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INTERIOR ESTIMATES FOR ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS IN THE $\mathscr{L}^{(q,\lambda)}$ SPACES OF STRONG TYPE

AKIRA ONO

Introduction. Recently the $\mathscr{L}^{(q,\lambda)}$ spaces have been investigated by many authors and the theory of these spaces has proved to be particularly important for research in partial differential equations (see for example [**15**], [**16**] and [**18**]).

The equations of elliptic type in these spaces were first studied by C. B. Morrey [**8**], [**9**], who applied his well-known imbedding theorems, and afterwards by S. Campanato [**3**], [**4**] with the aid of isomorphism theorems and the so-called fundamental inequalities due to him.

On the other hand, G. Stampacchia introduced the $\mathscr{L}^{(q,\lambda)}$ spaces of strong type [**17**], the structures of which are more general and complicated than those of $\mathscr{L}^{(q,\lambda)}$ spaces in the usual sense, and greater part of them were characterized by him, L. C. Piccinini, Y. Furusho, the author and others (see [**5**], [**11**]-[**14**], [**16**] and [**17**]).

Furthermore, M. Nakamura has given precise interior estimates for elliptic partial differential equations [**10**] by using theorems due to S. Agmon, A. Douglis and L. Nirenberg [**1**] and previous results obtained by the author.

In this paper, we will deduce more precise estimates for the solutions of elliptic partial differential equations in more general situations, that is, including equations of integral form.

This paper is organized as follows:

In Section 1 relevant definitions, fundamental assumptions on the equations and the first main theorems are stated. The proof of the theorems are given in Section 2.

In Section 3 we prove the Schauder estimates, that is, the strong Hölder continuity of the solutions.

The main tools for the proof of these theorems are theorems and techniques found in [**1**], [**3**], [**4**], [**2**], [**7**], [**6**], [**12**], [**13**], [**14**].

In Section 4 we apply the Morrey-Sobolev type imbedding theorems proved in [**12**] to deduce the regularity of the lower order derivatives of the solutions.

Additional comments on the above theorems are given in Section 5.

1. The strong $\mathscr{L}^{(q,\lambda)}$ estimates. Throughout this paper we denote by A,

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D_1, D_2 and D arbitrary fixed bounded domains in the Euclidean n-space E^n such that

$$A \subset \bar{A} \subset D_1 \subset \bar{D}_1 \subset D_2 \subset \bar{D}_2 \subset D$$

and such that all have sufficiently smooth boundaries.

We always consider subfamilies of integrable functions on D and an arbitrary subcube Q of the domain D with its sides parallel to the axes (from now on, subcube means such a parallel subcube). Furthermore, we denote the measure of a subcube Q by $|Q|$ and the mean value of a function u over Q by

$$u_Q : u_Q = |Q|^{-1} \int_Q u(x) dx.$$

Definition 1. A function $u \in L^q(D)$ is said to *belong to the space* $\mathcal{L}_p^{(q,\lambda)}(D)$ (the $\mathcal{L}^{(q,\lambda)}$ space of strong type p), if the following inequality holds for u:

$$(1) \quad [u]_{\mathcal{L}_p^{(q,\lambda)}(D)} = \sup_{\{Q_j\} \in \bar{S}} \left[\sum_j \left(|Q_j|^{\frac{\lambda}{n}-1} \int_{Q_j} |u(x) - u_{Q_j}|^q dx \right)^{\frac{p}{q}} \right]^{\frac{1}{p}} < \infty$$

where $1 \leqq p < \infty$, $1 \leqq q < \infty$, $-\infty < \lambda < \infty$ and $\bar{S}$ is the family of all systems of subcubes $\{Q_j : \cup_j Q_j \subset D\}$ of finite number, no two of which have common interior point. Taking as the norm of the space $\mathcal{L}_p^{(q,\lambda)}(D)$,

$$\|u\|_{\mathcal{L}_p^{(q,\lambda)}(D)} = [u]_{\mathcal{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^q(D)}$$

we obtain a Banach space.

Here, we note that the domain D is occasionally replaced by the subdomain A, D_1 or D_2.

Definition 2. The *Sobolev space* $H^{l,p}(D)$ is the completion of the space $C^l(\bar{D})$ with respect to the norm

$$\|u\|_{H^{l,p}(D)} = \sum_{|\beta| \leqq l} \|D^\beta u\|_{L^p(D)}.$$

Now, we consider the following linear elliptic partial differential equation:

$$(E) \quad \sum_{|\beta| \leqq 2m} a_\beta(x) D^\beta u = f(x), \quad u \in H^{2m,p}(D).$$

Or, of the integral form:

$$(E)' \quad \sum_{\substack{|\beta| \leqq l \\ |\gamma| \leqq 2m-l}} D^\gamma [a_{\beta\gamma}(x) D^\beta u] = \sum_{|\gamma| \leqq 2m-l} D^\gamma f_\gamma(x), \quad u \in H^{l,p}(D),$$

$$(l < 2m)$$

under the fundamental assumptions for the equation (E):

$(A)_{\mathscr{L}_p^{(q,\lambda)}}$ I. The coefficients $\{a_\beta\}$ and the $(l - 2m)$-th derivatives $(l \geqq 2m)$ of the function f belong to the spaces $C^{l-2m+\frac{n}{p}-\frac{\lambda}{q}}(D)$ and $\mathscr{L}_p^{(q,\lambda)}(D)$ respectively, where $0 < \lambda < n$ and $0 < \dfrac{n}{p} - \dfrac{\lambda}{q} < 1$.

II. (Uniform ellipticity). There exists a constant E greater than unity such that the following inequality holds:

$$(2) \qquad E^{-1}|\xi|^{2m} \leqq \sum_{|\beta|=2m} a_\beta(x)\xi^\beta \leqq E|\xi|^{2m} \quad \forall x \in D.$$

Or, for the equation $(E)'$:

$(A)'_{\mathscr{L}_p^{(q,\lambda)}}$ I. The coefficients $\{a_{\beta\gamma}\}$ and the functions $\{f_\gamma\}$ belong to the spaces $C^{\frac{n}{p}-\frac{\lambda}{q}}(D)$ and $\mathscr{L}_p^{(q,\lambda)}(D)$ respectively, where $0 < \lambda < n$ and $0 < \dfrac{n}{p} - \dfrac{\lambda}{q} < 1$.

II. (Uniform ellipticity). The leading coefficients satisfy the following condition: there exists a constant E greater than unity such that

$$(2)' \qquad E^{-1}|\xi|^{2m} \leqq \sum_{|\beta+\gamma|=2m} a_{\beta\gamma}(x)\xi^{\beta+\gamma} \leqq E|\xi|^{2m} \quad \forall x \in D.$$

Now, our first main results read as follows:

THEOREM 1. *Under the condition* $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ *the l-th derivatives of the solution u of the equation (E) belong to the space* $\mathscr{L}_p^{(q,\lambda)}(A)$ *and the following estimate holds for u*:

$$(3) \qquad \sum_{|\beta|\leqq l} \|D^\beta u\|_{\mathscr{L}_p^{(q,\lambda)}(A)} \leqq C\left(\sum_{|\gamma|\leqq l-2m} \|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^p(D)}\right)$$

where C is a constant independent of u.

THEOREM 2. *Under the condition* $(A)'_{\mathscr{L}_p^{(q,\lambda)}}$ *the l-th derivatives of the solution u of the equation $(E)'$ belong to the space* $\mathscr{L}_p^{(q,\lambda)}(A)$ *and the following estimate holds for u*:

$$(4) \qquad \sum_{|\beta|\leqq l} \|D^\beta u\|_{\mathscr{L}_p^{(q,\lambda)}(A)} \leqq C\left(\sum_{|\gamma|\leqq 2m-l} \|f_\gamma\|_{\mathscr{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^p(D)}\right)$$

where C is a constant independent of u.

Here, we remark that throughout this paper we denote by the same letter C constants possibly different but independent of the function u or sometimes v.

2. Proof of theorems 1, 2. Before proceeding to prove Theorem 1, we prepare some lemmas.

LEMMA 1. **[1]** *Let u be a $H^{2m,p}(D)$ solution of the equation (E) under the condition $(A)_{\mathscr{L}_p^{(q,\lambda)}}$. Then u belongs in fact to the space $H^{l,p}(A)$ and we have*

$$(5)\quad \|u\|_{H^{l,p}(A)} \leqq C\left(\sum_{|\gamma|\leqq l-2m} \|D^{\gamma}f\|_{L^p(D)} + \|u\|_{L^p(D)}\right).$$

Here, we note that the space $\mathscr{L}_p^{(q,\lambda)}(D)$ is imbedded into the space $L^p(D)$ and therefore the right hand side of (5) is finite (see Lemma 2 below).

Next, we make the following definition so as to state Lemma 2.

Definition 3. A function $u \in L^p(D)$ is said to *belong to the space* Lip (a, p, D), that is, to satisfy a Lipshitz condition of order a in $L^p(D)$, if the following inequality holds for u:

$$(6)\quad [u]_{\mathrm{Lip}(a,p,D)} =$$

$$\sup_{x,x+h\in D} |h|^{-a+\bar{a}} \left(\int_D |D^{\bar{a}}u(x+h) - D^{\bar{a}}u(x)|^p dx\right)^{1/p} < \infty$$

where $1 \leqq p < \infty$, $0 < a < \infty$ and $\bar{a}$ is the greatest integer less than a. We define a norm $\|u\|_{\mathrm{Lip}(a,p,D)}$ by

$$[u]_{\mathrm{Lip}(a,p,D)} + \|u\|_{L^p(D)}.$$

Endowed with this norm, the space Lip(a, p, D) is a Banach space.

Then, the second lemma which we need is the following:

LEMMA 2. **[12]**. *The space $\mathscr{L}_p^{(q,\lambda)}(D)$ is isomorphic to the space* $\mathrm{Lip}\left(\frac{n}{p} - \frac{\lambda}{q}, p, D\right)$ *and we have*

$$(7)\quad C^{-1}\|u\|_{\mathscr{L}_p^{(q,\lambda)}(D)} \leqq \|u\|_{\mathrm{Lip}\left(\frac{n}{p}-\frac{\lambda}{q},p,D\right)} \leqq C\,\|u\|_{\mathscr{L}_p^{(q,\lambda)}(D)}$$

where p, q and λ are arbitrary constants satisfying $1 < p < \infty$, $1 < q < \infty$, $-q < \lambda < n$ *and* $0 < \frac{n}{p} - \frac{\lambda}{q} < 1$.

Now, we are going to give the

Proof of Theorem 1. From the equation (E), we have

$$\sum_{|\beta|=2m} a_{\beta}(x)D^{\beta}u(x) = -\sum_{|\beta|\leqq 2m-1} a_{\beta}(x)D^{\beta}u(x) + f(x)$$

$$\sum_{|\beta|=2m} a_\beta(x+h)D^\beta u(x+h)$$

$$= - \sum_{|\beta|\leqq 2m-1} a_\beta(x+h)D^\beta u(x+h) + f(x+h)$$

$$\forall x, \forall x+h \in D.$$

Hence, we have easily

$$\sum_{|\beta|=2m} a_\beta(x)D^\beta(u(x+h) - u(x))$$

$$= - \sum_{|\beta|\leqq 2m} (a_\beta(x+h) - \alpha_\beta(x))D^\beta u(x+h)$$

$$- \sum_{|\beta|\leqq 2m-1} a_\beta(x)(D^\beta u(x+h) - D^\beta u(x)) + f(x+h)$$

$$- f(x).$$

Applying Lemma 1 we obtain the following estimate for $u(x+h) - u(x)$:

$$\|u(x+h) - u(x)\|_{H^{l,p}(A)}$$

$$\leqq C\Big\{ \sum_{\substack{|\beta|\leqq 2m \\ |\gamma|\leqq l-2m}} \|D^\gamma\{(a_\beta(x+h) - a_\beta(x))D^\beta u(x+h)\}\|_{L^p(D_1)}$$

$$+ \sum_{\substack{|\beta|\leqq 2m-1 \\ |\gamma|\leqq l-2m}} \|D^\gamma\{a_\beta(x)D^\beta(u(x+h) - u(x))\}\|_{L^p(D_1)}$$

$$+ \sum_{|\gamma|\leqq l-2m} \|D^\gamma(f(x+h) - f(x))\|_{L^p(D_1)}$$

$$+ \|u(x+h) - u(x)\|_{L^p(D_1)}\Big\}$$

for an arbitrary vector h with norm sufficiently small such that $x \in D_1$ and $x + h \in D_2$.

This means that the following inequality holds:

$$\|u(x+h) - u(x))\|_{H^{l,p}(A)}$$

$$\leqq C\Big\{ \sum_{\substack{|\beta|\leqq 2m \\ |\gamma|\leqq l-2m}} \|D^\gamma(a_\beta(x+h) - a_\beta(x))\|_{L^\infty(D_1)} \cdot \|u\|_{H^{l,p}(D_1)}$$

$$+ \sum_{|\beta| \leqq 2m-1} \|a_\beta\|_{C^{l-2m}(D)} \cdot \|u(x+h) - u(x)\|_{H^{l-1,p}(D_1)}$$

$$+ \|f(x+h) - f(x)\|_{H^{l-2m,p}(D_1)} + \|u(x+h) - u(x)\|_{L^p(D_1)} \Big\}.$$

Here, applying Lemma 2 to f, Poincaré's inequality to

$$\|u(x+h) - u(x)\|_{H^{l-1,p}(D_1)}$$

and afterwards Lemma 1 to

$$\|u\|_{H^{l,p}(D_2)},$$

we have

$$\leqq C \Big\{ |h|^a \Big(\sum_{|\beta| \leqq 2m} \|a_\beta\|_{C^{l-2m+a}(D)} + 1 \Big) \|u\|_{L^p(D)}$$

$$+ |h| \Big(\sum_{|\beta| \leqq 2m-1} \|a_\beta\|_{C^{l-2m+a}(D)} + 1 \Big) \|u\|_{L^p(D)}$$

$$+ (|h| + |h|^a) \Big(\sum_{|\beta| \leqq 2m} \|a_\beta\|_{C^{l-2m+a}(D)}$$

$$+ 1 \Big) \sum_{|\gamma| \leqq l-2m} \|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(D)} \Big\}$$

$$\leqq C|h|^a \Big(\sum_{|\gamma| \leqq l-2m} \|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^p(D)} \Big) \Big(a = \frac{n}{p} - \frac{\lambda}{q} \Big).$$

Applying Lemma 2 again to the left hand side, we can conclude that the following estimate holds:

$$\sum_{|\beta| \leqq l} \|D^\beta u\|_{\mathscr{L}_p^{(q,\lambda)}(A)} \leqq C \Big(\sum_{|\gamma| \leqq l-2m} \|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^p(D)} \Big).$$

This completes the proof of Theorem 1.

Next, by calculations which are similar to but need more precision than those of the proof of Theorem 1 we give the

Proof of Theorem 2. For this purpose, we need the following lemma instead of Lemma 1:

LEMMA 3. **[1]** *Let u be a $H^{l,p}(D)$ solution of the equation $(E)'$ under the condition $(A)'_{\mathscr{L}_p^{(q,\lambda)}}$. Then, we have*

$$(8) \qquad \|u\|_{H^{l,p}(A)} \leqq C \Big(\sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{L^p(D)} + \|u\|_{L^p(D)} \Big).$$

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Now, from the equation $(E)'$ we have

$$\sum_{\substack{|\beta|=l \\ |\gamma|\leqq 2m-l}} D^\gamma(a_{\beta\gamma}(x)D^\beta u(x)) = -\sum_{\substack{|\beta|\leqq l-1 \\ |\gamma|\leqq 2m-l}} D^\gamma(a_{\beta\gamma}(x)D^\beta u(x)) + \sum_{|\gamma|\leqq 2m-l} D^\gamma f_\gamma(x)$$

$$\sum_{\substack{|\beta|=l \\ |\gamma|\leqq 2m-l}} D^\gamma(a_{\beta\gamma}(x+h)D^\beta u(x+h))$$

$$= -\sum_{\substack{|\beta|\leqq l-1 \\ |\gamma|\leqq 2m-l}} D^\gamma(a_{\beta\gamma}(x+h)D^\beta u(x+h))$$

$$+ \sum_{|\gamma|\leqq 2m-l} D^\gamma f_\gamma(x+h) \quad \forall x, \forall x+h \in D$$

and therefore

$$\sum_{\substack{|\beta|=l \\ |\gamma|\leqq 2m-l}} D^\gamma\{a_{\beta\gamma}(x)D^\beta(u(x+h) - u(x))\}$$

$$= -\sum_{\substack{|\beta|\leqq l \\ |\gamma|\leqq 2m-l}} D^\gamma\{(a_{\beta\gamma}(x+h) - a_{\beta\gamma}(x))D^\beta u(x+h))\}$$

$$-\sum_{\substack{|\beta|\leqq l-1 \\ |\gamma|\leqq 2m-l}} D^\gamma\{a_{\beta\gamma}(x)D^\beta(u(x+h) - u(x))\}$$

$$+ \sum_{|\gamma|\leqq 2m-l} D^\gamma(f_\gamma(x+h) - f_\gamma(x)).$$

Applying Lemma 3 to $u(x+h) - u(x)$, we have

$$\sum_{|\beta|\leqq l} \|D^\beta(u(x+h) - u(x))\|_{L^p(A)}$$

$$\leqq C\left\{\sum_{\substack{|\beta|\leqq l \\ |\gamma|\leqq 2m-l}} \|(a_{\beta\gamma}(x+h) - a_{\beta\gamma}(x))D^\beta u(x+h)\|_{L^p(D_1)}\right.$$

$$+ \sum_{\substack{|\beta| \leqq l-1 \\ |\gamma| \leqq 2m-l}} \|a_{\beta\gamma}(x) D^\beta (u(x+h) - u(x))\|_{L^p(D_1)}$$

$$+ \sum_{|\gamma| \leqq 2m-l} \|f_\gamma(x+h) - f_\gamma(x)\|_{L^p(D_1)}$$

$$+ \|u(x+h) - u(x)\|_{L^p(D_1)} \Big\}.$$

Using condition $(A)'_{\mathscr{L}_p^{(q,\lambda)}}$ I for the coefficients $\{a_{\beta\gamma}\}$ and applying Poincaré's inequality to $u(x+h) - u(x)$, this is

$$\leqq C|h|^a \Bigg\{ \Bigg(\sum_{\substack{|\beta| \leqq l \\ |\gamma| \leqq 2m-l}} \|a_{\beta\gamma}\|_{C^a(D)} \Bigg) \cdot \|u\|_{H^{l,p}(D_2)}$$

$$+ |h|^{1-a} \Bigg(\sum_{\substack{|\beta| \leqq l \\ |\gamma| \leqq 2m-l}} \|a_{\beta\gamma}\|_{C^a(D)} \Bigg) \cdot \|u\|_{H^{l,p}(D_2)}$$

$$+ \sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{\mathrm{Lip}(a,p,D_2)} + |h|^{1-a} \|u\|_{H^{1,p}(D_2)} \Bigg\} \quad \left(a = \frac{n}{p} - \frac{\lambda}{q} \right)$$

$$\leqq C|h|^a \Bigg(\sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{\mathrm{Lip}(a,p,D_2)} + \|u\|_{H^{l,p}(D_2)} \Bigg).$$

Furthermore, applying Lemmas 2 and 3 to the first and second terms respectively, we have

$$\sum_{|\beta| \leqq l} \|D^\beta (u(x+h) - u(x))\|_{L^p(A)}$$

$$\leqq C|h|^a \Bigg(\sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{\mathscr{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^p(D)} \Bigg).$$

Applying Lemma 2 again to the left hand side of this inequality, we have

$$\sum_{|\beta| \leqq l} \|D^\beta u\|_{\mathscr{L}_p^{(q,\lambda)}(A)}$$

$$\leqq C \Bigg(\sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{\mathscr{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^p(D)} \Bigg).$$

Hence, the proof of Theorem 2 is complete.

3. Schauder estimates. We begin this section with the following:

Definition 4. We say that a function $u \in C^{k+\alpha}(D)$ *belongs to the space* $\mathscr{H}_p^{k+\alpha}(D)$ (the Hölder space of strong type p with exponent $k + \alpha$) if the following condition is satisfied:

$$[u]_{\mathscr{H}_p^{k+\alpha}(D)} = \sup_{\{Q_j\} \in \bar{S}} \left(\sum_j \sum_{|\beta|=k} [D^\beta u]^p_{C^\alpha(Q_j)} \right)^{1/p} < \infty$$

where k, α, p are constants such that $0 \leqq k =$ integer, $0 < \alpha < 1$, $1 < p < \infty$ and in addition $\bar{S}$ is the family of all systems of subcubes of D considered in Definition 1. We ensure that the Space $\mathscr{H}_p^{k+\alpha}(D)$ is a Banach space by taking as the norm

$$\|u\|_{\mathscr{H}_p^{k+\alpha}(D)} = [u]_{\mathscr{H}_p^{k+\alpha}(D)} + \max_{x \in D} |u(x)|.$$

Now, in place of the fundamental condition $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ on (E) or $(A)'_{\mathscr{L}_p^{(q,\lambda)}}$ on $(E)'$ we propose the following:

$(A)_{\mathscr{H}_p^\alpha}$ I. The coefficients $\{a_\beta\}$ and the function f belong to the space $\mathscr{H}_p^{l-2m+\alpha}(D)$.

II. Same as the condition $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ II.

$(A)'_{\mathscr{H}_p^\alpha}$ I. The coefficients $[a_{\beta\gamma}\}$ and the functions $\{f_\gamma\}$ belong to the space $\mathscr{H}_p^\alpha(D)$.

II. Same as the condition $(A)'_{\mathscr{L}_p^{(q,\lambda)}}$ II.

Then, the strong Hölder continuity of the derivatives of the solution read as follow:

THEOREM 3. *Under the condition* $(A)_{\mathscr{H}_p^\alpha}$, *the* $H^{2m,p}(D)$ *solution* u *of the equation* (E) *is in fact* $\mathscr{H}_p^{l+\alpha}(A)$ *solution of* (E) *and the following estimate holds for* u:

$$(9) \quad \|u\|_{\mathscr{H}_p^{l+\alpha}(A)} \leqq C(\|f\|_{\mathscr{H}_p^{l-2m+\alpha}(D)} + \|u\|_{L^p(D)}).$$

THEOREM 4. *Under the condition* $(A)'_{\mathscr{H}_p^\alpha}$, *the* $H^{l,p}(D)$ *solution* u *of the equation* $(E)'$ *is in fact* $\mathscr{H}_p^{l+\alpha}(A)$ *solution of* $(E)'$ *and the following estimate holds for* u:

$$(10) \quad \|u\|_{\mathscr{H}_p^{l+\alpha}(A)} \leqq C\left(\sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{\mathscr{H}_p^\alpha(D)} + \|u\|_{L^p(D)} \right).$$

For the proof of these theorems we prepare the following lemmas:

LEMMA 4. [**1**] *Let* u *be a* $H^{2m,p}(D)$ *solution of the equation* (E) *under the condition* $(A)_{\mathscr{H}_p^\alpha}$. *Then,* u *is in fact a* $C^{l+\alpha}(A)$ *solution of the equation* (E) *and we have*

$$(11) \quad \|u\|_{C^{l+\alpha}(A)} \leqq C(\|f\|_{C^{l-2m+\alpha}(D)} + \|u\|_{L^p(D)}).$$

LEMMA 5. [**1**] *Let u be a* $H^{l,p}(D)$ *solution of the equation* $(E)'$ *under the condition* $(A)'_{\mathscr{H}_p^\alpha}$. *Then, u is in fact a* $C^{l+\alpha}(A)$ *solution of the equation* $(E)'$ *and we have*

$$(12) \quad \|u\|_{C^{l+\alpha}(A)} \leqq C\Big(\sum_{|\gamma|\leqq 2m-l} \|f\gamma\|_{C^\alpha(D)} + \|u\|_{L^p(D)}\Big).$$

LEMMA 6. *The space* $\mathscr{H}_p^{l+\alpha}(D)$ *is isomorphic to the space* $\text{Lip}\left(l + \alpha + \frac{n}{p}, p, D\right)$ *and we have*

$$(13) \quad C^{-1}\,\|u\|_{\mathscr{H}_p^{l+\alpha}(D)} \leqq \|u\|_{\text{Lip}\left(l+\alpha+\frac{n}{p},p,D\right)} \leqq C\,\|u\|_{\mathscr{H}_p^{l+\alpha}(D)}$$

where

$$0 < \alpha < 1, \quad 1 < p < \infty \quad \text{and} \quad \alpha + \frac{n}{p} \leqq 1.$$

We can prove this lemma by a procedure analogous to the proof of Theorem 1 in [**12**] and with the aid of the following:

LEMMA 7. ($-q < \lambda < 0$ [**2**, **7**]; $\lambda = 0$ [**6**]; $0 < \lambda < n$ [**14**]). *The space* $\mathscr{L}_p^{(q,\lambda)}(D)$ *is isomorphic to the space* $\mathscr{L}_p^{\left(1,\frac{\lambda}{q}\right)}(D)$ *and we have*

$$(14) \quad C^{-1}\,\|v\|_{\mathscr{L}_p^{(q,\lambda)}(D)} \leqq \|v\|_{\mathscr{L}_p^{(1,(\lambda/q))}(D)} \leqq \|v\|_{\mathscr{L}_p^{(q,\lambda)}(D)}$$

where λ *is an arbitrary constant indicated above and p, q are constants satisfying*

$$1 \leqq p < \infty, \quad 1 < q < \infty \quad \text{and} \quad \frac{\lambda}{q} \leqq \frac{n}{p}.$$

In particular, for the case of $-q < \lambda < 0$ *we may take* $\mathscr{H}_p^\alpha(D)$ *in place of* $\mathscr{L}_p^{\left(1,\frac{\lambda}{q}\right)}(D)$; *where* $0 < \alpha = -\lambda/q < 1$, *and an analogous inequality to* (14) *holds.*

Now, the proof of Theorem 3 is analogous to and simpler than that of Theorem 4 and therefore we give only the

Proof of Theorem 4. By the same calculations as in the proof of Theorem 2 and applying Lemma 3 to $u(x + h) - u(x)$, we have

$$\|u(x + h) - u(x)\|_{H^{l,p}(A)}$$

$$\leqq C\Bigg\{\sum_{\substack{\|\beta\|\leqq l \\ |\gamma\|\leqq 2m-l}} \|(a_{\beta\gamma}(x + h) - a_{\beta\gamma}(x))D^\beta u\|_{L^p(D_1)}$$

$$+ \sum_{\substack{|\beta| \leqq l-1 \\ |\gamma| \leqq 2m-l}} \| a_{\beta\lambda}(x) D^{\beta}(u(x+h) - u(x)) \|_{L^p(D_1)}$$

$$+ \sum_{|\gamma| \leqq 2m-l} \| f_{\gamma}(x+h) - f_{\gamma}(x) \|_{L^p(D_1)}$$

$$+ \| u(x+h) - u(x) \|_{L^p(D_1)} \Big\}.$$

Furthermore, utilizing Poincaré's inequality and Lemma 5 to u this is

$$\leqq C(\|u\|_{C^{l+\alpha}(D_1)} \cdot \sum_{\substack{|\beta| \leqq l \\ |\gamma| \leqq 2m-l}} \| a_{\beta\gamma}(x+h) - a_{\beta\gamma}(x) \|_{L^p(D_1)}$$

$$+ |h| \, \|u\|_{C^{l+\alpha}(D_2)} \cdot \sum_{\substack{|\beta| \leqq l-1 \\ |\gamma| \leqq 2m-l}} \| a_{\beta\gamma} \|_{L^p(D_1)}$$

$$+ \sum_{|\gamma| \leqq 2m-l} \| f_{\gamma}(x+h) - f_{\gamma}(x) \|_{L^p(D_1)} + |h| \, \|u\|_{C^{l+\alpha}(D_2)}).$$

Applying Lemma 5 to $\|u\|_{C^{l+\alpha}}$ again we have

$$\leqq C[\Big\{ \sum_{|\gamma| \leqq 2m-l} \| f_{\gamma} \|_{C^{\alpha}(D)} + \|u\|_{L^p(D)} \Big\}$$

$$\times \Big\{ \sum_{\substack{|\beta| \leqq l \\ |\gamma| \leqq 2m-l}} \| a_{\beta\gamma}(x+h) - a_{\beta\gamma}(x) \|_{L^p(D_1)}$$

$$+ |h| \Big(\sum_{\substack{|\beta| \leqq l-1 \\ |\gamma| \leqq 2m-l}} \| a_{\beta\gamma} \|_{L^p(D_1)} + 1 \Big) \Big\}$$

$$+ \sum_{|\gamma| \leqq 2m-l} \| f_{\gamma}(x+h) - f_{\gamma}(x) \|_{L^p(D_1)}].$$

Applying Lemma 6 to $\{a_{\beta\gamma}\}$ and $\{f_{\gamma}\}$ taking account of the condition $(A)'_{\mathscr{H}_p^{\alpha}}$ I, this is

$$\leqq C\Big\{ \Big(\sum_{|\gamma| \leqq 2m-l} \| f_{\gamma} \|_{C^{\alpha}(D)} + \|u\|_{L^p(D)} \Big)$$

$$\times \left(\sum_{\substack{|\beta| \leqq l \\ |\gamma| \leqq 2m-l}} \|a_{\beta\gamma}\|_{\mathrm{Lip}\left(\alpha+\frac{n}{p},p,D_2\right)} + 1 \right)$$

$$\left. + \sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{\mathrm{Lip}\left(\alpha+\frac{n}{p},p,D_2\right)} \right\} |h|^{\alpha+\frac{n}{p}}.$$

Applying Lemma 6 to the last term again, and furthermore noting that

$$\{ \|f_\gamma\|_{C^\alpha(D)} \}_{|\gamma| \leqq 2m-l}$$

are obviously majorized by

$$\{ \|f_\gamma\|_{\mathscr{H}_p^\alpha(D)} \}_{|\gamma| \leqq m-l},$$

we obtain the following inequality:

$$\|u(x + h) - u(x)\|_{H^{l,p}(A)} \leqq$$

$$C\left(\sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{\mathscr{H}_p^\alpha(D)} + \|u\|_{L^p(D)} \right) |h|^{\alpha+\frac{n}{p}}.$$

Hence, we can conclude that u belongs to the space Lip $\left(l + \alpha + \frac{n}{p}\right.$, $\left.p, A\right)$ and therefore to the space $\mathscr{H}_p^{l+\alpha}(A)$ by Lemma 6, and the estimate (10) holds for u.

The proof of Theorem 4 is complete.

4. Applications of Morrey-Sobolev type imbedding theorems. We have proved the following theorem:

LEMMA 8. [**12**]. *Let v be a function such that the derivatives v_x belong to the space $\mathscr{L}_p^{(q,\lambda)}(A)$, where p, q and λ are constants such that* $1 < p < \infty$, $1 < q < \infty$, $0 < \lambda < n$ *and* $0 < \frac{n}{p} - \frac{\lambda}{q} < 1$.

Then, the following estimates hold for v:

(i) $q < \lambda$; *v belongs to the space $\mathscr{L}_r^{(\tilde{q},\lambda)}(A)$ and*

(15) $\quad [v]_{\mathscr{L}_r^{(\tilde{q},\lambda)}(A)} \leqq C \|v_x\|_{\mathscr{L}_p^{(q,\lambda)}(A)}$

where $\frac{1}{\tilde{q}} = \frac{1}{q} - \frac{1}{\lambda}$ *and r is an arbitrary constant greater than nq/λ.*

(ii) $q = \lambda$; *v belongs to the space $\mathscr{L}_r^{(1,0)}(A)$ and*

(16) $\quad [v]_{\mathscr{L}_r^{(1,0)}(A)} \leqq C \|v_x\|_{\mathscr{L}_p^{(q,\lambda)}(A)}$

where r is as in (i).

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(iii) $q > \lambda$; *v belongs to the space* $\mathscr{H}_r^{1-\frac{\lambda}{q}}(A)$ *and*

(17) $\quad [v]_{\mathscr{H}_r^{1-(\lambda/q)}(A)} \leqq C\, \|v_x\|_{\mathscr{L}_p^{(q,\lambda)}(A)}$

where r is as in (i).

Here, we remark that throughout the remainder of this paper, we denote always by $\tilde{q}$ the constant defined in (i) of this lemma.

Combining this lemma and Theorems 1, 2, we can deduce the following:

THEOREM 5. *Let u be an $H^{l,p}(D)$ solution of the equation (E) or $(E)'$ under the condition $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ or $(A)'_{\mathscr{L}_p^{(q,\lambda)}}$ respectively. Then, the following estimates hold for u*:

(i) $q < \lambda$; *the derivatives* $\{D^\beta u\}_{|\beta| \leqq l-1}$ *belong to the space* $\mathscr{L}_r^{(\tilde{q},\lambda)}(A)$ *and we have*

$$(18) \quad \sum_{|\beta| \leqq l-1} \|D^\beta u\|_{\mathscr{L}_r^{(\tilde{q},\lambda)}(A)}$$

$$\leqq C\left(\sum_{|\gamma| \leqq l-2m} \|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^p(D)} \right)$$

in the case of (E), *or*

$$\leqq C\left(\sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{\mathscr{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^p(D)} \right)$$

in the case of $(E)'$, *where $\tilde{q}$, r are as in* (i) *of the preceding lemma.*

(ii) $q = \lambda$; *the derivatives* $\{D^\beta u\}_{|\beta| \leqq l-1}$ *belong to the space* $\mathscr{L}_r^{(1,0)}(A)$ *and we have*

$$(19) \quad \sum_{|\beta| \leqq l-1} \|D^\beta u\|_{\mathscr{L}_r^{(1,0)}(A)} \leqq \text{the right hand side of } (18)$$

where r is as in (i).

(iii) $q > \lambda$; *u belongs to the space* $\mathscr{H}_r^{l-\frac{\lambda}{q}}(A)$ *and we have*

(20) $\quad \|u\|_{\mathscr{H}_r^{l-(\lambda/q)}(A)} \leqq$ *the right hand side of* (18)

where r is as in (i).

Proof. By taking $v_x = \{D^\beta u\}_{|\beta|=l}$ and applying the preceding lemma the conclusion is immediate with the aid of Theorems 1, 2.

Next, we shall prove that analogous results to Theorem 5 hold under weaker conditions than the condition $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ I or $(A)'_{\mathscr{L}_p^{(q,\lambda)}}$ I.

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For this purpose, we make at first the following condition on (E) instead of the condition $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ I:

$(A)_{\mathscr{L}_p^{(q,\lambda)}}$ I′. $l \geqq 2m + 1, \dfrac{n}{p} < \dfrac{\lambda}{q}$ and the coefficients $\{a_\beta\}_{|\beta| \leqq 2m}$ and the functions $\{D^\gamma f\}_{|\gamma| \leqq l-2m}$ belong to the spaces $C^{l-2m+\frac{n}{p}-\frac{\lambda}{q}}(D)$ and $\mathscr{L}_p^{(q,\lambda)}(D)$ respectively.

Although the condition $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ I′ is of the same type as the condition $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ I, these two conditions are essentially different: namely,

LEMMA 9. *The condition* $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ I′ *means the following*:

(i) $q < \lambda$ *and* $\dfrac{\lambda}{\tilde{q}} < \dfrac{n}{p} < \dfrac{\lambda}{q}$; $\{a_\beta\}_{|\beta| \leqq 2m}$ *and* $\{D^\gamma f\}_{|\gamma| \leqq l-2m-1}$ *belong to the spaces*

$$C^{l-2m-1+\frac{n}{p}-\frac{\lambda}{\tilde{q}}}(D) \textit{ and } \mathscr{L}_p^{(\tilde{q},\lambda)}(D)$$

respectively and we have

$$(21) \qquad \sum_{|\gamma| \leqq l-2m-1} \|D^\gamma f\|_{\mathscr{L}_p^{(\tilde{q},\lambda)}(D)} \leqq C \sum_{|\gamma| \leqq l-2m} \|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(D)}.$$

(ii) $q = \lambda$; $\{a_\beta\}_{|\beta| \leqq 2m}$ *and* $\{D^\gamma f\}_{|\gamma| \leqq l-2m-1}$ *belong to the spaces*

$$C^{l-2m-1+\frac{n}{p}}(D) \textit{ and } \mathscr{L}_p^{(1,0)}(D)$$

respectively and we have

$$(22) \qquad \sum_{|\gamma| \leqq l-2m-1} \|D^\gamma f\|_{\mathscr{L}_p^{(1,0)}(D)} \leqq C \sum_{|\gamma| \leqq l-2m} \|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(D)}.$$

(iii) $q > \lambda$; $\{a_\beta\}_{|\beta| \leqq 2m}$ *and* f *belong to the spaces*

$$C^{l-2m-1+\left(\frac{n}{p}+1-\frac{\lambda}{q}\right)}(D) \textit{ and } \mathscr{H}_p^{l-2m-\frac{\lambda}{q}}(D)$$

respectively and we have

$$(23) \qquad \|f\|_{\mathscr{H}_p^{l-2m-(\lambda/q)}(D)} \leqq C \sum_{|\gamma| \leqq l-2m} \|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(D)}.$$

Proof. We note at first the following equality:

$$l - 2m + \frac{n}{p} - \frac{\lambda}{q} = l - 2m - 1 + \left(\frac{n}{p} - \frac{\lambda}{q} + 1\right)$$

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$$= l - 2m - 1 + \frac{n}{p} - \frac{\lambda}{\tilde{q}} \quad \text{in the case of (i)}$$

$$= l - 2m - 1 + \frac{n}{p} \quad \text{in the case of (ii)}$$

and afterwards we refer the following theorem for the proof of the inequalities (21)-(23).

LEMMA 10. [**12**] *Let v be a function such that the derivatives v_x belong to the space $\mathscr{L}_p^{(q,\lambda)}(D)$, where p, q and λ are constants such that $1 < p < \infty$, $1 < q < \infty$, $0 < \lambda < n$ and $\frac{n}{p} < \frac{\lambda}{q}$. Then, the following estimates hold for v:*

(i) *$q < \lambda$ and $\frac{\lambda}{\tilde{q}} < \frac{n}{p} < \frac{\lambda}{q}$; v belongs to the space $\mathscr{L}_p^{(\tilde{q},\lambda)}(D)$ and we have*

(24) $[v]_{\mathscr{L}_p^{(\tilde{q},\lambda)}(D)} \leqq C\,\|v_x\|_{\mathscr{L}_p^{(q,\lambda)}(D)}.$

(ii) *$q = \lambda$; v belongs to the space $\mathscr{L}_p^{(1,0)}(D)$ and we have*

(25) $[v]_{\mathscr{L}_p^{(1,0)}(D)} \leqq C\,\|v_x\|_{\mathscr{L}_p^{(q,\lambda)}(D)}.$

(iii) *$q > \lambda$; v belongs to the space $\mathscr{H}_p^{1-\frac{\lambda}{q}}(D)$ and we have*

(26) $[v]_{\mathscr{H}_p^{1-(\lambda/q)}(D)} \leqq C\,\|v_x\|_{\mathscr{L}_p^{(q,\lambda)}(D)}.$

Now, our last main result is the following:

THEOREM 6. *Under the conditions $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ I′ and II, $H^{l,p}(D)$ solution ($l \geqq 2m + 1$) u of the equation (E) satisfies the following estimates:*

(i) *$q < \lambda$ and $\frac{\lambda}{\tilde{q}} < \frac{n}{p} < \frac{\lambda}{q}$; $\{D^\beta u\}_{|\beta| \leqq l-1}$ belong to the space $\mathscr{L}_p^{(\tilde{q},\lambda)}(A)$ and we have*

$$(27) \quad \sum_{|\beta| \leqq l-1} \|D^\beta u\|_{\mathscr{L}_p^{(\tilde{q},\lambda)}(A)} \leqq C \left(\sum_{|\gamma| \leqq l-2m} \|D^\gamma f\|_{\mathscr{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^p(D)} \right).$$

(ii) *$q = \lambda$; $\{D^\beta u\}_{|\beta| \leqq l-1}$ belong to the space $\mathscr{L}_p^{(1,0)}(A)$ and we have*

$$(28) \quad \sum_{|\beta| \leqq l-1} \|D^\beta u\|_{\mathscr{L}_p^{(1,0)}(A)} \leqq \text{the right hand side of (27).}$$

(iii) *$q > \lambda$; u belongs to the $\mathscr{H}_p^{l-\frac{\lambda}{q}}(A)$ and we have*

(29) $\|u\|_{\mathscr{H}_p^{l-(\lambda/q)}(A)} \leqq$ *the right hand side of* (27).

Proof. (i) By taking $l - 1$ ($\geqq 2m$), $\tilde{q}$ in place of l, q in Theorem 1 respectively, we can easily verify the following inequality:

$$\sum_{|\beta| \leqq l-1} ||D^{\beta}u||_{\mathscr{L}_p^{(\tilde{q},\lambda)}(A)} \leqq C\left(\sum_{|\gamma| \leqq l-2m-1} ||D^{\gamma}f||_{\mathscr{L}_p^{(\tilde{q},\lambda)}(D)} + ||u||_{L^p(D)}\right).$$

And applying Lemma 9 (i) to the first term of the right hand side of the above inequality, we can conclude that the estimate (27) holds for u, which completes the proof of the case (i).

By similar arguments as in the proof of the case (i), we can deduce the estimates (28), (29) for the cases (ii), (iii) respectively.

Hence, the proof of this theorem is complete.

Moreover, we can assert that there holds the following:

PROPOSITION 1. *Let u be an $H^{l,p}(D)$ solution ($l \geqq 2m + 1$) of the equation (E) under the condition $(A)_{\mathscr{L}_p^{(q,\lambda)}}$ with $\frac{n}{p} = \frac{\lambda}{q}$. Then, the estimates of the same type as in Theorem 5 hold for u.*

Proof. Let r be an arbitrary constant greater than $p = nq/\lambda$. Then, using Theorem 6, the estimates (27)-(29) hold for u with r in place of p.

Hence, the conclusion follows directly by using the well known inequality (see [**17**]):

$$||D^{\gamma}f||_{\mathscr{L}_r^{(q,\lambda)}(D)} \leqq ||D^{\gamma}f||_{\mathscr{L}_p^{(q,\lambda)}(D)}.$$

Remark 1. This proposition asserts that Theorem 5 concerning the equation (E) ($l \geqq 2m + 1$) holds as long as $\frac{\lambda}{q} \leqq \frac{n}{p} < \frac{\lambda}{q} + 1$, that is, it is still valid for the limiting case.

Now, we terminate this section by showing a proposition concerning the limiting case of the equation (E)′ as follows:

PROPOSITION 2. *Let u be an $H^{l,p}(D)$ solution of (E)′ ($1 \leqq l \leqq 2m$) under the condition $(A)'_{\mathscr{L}_p^{(q,\lambda)}}$ with $p = nq/\lambda$. Then, the following estimates hold for u:*

(i) $q \leqq \lambda$; *$\{D^{\beta}u\}_{|\beta| \leqq l-1}$ belong to the space $\mathscr{L}_{p_1}^{(\tilde{q}_1,\lambda)}(A)$ and*

$$(30) \quad \sum_{|\beta| \leqq l-1} ||D^{\beta}u||_{\mathscr{L}_{p_1}^{(\tilde{q}_1,\lambda)}(A)} \leqq C\left(\sum_{|\gamma| \leqq 2m-l} ||f_{\gamma}||_{\mathscr{L}_p^{(q,\lambda)}(D)} + ||u||_{L^p(D)}\right)$$

where p_1, q_1 and $\tilde{q}_1$ are constants such that

$$1 < p_1 < p = \frac{ng}{\lambda},\ q_1 = \frac{\lambda p_1}{n} > 1 \quad \text{and} \quad \frac{1}{\tilde{q}_1} = \frac{1}{q_1} - \frac{1}{\lambda}.$$

(ii) $q > \lambda$; *u belongs to the space $\mathscr{H}_{p_1}^{l-\frac{\lambda}{q_1}}(A)$ and*

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(31) $\|u\|_{\mathscr{H}_{p_1}^{l-(\lambda/q_1)}(A)} \leqq$ *the right hand side of* (30)

where p_1 *and* q_1 *are as in* (i).

For the proof of this proposition, we make use of the following lemmas.

LEMMA 11. [**6**]. *Let* v *be a function belonging to the space* $\mathscr{L}_p^{\left(1,\frac{\lambda}{q}\right)}(D)$ $(p = nq/\lambda)$. *Then, in fact* v *belongs to the space* $L^{p_1}(D)$ *for any constant such as* $1 < p_1 < p$ *and*

(32) $\|v\|_{L^{p}(D)} \leqq C\, \|v\|_{\mathscr{L}_p^{(1,(\lambda/q))}(D)}$.

LEMMA 12. [**11**]. *The space* $\mathscr{L}_{p_1}^{(p_1,n-p_1)}(A)$ *is isomorphic to the Sobolev space* $H^{1,p_1}(A)$ *and*

(33) $C^{-1}\|v\|_{H^{1,p_1}(A)} \leqq \|v\|_{\mathscr{L}_{p_1}^{(p_1,n-p_1)}(A)} \leqq C\|v\|_{H^{1,p_1}(A)}$.

Proof of Proposition 2. We give the proof for the cases (i) and (ii) simultaneously.

As the functions $\{f_\gamma\}_{|\gamma|\leqq 2m-l}$ belong to the space $L^{p_1}(D)$ by Lemma 11, the following estimate holds for u by Lemma 3:

$$\sum_{|\beta|\leqq l} \|D^\beta u\|_{L^{p_1}(A)} \leqq C\left(\sum_{|\lambda|\leqq 2m-l} \|f_\gamma\|_{L^{p_1}(D)} + \|u\|_{L^{p_1}(D)}\right).$$

This means that the functions $\{D^\beta u\}_{|\beta|\leqq l-1}$ belong to the Sobolev space $H^{1,p_1}(A)$ and therefore to the space $\mathscr{L}_{p_1}^{(p_1,n-p_1)}(A)$ by Lemma 12. Here, we note that p_1 is equal to nq_1/λ $(1 < q_1 < q)$ and therefore

$$\mathscr{L}_{p_1}^{(p_1,n-p_1)}(A) = \mathscr{L}_{p_1}^{\left(\frac{nq_1}{\lambda},\frac{nq_1}{\lambda}\left(\frac{\lambda}{q_1}-1\right)\right)}(A)$$

$$\cong \mathscr{L}_{p_1}^{(\tilde{q}_1,\lambda)}(A) \text{ in the case of (i)}$$

or

$$\cong \mathscr{H}_{p_1}^{1-\frac{\lambda}{q_1}}(A) \text{ in the case of (ii)}.$$

The last isomorphism relations follow from Lemma 7.

Hence, the proof is complete with the aid of the following inequality (see (32)):

$$\sum_{|\gamma|\leqq 2m-l} \|f_\gamma\|_{L^{p_1}(D)} + \|u\|_{L^{p_1}(D)}$$

$$\leqq C\left(\sum_{|\gamma|\leqq 2m-l} \|f_\gamma\|_{\mathscr{L}_p^{(q,\lambda)}(D)} + \|u\|_{L^{p}(D)}\right).$$

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Remark 2. If we take the strong Morrey space $L_p^{(q,\lambda)}(D)$ ($p = nq/\lambda$; for the definition see [**16**]) instead of space $\mathscr{L}_p^{(q,\lambda)}(D)$, then we can obtain slightly stronger results than those of Proposition 2, which are closely analogous to those of Theorem 5; namely, the following estimates hold for u:

(i) $q < \lambda$; $\{D^\beta u\}_{|\beta| \leqq l-1}$ belong to the space $\mathscr{L}_p^{(\tilde{q},\lambda)}(A)$ and

$$(34) \quad \sum_{|\beta| \leqq l-1} \|D^\beta u\|_{\mathscr{L}_p^{(\tilde{q},\lambda)}(A)} \leqq C \left(\sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{L_p^{(q,\lambda)}(D)} + \|u\|_{L^p(D)} \right).$$

(ii) $q = \lambda$; $\{D^\beta u\}_{|\beta| \leqq l-1}$ belong to the space $\mathscr{L}_p^{(1,0)}(A)$ and

$$(35) \quad \sum_{|\beta| \leqq l-1} \|D^\beta u\|_{\mathscr{L}_p^{(1,0)}(A)} \leqq \text{the right hand side of (34).}$$

(iii) $q > \lambda$; u belongs to the space $\mathscr{H}_p^{l-\frac{\lambda}{q}}(A)$ and

$$(36) \quad \|u\|_{\mathscr{H}_p^{(l-(\lambda/q))}(A)} \leqq \text{the right hand side of (34).}$$

Actually, the space $L_p^{(q,\lambda)}(D)$ ($p = nq/\lambda$) is isomorphic to the space $L^p(D)$ (for the proof, see [**16**] or [**17**]) and the conclusion is immediate by applying Lemmas 3, 12 and 7 successively, as in the proof of Proposition 2.

5. Comments on the theorems.

1. According to [**3**], we make the following:

Definition 5. Let X be a normed function space. Then, a function ζ is said to *belong to the multiplicator space on* X:$M(X)$, if the following inequality holds for an arbitrary function v belonging to the space X:

$$(37) \quad \|\zeta v\|_X \leqq C \|v\|_X.$$

Therefore, we take the multiplicator spaces fairly wide so as to deduce the preceding theorems as follows:

THEOREM 1. $X = \{u; \{D^\beta u\}_{|\beta| \leqq l} \in \mathscr{L}_p^{(q,\lambda)}(A)\}$; $M(X) = C^{l-2m+\frac{n}{p}-\frac{\lambda}{q}}(D)$.

THEOREM 2. $X =$ *same as Theorem* 1; $M(X) = C^{\frac{n}{p}-\frac{\lambda}{q}}(D)$.

THEOREM 3. $X = \mathscr{H}_p^{l+\alpha}(A)$; $M(X) = \mathscr{H}_p^{l-2m+\alpha}(D)$.

THEOREM 4. $X =$ *same as Theorem* 3; $M(X) = \mathscr{H}_p^{\alpha}(D)$.

THEOREM 5. *the same type as Theorems* 1, 2.

THEOREM 6. *the same type as Theorem* 1.

Here, we make the following:

Remark 3. If $-q < \lambda < 0$ in Theorems 1, 2, then we may set

$$\alpha = -\frac{\lambda}{q} \quad (0 < \alpha < 1)$$

and therefore

$$M(X) = C^{l-2m+\alpha+\frac{n}{p}}(D),\ C^{\alpha+\frac{n}{p}}(D) \quad \text{respectively.}$$

On the other hand, in Theorems 3 and 4 $M(X) = \mathscr{H}_p^{l-2m+\alpha}(D)$ and $\mathscr{H}_p^{\alpha}(D)$ which are isomorphic to the spaces $\mathrm{Lip}(l - 2m + \alpha + \frac{n}{p}, p, D)$ and $\mathrm{Lip}(\alpha + \frac{n}{p}, p, D)$ respectively by Lemma 6. Obviously, these spaces are wider than the spaces $C^{l-2m+\alpha+\frac{n}{p}}(D)$ and $C^{\alpha+\frac{n}{p}}(D)$ with their corresponding norms respectively. Hence, Theorems 3, 4 give more precise estimates for negative λ than Theorems 1, 2 because of Lemmas 4-6.

2. If $\frac{n}{p} - \frac{\lambda}{q}$ is equal to or greater than unity, then we can improve some of the preceding theorems as follow:

THEOREMS 1, 2. *The solution u belongs to the Sobolev space* $H^{l+1,p}(A)$ *and we have*

$$(38) \quad \|u\|_{H^{l+1,p}(A)} \leqq C\left(\|f\|_{H^{l-2m+1,p}(D)} + \|u\|_{L^p(D)}\right)$$

for the equation (E) *or*

$$\leqq C\left(\sum_{|\gamma| \leqq 2m-l} \|f_\gamma\|_{H^{1,p}(D)} + \|u\|_{L^p(D)}\right)$$

for the equation $(E)'$.

THEOREM 5. (i) $q < \lambda$; *the derivatives* $\{D^\beta u\}_{|\beta| \leqq l-1}$ *belong to the space* $\mathscr{L}_{p^*}^{(\bar{q},\lambda)}(A)$ *and we have*

$$(39) \quad \sum_{|\beta| \leqq l-1} \|D^\beta u\|_{\mathscr{L}_{p^*}^{(\bar{q},\lambda)}(A)} \leqq \textit{the right hand side of } (38)$$

where p^* *is the Sobolev's exponent; that is,*

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n} \geqq \frac{\lambda}{nq}.$$

(ii) $q = \lambda$; *the derivatives* $\{D^\beta u\}_{|\beta| \leqq l-1}$ *belong to the space* $\mathscr{L}_{p^*}^{(1,0)}(A)$ *and we have*

$$(40) \quad \sum_{|\beta| \leqq l-1} \|D^\beta u\|_{\mathscr{L}_{p^*}^{(1,0)}(A)} \leqq \textit{the right hand side of } (38)$$

where p^* *is as in* (i).

(iii) $q > \lambda$; *u belongs to the space* $\mathscr{H}_{p^*}^{l-\frac{\lambda}{q}}(A)$ *and we have*

(41) $\|u\|_{\mathscr{H}_{p^*}^{l-(\lambda/q)}(A)} \leqq$ *the right hand side of* (38)

where p^* *is as in* (i).

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Kyushu University,
Fukuoka, Japan