \in \text{bd}\, M_{n+1}(x)$. Therefore, $x \in \overline{\text{int}\, M_{n+1}(\mathfrak{K})}$, and

$$M_n(\mathfrak{K}) - M_\infty(\mathfrak{K}) \subset \overline{\text{int}\, M_{n+1}(\mathfrak{K})};$$

whence,

$$M_n(\mathfrak{K}) - M_{n+1}(\mathfrak{K}) \subset \text{bd int}\, M_{n+1}(\mathfrak{K}),$$

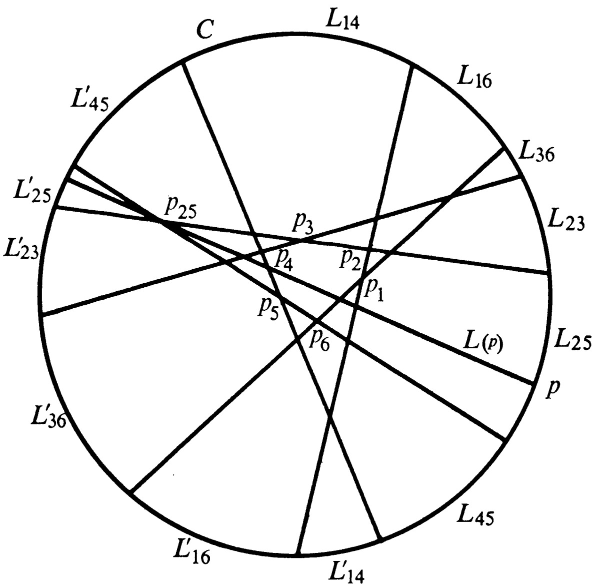
which proves part (1) of the theorem.

5. Condition implying non-empty $M_5(\mathfrak{K})$. When a non-degenerate triangle with sides on curves in $\mathfrak{K}$ exists, $\text{int}\, M_3(\mathfrak{K}) \neq \emptyset$. We seek a similar condition implying $\text{int}\, M_m(\mathfrak{K}) \neq \emptyset$ where, of course, $m > 3$ is odd. Now, let $m = 5$.

In this section we shall answer the following natural question: Which is the smallest number n such that the existence of an n-gon implies $\text{int}\, M_5(\mathfrak{K}) \neq \emptyset$? We are now concerned only with polygons having arcs of curves in $\mathfrak{K}$ as sides.

LEMMA 1. *The existence of a hexagon* (*a* 6-*gon*) *does not imply* $\text{int}\, M_5(\mathfrak{K}) \neq \emptyset$.

Proof. The proof will be given by means of an example. Let $p_1, p_2, p_3, p_4, p_5, p_6$ be the vertices of a customary convex hexagon (with segments as sides) in the plane, with the property that it is included in the triangle with vertices $\overline{p_1p_2} \cap \overline{p_4p_5}$, $\overline{p_2p_3} \cap \overline{p_5p_6}$, $\overline{p_3p_4} \cap \overline{p_6p_1}$ (see notation (5) in § 2). Now, construct the circle C containing the preceding triangle in its interior, denote by L_{ij} and L_{ij}' (say $i < j$) those components of $C - \bigcup_{k=1}^{6} \overline{p_kp_{k+1}}$ which have all their endpoints on $\overline{p_ip_{i+1}}$ and $\overline{p_jp_{j+1}}$ ($p_7 = p_1$), and associate with every point $p \in L_{ij}$ the chord $L(p)$ of C passing through p and $p_{ij} = \overline{p_ip_{i+1}} \cap \overline{p_jp_{j+1}}$.



Defined in this way, $\mathfrak{L} = \{L(p)\}$ is a continuous family of curves, and the union of all open triangles with sides on members of $\mathfrak{L}$ equals $M_3(\mathfrak{L}) - M_4(\mathfrak{L})$; thus, $M_5(\mathfrak{L}) = \{p_{ij}\}$ and $\operatorname{int} M_5(\mathfrak{L}) = \emptyset$.

Let C be again an arbitrary simple closed curve, D the bounded component of its complement, and $\mathfrak{L}$ a continuous family of curves in D.

LEMMA 2. *If $L_1, \ldots, L_n \in \mathfrak{L}$, then $D - \bigcup_{i=1}^n L_i$ has at most $\frac{1}{2}(n^2 + n + 2)$ components.*

Proof. If any three curves in $\mathfrak{L}$ have void intersection, then the planar graph formed by

$$C \cup \bigcup_{i=1}^{n} L_i$$

obviously has $\binom{n}{2} + 2n$ vertices, and $n^2 + 2n$ edges. Now, from Euler's formula, it follows that there are $\frac{1}{2}(n^2 + n + 2)$ components in D. If there exist triples of curves in $\mathfrak{L}$ with non-void intersection, then the number of components in D is clearly less than $\frac{1}{2}(n^2 + n + 2)$.

I am indebted to the referee for suggesting this more elegant proof.

Next we shall associate s "crosses" to each component of $D - \bigcup_{i=1}^n L(p_i)$, where $L(p_1), \ldots, L(p_n) \in \mathfrak{L}$, having incidence order s, i.e. included in $M_s(\mathfrak{L}')$, where $\mathfrak{L}'$ is any continuous family admitting $L(p_1), \ldots, L(p_n)$ as elements and s is the largest possible. It appears to be more useful to find "global" estimations of the sum of all such associated crosses than of the particular incidence orders of isolated components, in order to establish the existence of components of higher incidence order.

LEMMA 3. *Let $L(p_1), \ldots, L(p_n) \in \mathfrak{L}$, each quadruple of these curves having void intersection, and let $p_1, \ldots, p_n$ be consecutive points on C. If t is the number of triples of curves among $L(p_1), \ldots, L(p_n)$ having non-void intersection, then the total number of crosses is at least $n^2 - 3t$.*

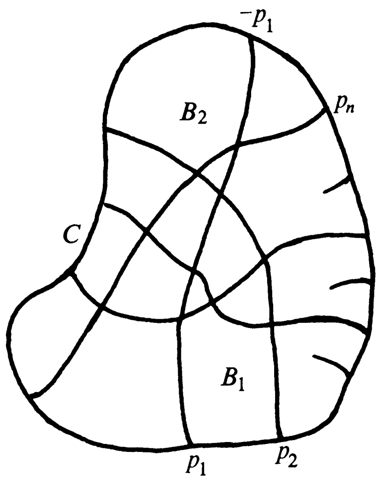
Proof. Let B_1, B_2 be the pair of opposite components of $D - (L(p_1) \cup L(p_2))$ whose boundaries contain p_1p_2 and $-p_1(-p_2)$, respectively (p_1p_2 and $-p_1(-p_2)$ are arcs on C not containing $-p_1$ and p_1). Suppose that

$$L(p_1) \cap L(p_2) \cap \bigcup_{i=3}^{n} L(p_i) = \emptyset.$$

If

$$\bigcup_{\substack{i,j=3;\\ i\neq j}}^{n} (L(p_i) \cap L(p_j)) \cap (B_1 \cup B_2) = \emptyset,$$

then there are exactly n components of $D - \bigcup_{i=1}^{n} L(p_i)$ in $B_1 \cup B_2$. If not, the number of these components is greater than n. Each point in $B_1 \cup B_2$ belongs to a curve of $\mathfrak{L}$ with an endpoint in the arc p_1p_2 of C. We may associate



with every component of $(B_1 \cup B_2) - \bigcup_{i=1}^{n} L(p_i)$ one cross and add in the same fashion other crosses, when the other $n - 1$ arcs $p_2p_3, \ldots, p_{n-1}p_n, -p_1p_n$ on C are considered. Thus, the total number of crosses is at least n^2 and one obviously loses three crosses when a triangle degenerates.

LEMMA 4. *If there exists a $2m$-gon, then the total number of crosses is at least $7m^2 - 9m + 6$.*

Proof. Let P be a $2m$-gon in D, with sides on $L_1, \ldots, L_{2m} \in \mathfrak{L}$. Consider again the pair of opposite components B_1, B_2 of $D - (L_1 \cup L_2)$, such that the boundary of B_1 contains the arc $p_1p_2 \subset C$ (p_1, p_2 are consecutive endpoints of L_1, L_2 on C).

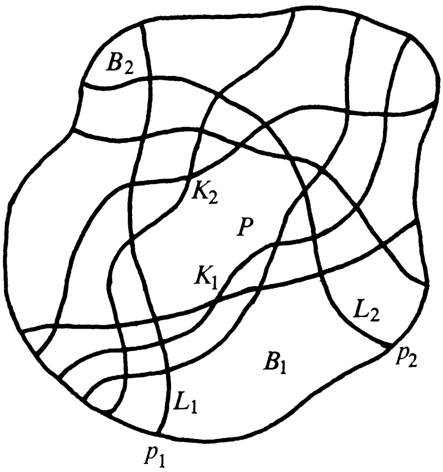
There are no triples of concurrent curves in $\{L_1, \ldots, L_{2m}\}$. Therefore,

$$L_1 \cap L_2 \cap \bigcup_{i=3}^{2m} L_i = \emptyset.$$

If

$$\operatorname{card} \bigcup_{\substack{i,j=3;\\ i\neq j}}^{2m} (L_i \cap L_j) \cap (B_1 \cup B_2) = q,$$

then there are exactly $2m + q$ components of $D - \bigcup_{i=1}^{2m} L_i$ in $B_1 \cup B_2$.



Since P has sides on curves in $\mathfrak{L}$, there exists a triangle containing it, and therefore, there are at least three domains like B_1 or B_2 that include P. Suppose that $P \subset B_1$; let K be the boundary of P and K_1, K_2 the components of $K - (L_1 \cup L_2)$, where K_2 may be void. Let

$$M_1 = \{\bigcup (L_i \cap L_j): i \neq j,\ L_i \cap K_1 \neq \emptyset,\ L_j \cap K_1 \neq \emptyset\},$$
$$M_2 = \{\bigcup (L_i \cap L_j): i \neq j,\ L_i \cap K_2 \neq \emptyset,\ L_j \cap K_2 \neq \emptyset\},$$

$$m_1 = \operatorname{card}\{i: L_i \cap K_1) \neq \emptyset\}, \quad \text{and} \quad m_2 = \operatorname{card}\{j: L_j \cap K_2 \neq \emptyset\}.$$

Since

$$m_1 + m_2 = 2m - 2,$$

we have

$$\operatorname{card} M_1 + \operatorname{card} M_2 = \binom{m_1}{2} + \binom{m_2}{2} \geqq 2\binom{m-1}{2}.$$

Repeating these arguments for the other two domains containing P and using Lemma 3, it follows that we have at least

$$(2m)^2 + 6\binom{m-1}{2} = 7m^2 - 9m + 6$$

crosses.

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LEMMA 5. *If there exists a* $(2m+1)$*-gon, then there are at least* $7m^2 - 2m + 4$ *crosses.*

Proof. All the arguments are just the same as in Lemma 4, except for

$$m_1 + m_2 = 2m - 1$$

and

$$\text{card } M_1 + \text{card } M_2 = \binom{m_1}{2} + \binom{m_2}{2} \geqq \binom{m}{2} + \binom{m-1}{2}.$$

THEOREM 3. *If there exists a heptagon (a 7-gon), then* int $M_5(\mathfrak{L}) \neq \emptyset$.

Proof. Let $L_1, \ldots, L_7$ be the curves in $\mathfrak{L}$ that form the heptagon. Let $D_1, \ldots, D_{14}$ be the components of $D - \bigcup_{i=1}^{7} L_i$ which have arcs of C on their boundaries. It is easily seen that one cross is associated with each of these fourteen domains. Following Lemma 2, $D - \bigcup_{i=1}^{7} L_i$ has at most 29 components (in fact, exactly 29 components, since no triple of concurrent curves in $\{L_1, \ldots, L_7\}$ exists). Following Lemma 5, the total number of crosses is at least 61. Since $D_1, \ldots, D_{14}$ have only fourteen crosses, it follows that the other fifteen components of $D - \bigcup_{i=1}^{7} L_i$ have at least 47 crosses, and therefore there exists at least one component, say D_{15}, with at least four crosses. Hence, $D_{15} \subset M_4(\mathfrak{L})$. If $D_{15} \subset M_5(\mathfrak{L})$, then obviously

$$\text{int } M_5(\mathfrak{L}) \supset D_{15} \neq \emptyset.$$

If $D_{15} \cap M_4(\mathfrak{L}) - M_5(\mathfrak{L}) \neq \emptyset$, then int $M_5(\mathfrak{L}) \neq \emptyset$ by Theorem 2.

Hence, following Lemma 1 and Theorem 3, the least number n such that the existence of an n-gon implies int $M_5(\mathfrak{L}) \neq \emptyset$ is $n = 7$.

6. More information about $M_3(\mathfrak{L}) - M_4(\mathfrak{L})$**.** Our next aim is to obtain some information about the structure of $M_3(\mathfrak{L}) - M_4(\mathfrak{L})$ when an n-gon exists and int $M_7(\mathfrak{L}) = \emptyset$.

LEMMA 6. *If* int $M_7(\mathfrak{L}) = \emptyset$ *and if there exists a* $4m$*-gon with sides on* $L_1, \ldots, L_{4m} \in \mathfrak{L}$, *then the number of disjoint polygons with sides on* $\bigcup_{i=1}^{4m} L_i$, *not included in* $M_4(\mathfrak{L})$, *is less than* $2(3m^2 - 1)$.

Proof. As in the proof of Theorem 3, it is easy to see that $8m$ domains formed by $L_1, \ldots, L_{4m}$ and C have only $8m$ crosses. Using Lemma 2, $D - \bigcup_{i=1}^{4m} L_i$ has $8m^2 + 2m + 1$ components, and following Lemma 4, there are at least $28m^2 - 18m + 6$ crosses.

Now, suppose that there are $2(3m^2 - 1)$ polygons not included in $M_4(\mathfrak{L})$. They have exactly $6(3m^2 - 1)$ crosses, and therefore $8m + 2(3m^2 - 1)$ domains have exactly $8m + 6(3m^2 - 1)$ crosses. Then the other $2m^2 - 6m + 3$ components of $D - \bigcup_{i=1}^{4m} L_i$ have at least $10m^2 - 26m + 12$ crosses. Since

$$\frac{10m^2 - 26m + 12}{2m^2 - 6m + 3} > 5,$$

there is a polygon with at least six crosses. It follows (as in the proof of Theorem 3) that int $M_7(\mathfrak{L}) \neq \emptyset$, contrary to the hypothesis.

LEMMA 7. *If* int $M_7(\mathfrak{L}) = \emptyset$ *and if there exists a* $(4m + 2)$*-gon with sides on* $L_1, \ldots, L_{4m+2} \in \mathfrak{L}$, *then the number of disjoint polygons with sides on* $\bigcup_{i=1}^{4m+2} L_i$, *not included in* $M_4(\mathfrak{L})$, *is less than* $6m^2 + 4m + 1$.

LEMMA 8. *If* int $M_7(\mathfrak{L}) = \emptyset$ *and if there exists a* $(2m + 1)$*-gon with sides on* $L_1, \ldots, L_{2m+1} \in \mathfrak{L}$, *then the number of disjoint polygons with sides on* $\bigcup_{i=1}^{2m+1} L_i$, *not included in* $M_4(\mathfrak{L})$, *is less than* $\frac{1}{2}m(3m + 1)$.

The proofs of Lemmas 7 and 8 being similar to that of Lemma 6 will be omitted.

Combining Lemmas 6, 7, and 8, we obtain the following theorem.

THEOREM 4. *Let*

$$f(n) = \begin{cases} \frac{1}{8}(n-1)(3n-1) & \text{if } n \text{ is odd,} \\ \frac{1}{8}(3n^2 - 16) & \text{if } n \text{ is a multiple of } 4, \\ \frac{1}{8}(3n^2 - 4n + 4) & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

If int $M_7(\mathfrak{L}) = \emptyset$ *and if there exists an n-gon with sides on* $L_1, \ldots, L_n \in \mathfrak{L}$, *then the number of disjoint polygons with sides on* $\bigcup_{i=1}^{n} L_i$, *not included in* $M_4(\mathfrak{L})$, *is less than* $f(n)$.

Remark. I believe that Theorem 4 provides a good evaluation for $f(n)$, but I have not proved that it is the best. To prove or disprove this in the odd case, it may be useful to note that there are a 5-gon and a 7-gon with sides on $L_1, \ldots, L_5 \in \mathfrak{L}$, and $L_1', \ldots, L_7' \in \mathfrak{L}'$, respectively, such that the number of disjoint polygons with sides on $\bigcup_{i=1}^{5} L_i$ $(\bigcup_{i=1}^{7} L_i')$, not included in $M_4(\mathfrak{L})(M_4(\mathfrak{L}'))$ is 6 (13), while $f(5) = 7$ and $f(7) = 15$.

7. Six-partite problems. As stated by Ceder (**2**), the so-called six-partite problems are involved in his work in a generalized manner, many earlier results being, in fact, corollaries of Ceder's theorems. We shall give here further generalizations in the same direction, replacing the families of lines admitting continuous selections by continuous families of curves. "Continuous selections" for families of curves could easily be introduced, but no essentially new facts would be obtained.

Let C be a simple closed curve and $\mathfrak{L}$ a continuous family of curves. All six-partite problems (including the results of this section) have as origin the following very simple theorem.

THEOREM 5. *If* $f_1, f_2\colon \mathfrak{L} \to \mathfrak{L}$ *are continuous functions, then there exists a curve* $L \in \mathfrak{L}$ *such that*

$$L \cap f_1(L) \cap f_2(L) \neq \emptyset.$$

Proof. If $f_1(L_0) = f_2(L_0)$ for some $L_0 \in \mathfrak{L}$, then obviously

$$L_0 \cap f_1(L_0) \cap f_2(L_0) \neq \emptyset.$$

If not, let p be a point varying on C. $L(p), f_1(L(p)), f_2(L(p))$, and $q(p) = f_1(L(p)) \cap f_2(L(p))$ are continuous functions of p. Suppose that $q(p)$ lies in the right component of $D - L(p)$, viewed from p. Then clearly $q(-p)$ lies in the left component of $D - L(-p)$, as viewed from $-p$. Thus (by the continuity of $q(p)$), there exists a point p^* on C such that $q(p^*) \in L(p^*)$.

THEOREM 6. *Let* $0 < a < b < c$ *and let*

$$g: (\mathfrak{L} \times \mathfrak{L} - \{(L, L): L \in \mathfrak{L}\}) \to (0, c)$$

be a real continuous function with the following properties:
(1) $g(L, L') \to 0$ *when* $L' \to L$ *clockwise;*
(2) $g(L, L'') \to c$ *when* $L'' \to L$ *counter-clockwise;*
(3) *if* A_L *is an arc of* C *having the same endpoints as* L, *then* $(gL, L(p))$ *is a strictly monotone function of* p *on* $A_L - L$.
Then there exists three concurrent curves $L_1, L_2, L_3 \in \mathfrak{L}$ *such that* $g(L_1, L_2) = a$ *and* $g(L_1, L_3) = b$.

Proof. The function $g(L, L(p))$ of p being continuous and one-to-one is a homeomorphism between $A_L - L$ and $(0, c)$. It follows that

$$h: (\mathfrak{L} \times \mathfrak{L} - \{(L, L): L \in \mathfrak{L}\}) \to \mathfrak{L} \times (0, c),$$

defined by $h(L_1, L_2) = (L_1, g(L_1, L_2))$ is also a homeomorphism. Now, obviously $h^{-1}(L, a)$ and $h^{-1}(L, b)$ are continuous functions on L.

Let $f_1, f_2: \mathfrak{L} \to \mathfrak{L}$ be defined by $h^{-1}(L, a) = (L, f_1(L))$ and $h^{-1}(L, b) = (L, f_2(L))$. Since f_1 and f_2 are continuous, we may use Theorem 5, and

$$L_1 \cap f_1(L_1) \cap f_2(L_1) \neq \emptyset$$

for a curve $L_1 \in \mathfrak{L}$.

Since

$$(L_1, g(L_1, f_1(L_1))) = h(L_1, f_1(L_1)) = (L_1, a)$$

and

$$(L_1, g(L_1, f_2(L_1))) = h(L_1, f_2(L_1)) = (L_1, b),$$

the concurrent curves L_1, $L_2 = f_1(L_1)$ and $L_3 = f_2(L_1)$ satisfy both conditions $g(L_1, L_2) = a$ and $g(L_1, L_3) = b$, and the theorem is proved.

Now, we formulate applications of Theorem 6, which are generalizations of (**2**, Theorems 1 and 2) and (partially) of (**2**, Theorem 3).

THEOREM 7. *If*
(1) D *has finite measure** c,

---

*By measure we mean a regular measure for which curves in $\mathfrak{L}$ have measure zero.

(2) *for any triangle T with sides on $L_1, L_2, L_3 \in \mathfrak{L}$, the measure of the component of $D - (L_1 \cup L_2)$ containing T plus the measure of the opposite component is greater than twice the measure of T,*

(3) $0 < a < b < c$,

then there exists three concurrent curves $L_1, L_2, L_3 \in \mathfrak{L}$, such that the union of two opposite components of $D - (L_1 \cup L_2)$ has measure a and the union of two opposite components of $D - (L_1 \cup L_3)$ has measure b.

Proof. We indeed have the continuous function

$$g: (\mathfrak{L} \times \mathfrak{L} - \{(L, L): L \in \mathfrak{L}\}) \to (0, c),$$

where $g(L_1, L_2)$ is the sum of measures of two components B_1 and B_2 of $D - (L_1 \cup L_2)$ chosen such that $L \cap C$ consists of two boundary points of $B_1 \cup B_2$, when L varies counter-clockwise from L_1 to L_2. Obviously, conditions (1) and (2) of the preceding theorem are verified. We prove next that $g(L, L(p))$ is strictly monotone on $A_L - L$ (using the notation of Theorem 6).

Let p, $-p$ be the first and the last point of A_L when C is described counter-clockwise and let $x \prec y$ if $x, y \in A_L$ and y can be obtained from x varying counter-clockwise on A_L. Also, consider $p_1, p_2 \in A_L$, $p_1 \prec p_2$, and put $q_1 = L(p) \cap L(p_1)$, $q_2 = L(p_1) \cap L(p_2)$, $q_3 = L(p_2) \cap L(p)$. Suppose, for example, that q_2 belongs to the bounded domain with boundary $L(p) \cup A_L$. Then, if μ is the given regular measure,

$$\begin{aligned}\mu[p, p_2, q_3] + \mu[-p, q_3, -p_2] &= \mu[p, p_1, q_1] + \mu[-p, -p_1, q_1] \\ &\quad + \mu[-p_1, q_1, q_3, -p_2] - \mu[q_1, q_2, q_3] + \mu[q_2, p_2, p_1] = \mu[p, p_1, q_1] \\ &\quad + \mu[-p, -p_1, q_1] + \mu[q_2, -p_2, -p_1] + \mu[q_2, p_2, p_1] \\ &\quad -2\mu[q_1, q_2, q_3] > \mu[p, p_1, q_1] + \mu[-p, -p_1, q_1],\end{aligned}$$

following condition (2) in the statement. Thus, $g(L, L(p))$ satisfies all conditions of Theorem 6, and hence there exists three concurrent curves $L_1, L_2, L_3 \in \mathfrak{L}$ such that $g(L_1, L_2) = a$ and $g(L_1, L_3) = b$, which proves Theorem 7.

Theorem 8 is an application of Theorem 7 and a generalization of case I in the proof of (**2**, Theorem 3).

THEOREM 8. *If*

(1) *D has finite measure c,*

(2) *all curves in $\mathfrak{L}$ are area bisectors (divide D in pairs of domains of measure $\frac{1}{2}c$),*

(3) *α_1, α_2, and α_3 are non-negative numbers whose sum is $\frac{1}{2}c$,*

then there exists three concurrent curves in $\mathfrak{L}$ dividing D into six parts of measures $\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2$, and α_3, respectively.

Proof. Show that the conditions of Theorem 7 are verified.

Conditions (1) of both Theorems 7 and 8 are identical.

Condition (2): For any triangle T with sides on L_1, L_2, L_3, the measure of the component of $D - (L_1 \cup L_2)$ containing T equals the measure of the opposite component; whence, their sum is greater than $2\mu T$, as required.

Condition (3): Take $a = 2\alpha_1$, $b = 2\alpha_1 + 2\alpha_2$.

The conclusion of Theorem 8 then follows from that of Theorem 7.

THEOREM 9. *If*

(1) *C is rectifiable, of length c,*

(2) $0 < a < b < c$,

then there exists three concurrent curves $L_1, L_2, L_3 \in \mathfrak{L}$ such that the sum of lengths of two opposite components of $C - (L_1 \cup L_2)$ equals a and the sum of lengths of two opposite components of $C - L_1 - L_3$ equals b.

Proof. This is another application of Theorem 6. For, we can consider the continuous function

$$g\colon (\mathfrak{L} \times \mathfrak{L} - \{(L, L)\colon L \in \mathfrak{L}\}) \rightarrow (0, c),$$

where $g(L_1, L_2)$ is the sum of lengths of two components C_1 and C_2 of $C - (L_1 \cup L_2)$ chosen such that $L \cap C \subset C_1 \cup C_2$ when L varies counterclockwise from L_1 to L_2. Obviously, all conditions of Theorem 6 are verified, and hence there are $L_1, L_2, L_3 \in \mathfrak{L}$ such that $L_1 \cap L_2 \cap L_3 \neq \emptyset$, $g(L_1, L_2) = a$, and $g(L_1, L_3) = b$.

The easy consequence of Theorem 9 below leads immediately to Ceder's result (**2**, Theorem 2).

THEOREM 10. *If*

(1) *C is rectifiable, of length c,*

(2) *all curves in $\mathfrak{L}$ are arc bisectors (divide C in pairs of arcs of length $\frac{1}{2}c$),*

(3) *α_1, α_2, and α_3 are non-negative numbers whose sum is $\frac{1}{2}c$,*

then there exists three concurrent curves in $\mathfrak{L}$ dividing C into six parts of lengths $\alpha_1, \alpha_2, \alpha_3, \alpha_1, \alpha_2$, and α_3, respectively.

Proof. It suffices to observe that the lengths of opposite components of $C - (L_1 \cup L_2)$ are equal ($L_1, L_2 \in \mathfrak{L}$), and to take $a = 2\alpha_1$, $b = 2\alpha_1 + 2\alpha_2$.

At the end of Ceder's paper (**2**), one finds the following unsolved six-partite problem: Given six numbers whose sum is the arc length of a convex curve, are there always three concurrent lines dividing the curve into six parts having lengths equal to the given six numbers?

This question posed for a circle receives a negative answer immediately. Consequently, by continuity, even for a larger class of convex curves, the answer remains negative.

8. Generalized continuous families of curves. As noticed in § 2, the notion of a continuous family of curves can be generalized by admitting connected intersections of couples of curves in $\mathfrak{L}$ instead of only single point intersections. Although a suitable further generalization could be obtained including in $\mathfrak{L}$ not only Jordan arcs but also arbitrary simply connected arcs, we shall limit ourselves here to treat generalized families of curves as defined in § 2.

Our aim in this section is to prove by an example how results on continuous families of curves can be extended to the generalized families.

Most of the results of (**3**) remain valid if continuous families of curves (in Grünbaum's sense) are replaced by generalized continuous families of curves. For instance, although the proof of (**3**, Theorem 1) uses axiom (iv), we shall now show that this theorem is still valid if (iv′) holds instead of (iv). Notice that, however, our proof is, in general, similar to that of Grünbaum.

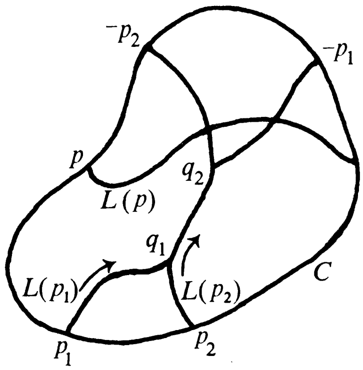
THEOREM 11. *Let $\mathfrak{L}$ be a generalized continuous family of curves. With at most one exception, each curve in $\mathfrak{L}$ contains a point of $M_3(\mathfrak{L})$.*

Proof. Let $L_0, L(p) \in \mathfrak{L}$, the endpoint p of $L(p)$ varying on one of the two arcs in which L_0 divides C.

Choose arbitrarily the point $q(L_0, p) \in L_0 \cap L(p)$; if $q(L_0, p)$ is a strictly monotone function on p (for every choice of $q(L_0, p)$ in $L_0 \cap L(p)$), then L_0 is said to be of type (a). It is easily seen that if $L \in \mathfrak{L}$ is not of type (a), then

$$L \cap M_3(\mathfrak{L}) \neq \emptyset.$$

We shall prove that two different curves of type (a) do not exist in $\mathfrak{L}$.



Assume, on the contrary, that the curves $L(p_1)$ and $L(p_2)$ are both of type (a). Let us say, for instance, that if p varies counter-clockwise on C, then $q(L(p_1), p)$ varies on $L(p_1)$ from p_1 to $-p_1$ and $q(L(p_2), p)$ on $L(p_2)$ from p_2 to $-p_2$. With these orderings ω_1 and ω_2 of $L(p_1)$ and $L(p_2)$ (the increasing sense being from p_i to $-p_i$), we set

$$q_1 = \min_{\omega_1} L(p_1) \cap L(p_2), \qquad q_2 = \max_{\omega_1} L(p_1) \cap L(p_2).$$

If p is a point on the component $-p_2p_1$ of $C - (L(p_1) \cup L(p_2))$, then $L(p)$

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must intersect at least one of the arcs $p_1q_1 \subset L(p_1)$ and $-p_2q_1 \subset L(p_2)$. However, the inequalities

$$L(p) \cap p_1q_1 \neq \emptyset, \qquad L(p) \cap -p_2q_1 \neq \emptyset$$

contradict

$$L(p) \cap L(p_1) \underset{\omega_1}{>} q_2, \qquad L(p) \cap L(p_2) \underset{\omega_2}{<} q_1,$$

respectively. The proof is now complete; it is valid for both cases:

$$q_1 \underset{\omega_2}{<} q_2 \quad \text{and} \quad q_1 \underset{\omega_2}{>} q_2.$$

9. Some types of connectedness in D. In this section, $\mathfrak{L}$ will be a generalized continuous family of curves in D. We shall be concerned with the notion of polygonal $\mathfrak{L}$-connectedness introduced by Grünbaum (**3**). This is an appropriate modification of a generalization of convex sets due to Horn and Valentine (**4**). A set $P \subset D$ is said to be polygonally $\mathfrak{L}$-connected if for each couple $x, y \in P$, there is an m-curve, with arbitrary m (see the definitions in § 2) in P, connecting x and y. P is an $L_n(\mathfrak{L})$-set provided that for each pair $x, y \in P$, there is an n-curve in P connecting x and y (note that every m-curve with $m < n$, is also an n-curve). P is called $\mathfrak{L}$-convex if and only if $L \cap P$ is connected (or void) for each curve $L \in \mathfrak{L}$.

Now, let us add the following definitions: The subset P of D is finitely $\mathfrak{L}$-convexly connected provided that for every curve $L \in \mathfrak{L}$, $L \cap P$ has a finite number of components (or is void); P is said to be nth order $\mathfrak{L}$-convexly connected if $L \cap P$ has at most n components for each $L \in \mathfrak{L}$. First-order $\mathfrak{L}$-convexly connectedness means precisely $\mathfrak{L}$-convexity.

We shall denote by $\mathfrak{K}_x(P)$ the set of all points in P that can be joined to the point $x \in P$ by an m-curve (m depending on x) lying entirely in P, and by $\mathfrak{K}_x^n(P)$ the set of all points in P that can be joined to x by an n-curve included in P. Furthermore, let us define the $\mathfrak{L}$-kernel of a set P in D as the subset of P of all points that can be joined to every point x of P by an m-curve (m depending on x). The nth order $\mathfrak{L}$-kernel of $P \subset D$ is the subset of P of all points that can be joined to every $x \in P$ by an n-curve.

It is interesting to compare our results below with the following lemma of A. M. Bruckner and J. B. Bruckner (**1**), in which $K_x^n(P)$ and L_n-sets are defined in a similar way, replacing only "n-curve" by "n-sided polygonal line" (**1**).

If P is a compact simply connected set in the plane and $x \in P$, then $K_x^n(P)$ is a compact, simply connected L_{2m}-set.

Before giving other results, we note that $\mathfrak{K}_x^n(P)$ and $\mathfrak{K}_x(P)$ are, respectively, an $L_{2n}(\mathfrak{L})$-set and polygonally $\mathfrak{L}$-connected.

The proof of the following lemma is straightforward.

Lemma 9. *If $P \subset D$ is closed, then $\mathfrak{K}_x^n(P)$ is compact.*

CONJECTURE. *If $P \subset D$ is compact and simply connected, then $\mathfrak{K}_x^n(P)$ is simply connected.*

The following theorem results immediately from Lemma 9, since the nth order $\mathfrak{L}$-kernel of P is $\bigcap_{x \in P} \mathfrak{K}_x^n(P)$.

THEOREM 12. *The nth order $\mathfrak{L}$-kernel of a compact set $P \subset D$ is compact.*

LEMMA 10. *If $P \subset D$ is nth order $\mathfrak{L}$-convexly connected, then $\mathfrak{K}_x(p)$ is nth order $\mathfrak{L}$-convexly connected.*

Proof. Suppose that there exists a curve $L \in \mathfrak{L}$ such that $L \cap \mathfrak{K}_x(P)$ has at least $n + 1$ components. Then there are the consecutive points

$$a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_{n+1} \in L$$

such that all $a_i \in \mathfrak{K}_x(P)$ and all $b_j \notin \mathfrak{K}_x(P)$. Since $a_i \in P$ and P is nth order $\mathfrak{L}$-convexly connected, there is an arc $a_l a_{l+1}$ of L included in P. Now, the union of $a_l b_l$ with the m-curve joining a_l and x is an $(m + 1)$-curve with endpoints b_l and x. Hence, $b_l \in \mathfrak{K}_x(P)$ and the contradiction shows that $\mathfrak{K}_x(P)$ is nth order $\mathfrak{L}$-convexly connected.

Let us note, without proof, the following simple remarks.

Remarks. (1) Let $P_1, \ldots, P_n$ be $\mathfrak{L}$-convexly connected sets of finite order (not only finitely $\mathfrak{L}$-convexly connected), the orders being not necessarily equal. Then there exists a natural number m such that $\bigcap_{i=1}^n P_i$ is mth order $\mathfrak{L}$-convexly connected.

(2) If $\{P_n\}_{n=1}^\infty$ is a sequence of second-order $\mathfrak{L}$-convexly connected sets, then $\bigcap_{n=1}^\infty P_n$ is not necessarily finitely $\mathfrak{L}$-convexly connected.

LEMMA 11. *If $P \subset D$ is simply connected, and finitely $\mathfrak{L}$-convexly connected, then $\mathfrak{K}_x(P)$ is simply connected. $\mathfrak{K}_x(P)$ is not necessarily compact, even if P is compact.*

Proof. Suppose that there exists a bounded component A of the complement of $\mathfrak{K}_x(P)$ in the plane. Since $\mathfrak{K}_x(P) \subset P$ and P is simply connected, $A \subset P$. Take $a \in A$. Since

$$P \subset D = M_1(\mathfrak{L}),$$

there exists a curve $L \in \mathfrak{L}$ passing through a.

Let $B_1, \ldots, B_n$ be the components of $L - P$, and $b_i \in B_i$. There is a point $y_i \in \mathfrak{K}_x(P)$ between a and b_i on L, since otherwise, owing to the connectedness and maximality of A,

$$ab_i \cap \mathfrak{K}_x(P) = \emptyset$$

would imply that $ab_i \subset A$, and A would include the (unbounded) complement of P. Choose b_j from $\{b_1, \ldots, b_n\}$ such that there is no b_i between a and b_j on L. In this case, obviously $ab_j \cap P$ is connected; whence $ay_j \subset P$.

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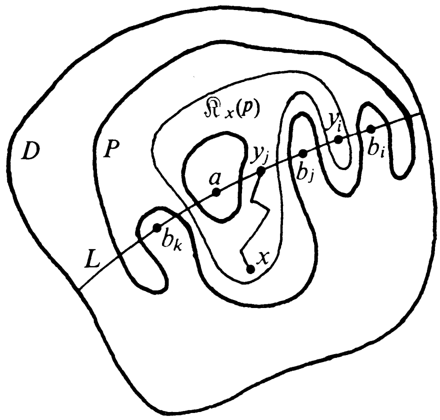

Now, since $y_i \in \mathfrak{K}_x(P)$, we also have $a \in \mathfrak{K}_x(P)$, contradicting $a \in A$. Hence, $\mathfrak{K}_x(P)$ is simply connected. It is easy to find an example proving the existence of a compact set P such that $\mathfrak{K}_x(P)$ is not compact for a suitable point $x \in P$, even in the case of a continuous family of curves in Grünbaum's sense.

The following theorem results from Lemma 10 for $n = 1$, since any intersection of $\mathfrak{L}$-convex sets in D is $\mathfrak{L}$-convex, and the $\mathfrak{L}$-kernel of $P \subset D$ equals $\bigcap_{x \in P} \mathfrak{K}_x(P)$.

THEOREM 13. *The $\mathfrak{L}$-kernel of an $\mathfrak{L}$-convex set in D is $\mathfrak{L}$-convex.*

Theorem 14 below is a consequence of Lemma 11, of Theorem 13, and of the obvious fact that each $\mathfrak{L}$-convex set is finitely $\mathfrak{L}$-convexly connected.

THEOREM 14. *The $\mathfrak{L}$-kernel of a simply connected, $\mathfrak{L}$-convex set in D is simply connected and $\mathfrak{L}$-convex.*

The following theorem follows from Lemma 11.

THEOREM 15. *The $\mathfrak{L}$-kernel of a simply connected, finitely $\mathfrak{L}$-convexly connected set in D is simply connected.*

Remark that in Theorems 13, 14, and 15, the $\mathfrak{L}$-kernel may well be void.

10. Final remarks. The study of these families of curves is of obvious interest in connection with planar convex bodies and families of lines associated with them. A generalization of continuous families of curves applying to higher dimensional convex bodies would be most desirable, but this seems to be rather difficult, even for the three-dimensional case. On the other hand, it seems possible to give further topological generalizations of concepts first appearing in plane convexity.

During the publication of the present paper the following result (**5**) was added to (**3**, Theorem **1**):

If the intersection of all curves in the continuous family $\mathfrak{L}$ *is void, then on all curves of* $\mathfrak{L}$, *with at most three exceptions, each of their dense subsets intersects* $M_3(\mathfrak{L})$.

Further, some kinds of closed curves can be associated with a continuous family, generalizing analogous curves already introduced for convex bodies in the plane.

References

1. A. M. Bruckner and J. B. Bruckner, *Generalized convex kernels*, Israel J. Math. *2* (1964), 27–32.

2. J. Ceder, *Generalized sixpartite problems*, Bol. Soc. Mat. Mexicana *9* (1964), 28–32.

3. B. Grünbaum, *Continuous families of curves*, Can. J. Math. *18* (1966), 529–537.

4. A. Horn and F. A. Valentine, *Some properties of L-sets in the plane*, Duke Math. J. *16* (1949), 131–140.

5. T. Zamfirescu, *Sur les familles continues de courbes* (Note I), Rend. Classe Sci. Fis. Mat. Naturali Accad. Naz. dei Lincei *42* (1967), 771–774.

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