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A NOTE ON A PAPER BY S. LAL

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Abstract

In this paper, the questions about bitopological spaces proposed by S. Lal are solved and one of his counterexamples rectified.

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In a recent paper, Lal [1] studies the relationship among pairwise properties in a bitopological space $(X, \mathcal{P}, \mathcal{Q})$, equivalent topological properties in $(X, \mathcal{P} \vee \mathcal{Q})$ and the same properties in spaces $(X, \mathcal{P})$ and $(X, \mathcal{Q})$ by means of the following scheme:

$$\text{sup-}P \rightleftarrows \text{p-}P \rightleftarrows \text{bi-}P.$$

In this paper, we solve the two implications left without solution by Lal [1] and we modify a wrong example from the same paper [1].

Definition 1 (Saegrove [2]). A bitopological space $(X, \mathcal{P}, \mathcal{Q})$ is *pairwise pseudocompact* if every pairwise continuous function $f\colon (X, \mathcal{P}, \mathcal{Q}) \to (R, \mathcal{U}, \mathcal{L})$ is bounded, where $\mathcal{U} = \{]a, +\infty[\colon a \in R\} \cup \{\varnothing, R\}$ and $L = \{]-\infty, a[\colon a \in R\} \cup \{\varnothing, R\}$.

If we take $\mathcal{P} = \mathcal{Q}$, we have the definition of pseudocompact topological space.

Lal [1], Theorem 8, proves, regarding pseudocompactness, that

$$\text{sup-}P \rightleftarrows \text{p-}P \nleftrightarrow \text{bi-}P.$$

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PROPOSITION 1. *Concerning pseudocompactness, we have*

$$\text{bi-}P \nrightarrow r\text{p-}P.$$

PROOF. Let $(R, \mathcal{P}, \mathcal{Q})$ be a bitopological space, where $\mathcal{P} = \{A \subset R\colon 0 \in A\} \cup \{\varnothing\}$ and $\mathcal{Q} = \{A \subset R\colon 0 \notin A\} \cup \{R\}$. $(R, \mathcal{P})$ and $(R, \mathcal{Q})$ are pseudocompact spaces (Steen, Seebach [3], pages 44 and 47), but $(R, \mathcal{P}, \mathcal{Q})$ is not pairwise pseudocompact because $f(x) = \min(x, 0)$ is a pairwise continuous nonbounded function:

If $x \geqslant 0, f^{-1}(]x, +\infty[) = \varnothing \in \mathcal{P}$ and if $x < 0, f^{-1}(]x, +\infty[) =]x, +\infty[\in \mathcal{P}$.

If $x > 0, f^{-1}(]-\infty, x[) = R \in \mathcal{Q}$ and if $x \leqslant 0, f^{-1}(]-\infty, x[) =]-\infty, x[\in \mathcal{Q}$.

Then, f is a continuous function from $(R, \mathcal{P})$ to $(R, \mathcal{U})$ and from $(R, \mathcal{Q})$ to $(R, \mathcal{L})$.

DEFINITION 2 (Lal [1]). A bitopological space $(X, \mathcal{P}, \mathcal{Q})$ is *pairwise extremally disconnected* if given a $\mathcal{P}$-open set U and a $\mathcal{Q}$-open set V with $U \cap V = \varnothing$, we have $(\mathcal{Q}\text{-cl}\, U) \cap (\mathcal{P}\text{-cl}\, V) = \varnothing$.

If we take $\mathcal{P} = \mathcal{Q}$, we have the definition of an extremally disconnected topological space.

Lal [1] shows, concerning extremally disconnected spaces, that

$$\sup\text{-}P \nrightarrow \text{p-}P \not\rightleftarrows \text{bi-}P.$$

PROPOSITION 2. *Concerning extremally disconnected spaces, we have*

$$\text{p-}P \nrightarrow \sup\text{-}P.$$

PROOF. Let $(X, \mathcal{P}, \mathcal{Q})$ be a bitopological space, where $X = \{a, b, c, d\}$, $\mathcal{P} = \{\varnothing, X, \{a, b, c\}\}$ and $\mathcal{Q} = \{\varnothing, X, \{a\}, \{b, c\}, \{a, b, c\}\}$.

One proves easily that $(X, \mathcal{P}, \mathcal{Q})$ is pairwise extremally disconnected, but $\mathcal{P} \vee \mathcal{Q} = \mathcal{Q}$ and $(X, \mathcal{Q})$ is not extremally disconnected, because if $U = \{a\}$ and $V = \{b, c\}$, we have $(\mathcal{Q}\text{-cl}\, U) \cap (\mathcal{Q}\text{-cl}\, V) = \{d\} \neq \varnothing$.

Finally Lal [1] says that $(X, \mathcal{P}, \mathcal{Q})$ is a bi-zero-dimensional not pairwise zero-dimensional space, where X is the real line, $\mathcal{P}$ the topology whose base is $\{[a, b[\colon a, b \in R\}$ and $\mathcal{Q}$ the discrete topology. But $(X, \mathcal{P}, \mathcal{Q})$ is a pairwise zero-dimensional space because $\{[a, b[\colon a, b \in R\}$ is a $\mathcal{P}$-base of $\mathcal{Q}$-closed sets and $\{\{x\}\colon x \in R\} \cup \{\varnothing\}$ is a $\mathcal{Q}$-base of $\mathcal{P}$-closed sets.

EXAMPLE 1. A bi-zero-dimensional space which is not pairwise zero-dimensional.

Let X be the set of integers. Let $\mathcal{P}$ be the topology whose base is $\{\{2n, 2n + 1\}\colon n \in X\}$ and let $\mathcal{Q}$ be the discrete topology.

Obviously, $\{2n, 2n+1\}$ are $\mathcal{P}$-open and $\mathcal{Q}$-closed sets, therefore $(X, \mathcal{P}, \mathcal{Q})$ is bi-zero-dimensional, but it is not a pairwise zero-dimensional space because the only $\mathcal{Q}$-base, $\{\{x\}: x \in X\} \cup \{\varnothing\}$ is not a family of $\mathcal{P}$-closed sets.

References

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[3] L. A. Steen and J. A. Seebach, *Counterexamples in Topology* (Springer-Verlag, New York, 1978).

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