ON THE DISTRIBUTION OF PRIMITIVE LATTICE POINTS IN THE PLANE

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1. Introduction. Let 1, $\theta_1, \theta_2, \ldots, \theta_n$ be real numbers linearly independent over the rational field and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be arbitrary real numbers. Then, to each $N > 0$ and $\varepsilon > 0$, there correspond integers

$$x > N, \; y_1, \; y_2, \ldots, y_n$$

which satisfy the set of inequalities

$$\text{(A)} \qquad |y_i - \theta_i x + \alpha_i| < \varepsilon \;, \; (i = 1, 2, \ldots, n).$$

This is one form of Kronecker's theorem [4] and, since N can be chosen arbitrarily large, it follows that there are an infinity of integer sets $(x, y_1, \ldots, y_n)$ with $x > 0$ satisfying (A). For $n \geqslant 2$, it is not possible to strengthen this result by replacing the ε in (A), throughout, by any function $\psi(x)$ which tends to zero as $x \to \infty$ (see, e.g., [5], Kap VII, §7, Satz 6). But, in the case $n = 1$, it is well known that there are an infinity of integer pairs (x, y) satisfying

$$\text{(B)} \qquad |y - \theta x + \alpha| < \frac{c}{x}, \; x > 0.$$

Here, the approximating function c/x is of the correct order of magnitude and indeed the exact value for c has been determined (see [1], for details and references to earlier work). However, an elementary geometrical argument [2], shows that we can, in fact, solve (A) with an infinity of integer sets satisfying the additional condition

$$(x, y_1, y_2, \ldots, y_n) = 1.$$

It is natural then to raise the question, analogous to (B), of finding an approximating function $\psi(x)$ such that the inequality

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$$|y - \theta x + \alpha| < \psi(x)$$

is satisfied by an infinity of coprime integers (x, y) with $x > 0$. We remark that it is easy to obtain $\psi(x) = O(x^{1-\delta})$, for any positive δ, and that by Brun's method, one can improve this to

$$\psi(x) = O\left(\frac{(\log x)^c}{x}\right),$$

for some positive constant c. As we do not know the correct order of magnitude for $\psi(x)$ as $x \to \infty$, the following estimate is of some interest:

THEOREM. For any given irrational number θ and any real number α, there exists an absolute constant λ such that

$$X|Y - \theta X + \alpha| < \lambda \left(\frac{\log X}{\log \log X}\right)^2 \tag{1}$$

is satisfied by infinitely many coprime integers X, Y with $X > 0$.

We observe that the result is significant only when $\alpha \neq 0$, since if $\alpha = 0$ we can take $Y = p_n$, $X = q_n$, where p_n/q_n is any convergent to the continued fraction for θ, and then $X|Y - \theta X| < 1$, $(X, Y) = 1$ for each n. Now, the interest of our result lies mainly in the condition imposed upon X and Y. Without the restriction $(X, Y) = 1$, the approach by continued fractions, for instance, would give $O(1)$ as $X \to \infty$, on the right of (1). This, in fact, is the starting point for our proof of the theorem and we introduce the relative primality condition by means of the following lemma.

LEMMA. Let x, y be given integers with $0 < x \leqslant y$ and θ' denote any real number satisfying $0 \leqslant \theta' < 1$. Let $\varepsilon_i = \pm 1$ $(i = 1, 2)$ be specified. Then, for certain increasing functions $m = m(x)$, $n = n(x)$ with $1 \leqslant m \leqslant n$, $m(x) \to \infty$ as $x \to \infty$, there exist integers u, v satisfying

$$0 \leqslant u - \theta' v < m, \qquad 0 \leqslant v < n \tag{2}$$

and

$$(x + \varepsilon_1 u, \ y + \varepsilon_2 v) = 1. \tag{3}$$

A result of this kind has been obtained recently by Erdős [3] with

$$m = n = C \log x/\log \log x; \tag{4}$$

C being a suitably large absolute constant. It seems likely that this result can be improved but not, so far as we can see, by the same method. To illustrate the scope of the argument, our proof is presented in terms of m and n; it gives

$$O(m(X)n(X))$$

in place of the function on the right of (1).

2. Proof of the theorem. By replacing Y by -Y, if necessary, we may suppose that $\theta > 0$. We can assume that $\alpha \neq 0$, since otherwise there is nothing to prove. Let p_{n-1}/q_{n-1}, p_n/q_n denote consecutive convergents to θ and put

$$\theta' = -(p_n - q_n\theta)/(p_{n-1} - q_{n-1}\theta). \tag{5}$$

Then

$$p_n q_{n-1} - p_{n-1} q_n = \delta_n = (-1)^{n-1}, \tag{6}$$

$$\frac{1}{2q_{n+1}} < |p_n - q_n\theta| < \frac{1}{q_{n+1}} \tag{7}$$

and

$$0 < \theta' < 1, \tag{8}$$

by properties of regular continued fractions. Let

$$\alpha' = \max(2, [|\alpha|] + 3) \tag{9}$$

and put

$$Q_n' = [q_n\alpha], \quad Q_n = [q_{n-1}\alpha] - \alpha'\delta_n; \tag{10}$$

then, since $p_n q_{n-1} - p_{n-1} q_n = \pm 1$, we can solve the equations

$$p_n\eta - q_n\xi = Q_n'$$

$$p_{n-1}\eta - q_{n-1}\xi = Q_n$$

with integers ξ, η. In particular

$$\eta = \delta_n(q_{n-1}Q_n' - q_n Q_n)$$

$$= \delta_n(q_{n-1}[q_n\alpha] - q_n[q_{n-1}\alpha]) + \alpha' q_n$$

and since $\left|q_{n-1}[q_n\alpha] - q_n[q_{n-1}\alpha]\right| < q_n$, we have

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$$(11) \qquad (\alpha' - 1)q_n < \eta < (\alpha' + 1)q_n.$$

We now take X, Y to be of the form

$$(12) \qquad X = \eta + uq_{n-1} + vq_n,$$

$$(13) \qquad Y = \xi + up_{n-1} + vp_n,$$

where u, v are non-negative integers. Observe that X, Y are relatively prime if, and only if,

$$(14) \qquad \begin{aligned} 1 &= (p_n X - q_n Y,\ p_{n-1}X - q_{n-1}Y) \\ &= (p_n\eta - q_n\xi + \delta_n u,\ p_{n-1}\eta - q_{n-1}\xi - \delta_n v) \\ &= (|Q'_n| \pm u,\ |Q_n| \pm v) \end{aligned}$$

for a certain choice of the $\pm$ signs. Now, by (8) and the lemma we can choose non-negative integers u, v to satisfy (14) and

$$(15) \qquad 0 \leqslant u - \theta' v < m(|Q_n|),\ 0 \leqslant v < n(|Q_n|),$$

provided that $0 < |Q_n| \leqslant |Q'_n|$. Since $q_n > n$ for all n, we have

$$\begin{aligned} |Q'_n| &= q_n|\alpha| + O(1) \\ &= a_n q_{n-1}|\alpha| + q_{n-2}|\alpha| + O(1) \\ &\geqslant q_{n-1}|\alpha| + q_{n-2}|\alpha| + O(1) \\ &> |Q_n| \end{aligned}$$

for all sufficiently large n. By (5), (7), (12), (13), we have

$$X|Y - \theta X + \alpha| = q_n|p_{n-1} - q_{n-1}\theta||q_{n-1}q_n^{-1}u + v + \varphi||u - \theta' v + \varphi'|,$$

$$\leqslant |q_{n-1}q_n^{-1}u + v + \varphi||u - \theta' v + \varphi'|,$$

where

$$0 < \varphi = \eta q_n^{-1} < \alpha' + 1,$$

and

$$|\varphi'| = \left|\frac{\xi - \theta\eta + \alpha}{p_{n-1} - q_{n-1}\theta}\right|$$

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$$< \quad 2q_n \left| \frac{q_n\alpha - Q_n'}{q_n} + \frac{(p_n - q_n\,\theta)\eta}{q_n} \right|$$

$$< \quad 2\left|q_n\alpha - Q_n'\right| + 2\left|(p_n - q_n\,\theta)\eta\right|$$

$$< \quad 2 + 2(\alpha' + 1),$$

by (7), (10) and (11). Hence, by (15),

$$X\left|Y - \theta X + \alpha\right| < (u + v + \alpha' + 1)(u - \theta' v + 2\alpha' + 4)$$

$$= O\left\{\left(m(|Q_n|) + n(|Q_n|)\right) \; m(|Q_n|)\right\},$$

$$= O\left\{m(|Q_n|)\, n(|Q_n|)\right\},$$

$$= O\left(m(X)\, n(X)\right), \text{ as } n \to \infty,$$

since (12), (11), (9) and (10) give, successively,

$$X \geqslant \eta \geqslant (\alpha' - 1)q_n$$

$$\geqslant (|\alpha| + 2)q_n$$

$$\geqslant q_{n-1}|\alpha| + 1 + \alpha'$$

$$\geqslant \left| \; [q_{n-1}\alpha] - \alpha'\delta_n \right|$$

$$= |Q_n|,$$

for all sufficiently large n. We remark that the constant implied by the O- symbol does not depend on α and the theorem itself follows immediately on substituting the values for m and n, given in (4).

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