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Theorems in the Products of Related Quantities.

By F. H. JACKSON, M.A.

§ 1. The object of this Note is to prove the following theorems:

$$\left.\begin{aligned}(a+b)_{-n} = a_{-n} - na_{-n-1}b_1 + \frac{n\cdot n+1}{2\,!}a_{-n-2}b_2 \\ - \frac{n\cdot n+1\cdot n+2}{3\,!}a_{-n-3}b_3 + \cdots\end{aligned}\right\} \quad (1)$$

in which $a_{-n} = \dfrac{1}{a+n\cdot a+n-1\cdots\cdot a+1\cdot}$ and the series is subject to conditions for convergence.

$$\left.\begin{aligned}\frac{P(0)}{x+n} - {}_nC_1\frac{P(y)}{x+n-1} + {}_nC_2\frac{P(2y)}{x+n-2} - \cdots \text{ to } n+1 \text{ terms} \\ \equiv (-1)^n\frac{x\,!\,n\,!}{x+n\,!}\cdot\frac{P(xy+ny)}{x}\end{aligned}\right\} \quad (2)$$

$$P(0) - {}_nC_1P(y) + {}_nC_2P(2y) - \cdots \text{ to } n+1 \text{ terms} \quad \equiv 0 \quad (3)$$

$$N(0) - {}_nC_1N(y) + {}_nC_2N(2y) - \cdots \quad ,, \quad ,, \quad ,, \quad \equiv (-y)^n.\,n\,! \quad (4)$$

in which $P(y) \equiv (a+y)(b+y)(c+y) \cdots$ to p factors

$N(y) \equiv (a+y)(b+y)(c+y) \ldots$ to n factors

and the quantities $abcd\cdots x$ are unrestricted. In theorem (2) $p \lesseqgtr n$, but in (3) $p<n$

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The following theorems are derived from the above

$$\left.\begin{aligned}\frac{(a)_p}{x} - {}_nC_1\frac{(a-1)_p}{x-1} + {}_nC_2\frac{(a-2)_p}{x+2} - \cdots\cdots\cdots + (-1)^n\frac{(a-n)_p}{x+n} \\ \equiv \frac{x\,!\,n\,!}{x+n\,!}\;\frac{(x+a)_p}{x}\end{aligned}\right\} \quad (5)$$

$$\frac{(a)_p}{x} - {}_nC_1\frac{(a+1)_p}{x-1} + \cdots\cdots\cdots\cdots \equiv (-1)^n\frac{x-n\,!\,n\,!}{x\,!}\;\frac{(x+a)_p}{x-n} \quad (6)$$

$$(p \lessgtr n)$$

$$(x)_n - {}_nC_1(x+y)_n + \cdots \;\; + (-1)^n(x+ny)_n \equiv (-y)^n n\,! \quad (7)$$

$$(x)_p - {}_nC_1(x+y)_p + \cdots + (-1)^n(x+ny)_p \equiv \quad 0 \qquad (p<n) \quad (8)$$

These correspond in form with the following theorems in the products of equal quantities (powers)

$$\frac{(a)^p}{x} - {}_nC_1\frac{(a-1)^p}{x+1} + \cdots\cdots + (-1)_n\frac{(a-n)^p}{x+n} \equiv \frac{x\,!\,n\,!}{x+n\,!}\;\frac{(x+a)^p}{x} \quad (9)$$

and three others formed from (6) (7) and (8) by changing subscript letters to indices, thus $(x+y)_n$ to $(x+y)^n$.

§2. The theorems in powers corresponding to (7) and (8) are given on page 372 of C. Smith's "Treatise on Algebra," and are mentioned here because of their similarity to (7) and (8). All the other identities are new to me. It would appear that, from most of the algebraical identities involving positive integral powers, other identities may be derived by substituting suffixes for indices. Thus Vandermonde's Theorem corresponds to the Binomial Theorem, and it is easily deduced that

$$(a+b+c+\cdots \text{ to } m \text{ terms})_n \equiv \Sigma\left\{\frac{(a)_p(b)_q(c)_r\cdots}{p\,!\,q\,!\,r\,!\cdots}\text{to } m \text{ factors}\right\},$$

$pqrs \cdots\cdots n$ being positive integers subject to $p+q+r+s+\cdots=n$. This corresponds to the Multinomial Theorem. It is possible to extend this for negative values of n, a_{-n} being interpreted as in theorem (1). Both the theorems (5) and (9) are particular cases of (2). May not the Binomial Theorem and Vandermonde's Theorem be special cases of some general theorem?

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Vandermonde's Theorem.

§ 3. Can any meaning be attached to the theorem

$$(a+b)_n = a_n + n_1 a_{n-1} b_1 + \frac{n_2}{2\,!} a_{n-2} b_2 + \cdots$$

when n is not restricted, as hitherto, to being a positive integer? In Vandermonde's Theorem a_n represents the product of n related factors $a \cdot a-1 \cdot a-2 \cdots \cdot a-n+1$, and certainly so long as we regard a_n as the product of n factors such expressions as

$$a_{-n}, \quad a_{\frac{1}{2}}, \quad a_{\frac{p}{q}},$$

seem beyond our comprehension. Exactly the same might have been written of the quantities

$$a^{-n}, \quad a^{\frac{p}{q}}, \quad a^{\frac{1}{2}},$$

so long as a^m was regarded as the product of m factors each equal to a. The Binomial Theorem, until fractional and negative indices were interpreted, was a finite algebraical identity; but as soon as a fundamental law $a^m \times a^n = a^{m+n}$, was assumed in the Theory of Indices, then the expressions

$$a^{-m}, \quad a^{\frac{p}{q}},$$

were interpreted, and the Binomial Theorem was shown to hold (with certain restrictions) for fractional and negative powers.

§ 4. Now in the expressions a_m, a_n, (the usual meanings being attached) we have these relations

$$a_n \times (a-n)_m = a_{m+n} \qquad (\alpha)$$

$$a_m \times (a-m)_n = a_{m+n} \qquad (\beta)$$

$$a_n \times (a-n)_{m-n} = a_m \qquad (\gamma)$$

These are all expressions of one law. Let us assume this law as general and interpret a_{-n}, a_o, in accordance with our assumption.

In the relation (α) put $n=0$ then we have $a_o \times a_m = a_m$ whence $a_o = 1$ and this is analogous to $a^o = 1$.

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In the relation (γ) change n to $-r$ then we obtain $a_{-r} \times (a+r)_{m+r} = a_m$

$$\therefore \quad a_{-r} = \frac{a_m}{(a+r)_{m+r}}$$

If m and r be integers $a_{-r} = \dfrac{1}{(a+r)(a+r-1)\dots(a+1)}$ which is analogous to $a^{-r} = \dfrac{1}{a \cdot a \cdot a \cdots}$ to r factors.

From the relations (α) and (β) we get

$$a_n \times (a-n)_m = a_m \times (a-m)_n$$

make $n = \dfrac{p}{q}$ then we have $\dfrac{a_{\frac{p}{q}}}{(a-m)_{\frac{p}{q}}} = \dfrac{a_m}{\left(a - \frac{p}{q}\right)_m}$

Supposing that m is a positive integer, this equation gives the ratio of any two functions $a_{\frac{p}{q}}$, $b_{\frac{p}{q}}$ in which a and b differ by an integer m, viz. :

$$a_{\frac{p}{q}} = b_{\frac{p}{q}} \cdot \frac{a \cdot a-1 \cdot a-2 \cdots a-m+1}{a - \frac{p}{q} \cdot a - \frac{p}{q} - 1 \dots a - \frac{p}{q} - m + 1}$$

and again when a is an integer

$$a_{\frac{p}{q}} = (0)_{\frac{p}{q}} \cdot \frac{a\,!}{\left(a - \frac{p}{q}\right)\left(a - \frac{p}{q} - 1\right)\dots\left(1 - \frac{p}{q}\right)}\cdot$$

The function $(a)_{\frac{p}{q}}$ will be discussed in another paper.

$a_{\frac{1}{n}}$ must be a function of a possessing the property

$$a_{\frac{1}{n}} \times \left(a - \frac{1}{n}\right)_{\frac{1}{n}} \times \left(a - \frac{2}{n}\right)_{\frac{1}{n}} \dots \times \left(a - \frac{n-1}{n}\right)_{\frac{1}{n}} = a\,.$$

We now proceed to prove Theorem (1).

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§ 5. Denote the infinite series (1) by $f(-n)$

thus $$f(-n) \equiv a_{-n} + (-n)_1 a_{-n-1} b_1 + \frac{(-n)_2}{2\,!} a_{-n-2} b_2 + \cdots \qquad \text{(A)}$$

Now $$\left.\begin{aligned}(a+b+n)_m = (a+n+r)_m + m_1(a+n+r)_{m-1}(b-r)_1 \\ + \frac{m_2}{2\,!}(a+n+r)_{m-2}(b-r)_2 + \cdots\cdots + (b-r)_m\end{aligned}\right\} \qquad \text{(B)}$$

m being a positive integer $> n$

Multiply $f(-n)$ by $(a+b+n)_m$ in the following manner:

a_{-n} by the series on the right side of B putting $r=0$

$(-n)_1 a_{-n-1} b_1$ ... ... $r=1$, etc.

Then we obtain

$$\left.\begin{aligned}&f(-n) \times (a+b+n)_m \\ &\equiv a_{-n}\Big[(a+n)_m + m_1(a+n)_{m-1}b_1 + \frac{m_2}{2\,!}(a+n)_{m-2}b_2 + \ldots + b_m\Big] \\ &+ (-n)_1 a_{-n-1} b_1 \\ &\quad \times \Big[(a+n+1)_m + \quad \ldots \quad \ldots \quad \ldots \quad + (b-1)_m\Big] \\ &+ \frac{(-n)_2}{2\,!} a_{-n-2} b_2 \\ &\quad \times \Big[(a+n+2)_m + \quad \ldots \quad \ldots \quad \ldots \quad + (b-2)_m\Big]\end{aligned}\right\} \qquad \text{(C)}$$

+ an infinite number of brackets similar to the above.

Now with the interpretation of symbols in §§ (4) and (1).

$$a_{-n-s} \times (a+n+s)_m = a_{m-n-s} \quad \text{and} \quad b_s \times (b-s)_r = b_r$$

∴ The expression (C) becomes

$$\begin{aligned}&\Big[a_{m-n} + m_1 a_{m-n-1} b_1 + \frac{m_2}{2\,!} a_{m-n-2} b_2 + \ldots + a_{-n} b_m\Big] \\ &+ (-n)_1 \cdot \Big[a_{m-n-1} b_1 + m_1 a_{m-n-2} b_2 + \quad \ldots \quad + a_{-n-1} b_{m+1}\Big] \\ &+ \frac{(-n)_2}{2\,!}\Big[a_{m-n-2} b_2 + \ \ldots \quad \ldots \quad \ldots \ + a_{-n-2} b_{m+1}\Big]\end{aligned}$$

+ etc. to infinity.

Collect the resulting terms diagonally.

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We obtain a_{m-n}

$$+a_{m-n-1}b_1\left[m_1+(-n)_1\right]$$

$$+a_{m-n-2}b_2\left[\frac{m_2}{2\,!}+\frac{m_1(-n)_1}{1\,!\,1\,!}+\frac{(-n)_2}{2\,!}\right]$$

$$\cdots\cdots\cdots\cdots$$

$$+\quad a_0b_{m-n}\left[\frac{m_{m-n}}{m-n\,!}+\cdots+\frac{(-r)_{m-n}}{m-n\,!}\right]$$

$+\cdots\cdot$ an infinite number of brackets similar to the above.

Now all brackets after the $m-n+1^{th}$ vanish identically by Vandermonde's Theorem since $m-n$ is a positive integer, and the expression becomes

$$a_{m-n}+(m-n)_1a_{m-n-1}b_1+\frac{(m-n)_2}{2\,!}a_{m-n-2}b_2+\quad\cdots+b_{m-n}\equiv(a+b)_{m-n}$$

$\therefore$ we have proved $(a+b)_{m-n}=(a+b+n)_m\times f(-n)$

whence $f(-n)=\dfrac{(a+b)_{m-n}}{(a+b+n)_m}=\dfrac{1}{(a+b+n)(a+b+n-1)\cdots(a+b+1)}$

which in our notation $=$ $(a+b)_{-n}$

Conditions for the convergence of $f(-n)$ can easily be obtained.

§ 6. To prove Theorems (2), (3), (4), etc.

It is well known that

$$1-\frac{x}{x+1}\,{}_nC_1+\frac{x}{x+2}\,{}_nC_2-\text{to } n+1 \text{ terms}\equiv\frac{n!}{x+n\cdot x+n-1\cdots x+1\cdot}$$

we shall write the expression on the right as $\dfrac{x\,!\,n\,!}{x+n\,!}$ remembering that when x is not an integer it must be written in the long form. Multiply both sides of the identity by $x+a$ in the following way.

The first term by $x+a$

,, second ,, ,, $(x+1)+(a-1)$

,, $r+1^{th}$,, ,, $(x+r)+(a-r)$

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we then obtain

$$x\left[1-{}_nC_1+{}_nC_2-\cdots\right]+\left[a-\frac{x}{x+1}{}_nC_1(a-1)+\cdots\right]\equiv\frac{x\,!\,n\,!}{x+n\,!}(x+a)\,.$$

Now the first bracket on the left is identically equal to zero

$$\therefore\quad \frac{a}{x}-{}_nC_1\frac{a-1}{x+1}+{}_nC_2\frac{a-2}{x+2}-\cdots\text{ to } n+1 \text{ terms}\equiv\frac{x\,!\,n\,!}{x+n\,!}\,\frac{(x+a)}{x}$$

Proceeding in this way we shall finally obtain

$$\left[a^s-{}_nC_1(a-1)^s+\cdots\cdots+(-1)^n(a-n)^s\right]$$
$$+\left[\frac{a^{s+1}}{x}-{}_nC_1\frac{(a-1)^{s+1}}{x+1}+\cdots\cdots\right]\equiv\frac{x\,!\,n\,!}{x+n\,!}\,\frac{(x+a)^{s+1}}{x}$$

Now it is well known that the first bracket on the left $\equiv 0$ so long as s is an integer $<n$. $\therefore$ we have

$$\frac{a^p}{x}-{}_nC_1\frac{(a-1)^p}{x+1}+{}_nC_2\frac{(a-2)^p}{x+2}-\cdots\cdots\equiv\frac{x\,!\,n\,!}{x+n\,!}\,\frac{(x+a)^p}{x}\qquad\text{(D)}$$

p being an integer $\lessgtr n$.

This proves Theorem (9). Replacing $x+n$ by x and $a-n$ by a we obtain

$$\frac{a^p}{x}\quad{}_nC_1\frac{(a+1)^p}{x-1}+\cdots\cdots\text{ to } n+1 \text{ terms}\equiv\frac{x-n\,!\,n\,!}{x\,!}\cdot\frac{(x+a)^p}{x-n}\qquad\text{(E)}$$

§7. Take $S_1=a+b+c+\cdots$ to p terms

$S_2=$ the sum of all products of the letters, two at a time.

. . . .

$S_r=$ the sum of all products, r at a time.

Then

$$P(x+n)=S_p+(x+n)S_{p-1}+(x+n)^2S_{p-2}+\cdots+(x+n)^pS_o\qquad\text{(F)}$$

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From Theorem (9)

$$(x+n)^p \equiv x \cdot \frac{x+n\,!}{x\,!\,n\,!}\left\{\frac{n^p}{x} - n\mathrm{C}_1\frac{(n-1)^p}{x+1} + \cdots\cdots + (-1)^n\frac{0^p}{x+n}\right\}$$

so long as $p \lesseqgtr n$. Substitute in (F) then

$$\mathrm{P}(x+n) \equiv \mathrm{S}_p\left[n^\circ - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1)^\circ + \frac{x}{x+2}{}_n\mathrm{C}_2(n-2)^\circ - \cdots\cdots \right.$$
$$\left. \cdots\cdots + (-1)^n\frac{x}{x+n}0^\circ\right]\frac{x+n\,!}{x\,!\,n\,!}$$
$$+\mathrm{S}_{p-1}\left[n - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1) + \cdots\cdots \right.$$
$$\left. \cdots\cdots + (-1)^n\frac{x}{x+n}\cdot 0\right]\frac{x+n\,!}{x\,!\,n\,!}$$
$$+\mathrm{S}_{p-2}\left[n^2 - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1)^2 + \cdots\cdots \right.$$
$$\left. \cdots\cdots + (-1)^n\frac{x}{x+n}\cdot 0^2\right]\frac{x+n\,!}{x\,!\,n\,!}$$
$$\cdots\cdots\cdots\cdots$$
$$+\ \mathrm{S}_0\left[n^p - \frac{x}{x+1}{}_n\mathrm{C}_1(n-1)^p + \cdots\cdots \right.$$
$$\left. \cdots\cdots + (-1)^n\frac{x}{x+n}0^p\right]\frac{x+n\,!}{x\,!\,n\,!}$$

This expression may be written

$$\frac{x+n\,!}{x\,!\,n\,!}\left\{\left[\mathrm{S}_p + n\mathrm{S}_{p-1} + n^2\mathrm{S}_{p-2} + \cdots\cdots + n^p\mathrm{S}_0\right]\right.$$
$$+\left[\mathrm{S}_p + (n-1)\mathrm{S}_{p-1} + (n-1)^2\mathrm{S}_{p-2} + \cdots\cdots\right]\frac{x+1}{x}{}_n\mathrm{C}_1$$
$$+\left[\mathrm{S}_p + (n-2)\mathrm{S}_{p-1} + (n-2)^2\mathrm{S}_{p-2} + \cdots\cdots\right]\frac{x+2}{x}{}_n\mathrm{C}_1$$
$$\left. +\left[\cdots\cdots\right] + \cdots\cdots\cdots\cdots \right\}$$
$$\equiv \frac{x+n\,!}{x\,!\,n\,!}\left\{\mathrm{P}(n) - n\mathrm{C}_1\frac{x}{x+1}\mathrm{P}(n-1) + \cdots\cdots \text{ to } n+1 \text{ terms}\right\}$$

whence $$\frac{x\,!\,n\,!}{x+n\,!}\mathrm{P}(x+n) \equiv \frac{\mathrm{P}(0)}{x+n} - n\mathrm{C}_1\frac{\mathrm{P}(1)}{x+n-1} + \cdots + (-1)^n\frac{\mathrm{P}(n)}{x}$$

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If we replace a. by $\frac{a}{y}$, b by $\frac{b}{y}$, etc., then

$$P(r) = \frac{(a+ry)(b+ry)\cdots\cdot}{y^p} = \frac{P(ry)}{y^p}$$

and we obtain the theorem in form (2), since y^p is common to all denominators and so divides out.

Replacing $x+n$ by r, we obtain

$$\frac{P(0)}{r} - nC_1\frac{P(y)}{r-1} + nC_2\frac{P(2y)}{r-2} - \cdots \equiv (-1)^n\frac{r-n\,!\,n\,!}{r\,!}\;\frac{P(ry)}{r-n} \qquad \text{(G)}$$

§ 8. Again in Theorem (G) make

$$y=1,\quad r=x,\quad b=a-1,\quad c=a-2,\quad \text{etc.,}$$

then

$$P(ry) \equiv (a+r)(a+r-1)\cdots\cdot(a+r-p+1) \equiv (a+r)_p \equiv (a+x)_p$$

and we have

$$\frac{(a)_p}{x} - nC_1\frac{(a+1)_p}{x+1} + \text{ etc. } \cdots\cdot \equiv (-1)^n\frac{x-n\,!\,n\,!}{x\,!}\;\frac{(x+a)_p}{x-n}$$

This is the Identity (6).

If we replace $a+n$ by a, $b+n$ by $a-1$, $c+n$ by $a-2$, etc., we obtain

$$\frac{(a)_p}{x} - nC_1\frac{(a-1)_p}{x+1} + nC_2\frac{(a+2)_p}{x+2} - \cdots\cdot \equiv \frac{x\,!\,n\,!}{x+n\,!}\;\frac{(x+a)_p}{x}$$

This is the Identity (5).

The left side of (3) may be written

$$\begin{aligned} &[S_p + 0\cdot S_{p-1} + 0^2S_{p-2} + \cdots\cdots + 0^pS_o] \\ &- nC_1[S_p + yS_{p-1} + y^2S_{p-2} + \cdots\cdots + y_pS_o] \\ &+ (-1)^n nC_n[S_p + nyS_{p-1} + n^2y^2S_{p-2} + \cdots + n^py^pS_o] \\ \equiv\ & S_p[1 - nC_1 + nC_2 - \cdots] + S_{p-1}[0 - nC_1\cdot y + nC_2\,.\,2y - \cdots] + \cdots\cdots \\ &\cdots\cdots + S_o[0^p - nC_1y^p + nC_2 2^py^p - \cdots] \end{aligned}$$

when p is an integer $<n$ each bracket $\equiv 0$

$$\therefore\ P(0) - nC_1P(y) + nC_2P(2y) - \cdots \equiv 0. \qquad \text{Theorem (3)}$$

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If $p=n$ all the brackets vanish except the last, which becomes

$$S_o[0^n - nC_1y^n + nC_2 2^n y^n - \cdots + (-1)^n n^n y^n] = (-y)^n n! \quad \text{Theorem (4)}$$

making $a=x$, $b=x-1$, $c=x-2$, etc., we have

$$(x)_n - nC_1(x+y)_n + nC_2(x+2y)_n - \cdots \equiv (-y)^n n!$$

$$(x)_p - nC_1(x+y)_p + \cdots\cdots \equiv 0. \quad \text{Theorems (7) and (8)}$$

Many other theorems can be obtained by varying the constants in (2) and (3).

§ 9. The Differential Equation of the n^{th} order

$$\left.\begin{aligned}(q_o - p_o x)x^{n-1}\frac{d^n y}{dx^n} + (q_1 - p_1 x)x^{n-2}\frac{d^{n-1}y}{dx^{n-1}} + \cdots\cdots \\ \cdots\cdots + (q_{n-1} - p_{n-1}x)\frac{dy}{dx} - p_n y = 0\end{aligned}\right\} \quad \text{(E)}$$

affords another very interesting analogy between powers, and products of related quantities.

When the constants p and q have the following values

$$p_n = a^n. \qquad q_{n-1} = \gamma^{n-1}$$

$$p_{n-1} = (a+1)^n - a^n. \qquad q_{n-2} = (\gamma+1)^{n-1} - \gamma^{n-1}$$

$$p_{n-2} = \frac{(a+2)^n}{2!} - \frac{(a+1)^n}{1!1!} + \frac{a^n}{2!}. \qquad \cdots\cdots\cdots\cdots$$

$$p_{n-3} = \frac{(a+3)^n}{3!} - \frac{(a+2)^n}{2!1!} + \frac{(a+1)^n}{1!2!} - \frac{a^n}{3!}. \qquad \cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$p_o = 1 \qquad q_o = 1$$

The equation has a particular solution

$$y = A\left\{1 + \frac{a^n}{1!\gamma^{n-1}}x + \frac{a^n(a+1)^n}{2!\gamma^{n-1}(\gamma+1)^{n-1}}x^2 + \cdots\right\} \quad \cdots \quad \text{(F)}$$

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When the constants in the differential equation have the following values

$$p_n = (a)_n \cdot \qquad\qquad q_{n-1} = (\gamma)_{n-1}$$

$$p_{n-1} = (a+1)_n - (a)_n \cdot \qquad\qquad q_{n-2} = (\gamma+1)_{n-1} - (\gamma)_{n-1}$$

$$p_{n-2} = \frac{(a+2)_n}{2\,!} - \frac{(a+1)_n}{1\,!\,1\,!} + \frac{(a)_n}{2\,!} \cdot \qquad\qquad \cdots\cdots\cdots\cdots$$

Etc. Etc.

The equation (E) has a solution

$$y = \mathrm{A}\left\{1 + \frac{(a)_n}{1\,!\,(\gamma)_{n-1}}x + \frac{(a)_n \cdot (a+1)_n}{2\,!\,(\gamma)_{n-1}(\gamma+1)_{n-1}}x^2 + \cdots\right\} \qquad \cdots \qquad \text{(G)}$$

The series (F) and (G) are particular cases of the Hypergeometric series of the n^{th} order.

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