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ON A PROPERTY OF BASES IN A HILBERT SPACE

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Abstract. In this paper we study a seemingly unnoticed property of bases in a Hilbert space that falls in the general area of constructing new bases from old, yet is quite atypical of others in this regard. Namely, if $\{x_n\}$ is any normalized basis for a Hilbert space H and $\{f_n\}$ the associated basis of coefficient functionals, then the sequence $\{x_n + f_n\}$ is again a basis for H. The unusual aspect of this observation is that the basis $\{x_n + f_n\}$ obtained in this way from $\{x_n\}$ and $\{f_n\}$ need not be equivalent to either, in contrast to the standard techniques of constructing new bases from given ones by means of an isomorphism on H. In this paper we study bases of this form and their relation to the component bases $\{x_n\}$ and $\{f_n\}$.

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1. Throughout this paper H will denote a separable, real Hilbert space, $\{x_n\}$ some normalized basis for H, and $\{f_n\}$ the sequence of coefficient functionals associated with $\{x_n\}$, another basis for H. Recall that the basis $\{x_n\}$ is said to be

(a) *Besselian* [**1**, p. 338] if $\sum a_n x_n$ converges $\Longrightarrow \{a_n\} \in l^2$,
(b) *Hilbertian* [**1**, p. 338] if $\{a_n\} \in l^2 \Longrightarrow \sum a_n x_n$ converges,
(c) *a Riesz basis* if it is both Besselian and Hilbertian (equivalently, the image of an orthonormal basis under an invertible linear operator on H).

Also, a basis $\{x_n\}$ is said to *dominate* another basis $\{y_n\}$ if the convergence of $\sum a_n x_n$ implies the convergence of $\sum a_n y_n$, for every sequence $\{a_n\}$ of real numbers.

2. We begin with the result concerning the formation of a certain type of basis from a given one that is the focus of this paper.

THEOREM 1. *If* $\{x_n, f_n\}$ *is any normalized basis for the Hilbert space* H*, the sequence* $\{x_n + f_n\}$ *is a Besselian basis that dominates both* $\{x_n\}$ *and* $\{f_n\}$.

Proof. Note first that the sequence $\{x_n + f_n\}$ is complete in H, since if $\langle x_n + f_n, x\rangle = 0$ for some vector x in H, then from the fact that $x = \sum \langle f_n, x\rangle x_n$, where by assumption $\langle f_n, x\rangle = (-1)\langle x_n, x\rangle$, it follows that $\|x\|^2 = (-1)\sum |\langle x_n, x\rangle|^2 \leq 0$, hence that $\|x\|^2 = 0$, and so x must be 0. Therefore, to show that $\{x_n + f_n\}$ is a basis for H we need only show that it satisfies the K-condition [**1**, p. 58]. If $1 \leq m < n$ and $\{a_i\}$ is any sequence of constants,

$$\left\|\sum_{i=1}^{m} a_i(x_i+f_i)\right\|^2 = \left\langle \sum_{i=1}^{m} a_i x_i, \sum_{i=1}^{m} a_i f_i \right\rangle = \left\|\sum_{i=1}^{m} a_i x_i\right\|^2 + \left\|\sum_{i=1}^{m} a_i f_i\right\|^2 + 2\sum_{i=1}^{m} a_i^2$$
$$\leq K_1^2 \left\|\sum_{i=1}^{n} a_i x_i\right\|^2 + K_2^2 \left\|\sum_{i=1}^{n} a_i f_i\right\|^2 + 2\sum_{i=1}^{n} a_i^2 \leq K^2 \left\|\sum_{i=1}^{n} a_i(x_i+f_i)\right\|^2,$$

where $\{x_i\}$ and $\{f_i\}$ satisfy the K-condition for $K = K_1 \geq 1$ and $K = K_2 \geq 1$, respectively, and where $K^2 = K_1^2 + K_2^2$. That is,

$$\left\|\sum_{i=1}^{m} a_i(x_i+f_i)\right\|^2 \leq K^2 \left\|\sum_{i=1}^{n} a_i(x_i+f_i)\right\|^2$$

(for all such $1 \leq m < n$ and for any $\{a_i\}$), implying $\{x_i + f_i\}$ is a basis for H. Moreover, as the proof above shows, for $n \geq 1$ and any sequence $\{a_i\}$, $\| \sum_{i=1}^n a_i x_i \|$, $\| \sum_{i=1}^n a_i f_i \|$, and $2\sum_{i=1}^n a_i^2$ are all at most $\| \sum_{i=1}^n a_i(x_i+f_i) \|$, implying that the basis $\{x_i + f_i\}$ dominates the bases $\{x_i\}$ and $\{f_i\}$ and is Besselian, thereby completing the proof of Theorem 1.

In general, the basis $\{x_n + f_n\}$ need not be equivalent to $\{x_n\}$ nor $\{f_n\}$ (an interesting fact in itself, since most techniques for constructing a new basis from a given one do so by producing a basis equivalent to the original one). The following result shows several aspects of the relationship of the basis $\{x_n + f_n\}$ to the "component" bases $\{x_n\}$ and $\{f_n\}$ besides those given by Theorem 1.

THEOREM 2. *If $\{x_n, f_n\}$ is a normalized basis for a Hilbert space H, the basis $\{x_n + f_n\}$ is equivalent to $\{x_n\}$ if and only if $\{x_n\}$ is Besselian, and is a Riesz basis if and only if $\{x_n\}$ is.*

Proof. By Theorem 1, if $\sum a_n(x_n + f_n)$ converges then so does $\sum a_n x_n$. If $\{x_n\}$ is a Besselian basis for H then the convergence of $\sum a_n x_n$ implies the convergence of $\sum a_n f_n$ (since $\{f_n\}$ is Hilbertian) and hence that of $\sum a_n(x_n + f_n)$. It follows that $\{x_n\}$ and $\{x_n + f_n\}$ are equivalent. On the other hand, if these are equivalent then, since $\{x_n + f_n\}$ is Besselian (Theorem 1), the basis $\{x_n\}$ must also be.

Conversely if $\{x_n + f_n\}$ is a Riesz basis for H then, in particular, it is Hilbertian. Therefore, whenever $\{a_n\}$ is in l^2, $\sum_n a_n(x_n + f_n)$ converges, implying that both $\sum a_n x_n$ and $\sum a_n f_n$ do, from which it follows that $\{x_n\}$ and $\{f_n\}$ are both Hilbertian. But if $\{f_n\}$ is Hilbertian, then $\{x_n\}$ is Besselian [**1**, p. 339] as well and is therefore equivalent to an orthonormal basis [**1**, p. 341]-i.e. $\{x_n\}$ is a Riesz basis, hence certainly Besselian, and (by the first part of the proof) consequently equivalent to $\{x_n + f_n\}$. It then follows that $\{x_n + f_n\}$ must also be a Riesz basis.

In the same vein, the following result shows another relationship between the bases $\{x_n\}$ and $\{x_n + f_n\}$, one involving the operator U on H that maps $\{x_n + f_n\}$ to $\{f_n\}$ for all n (a well-defined bounded linear operator on H as a consequence of Theorem 1).

THEOREM 3. *A normalized basis $\{x_n, f_n\}$ for a Hilbert space H is Besselian if and only if the operator U on H mapping $\{x_n + f_n\}$ to $\{f_n\}$ has $\|U\| < 1$.*

Proof. Suppose that $\{x_n, f_n\}$ is a normalized Besselian basis for H, and let x be a unit vector in H having the basis expansion $x = \sum a_n(x_n + f_n)$. Then, as noted in the proof of Theorem 1,

$$1 = \|x\|^2 = \left\|\sum a_n x_n\right\|^2 + \left\|\sum a_n f_n\right\|^2 + 2\sum |a_n|^2,$$

so that

$$\|Ux\|^2 = \left\|\sum a_n f_n\right\|^2 \leq 1 - \left\|\sum a_n x_n\right\|^2,$$

(implying, in particular, that $\|U\| \leq 1$ for any basis $\{x_n\}$).

Since we are assuming that $\{x_n\}$ is Besselian, there is an operator T on H mapping $\{x_n\}$ to $\{f_n\}$ [**1**, p. 339] and hence for which

$$\|Ux\|^2 = \left\|\sum a_n f_n\right\|^2 = \left\|T\left(\sum a_n x_n\right)\right\|^2,$$

(where x in H is as above). It follows that $\|Ux\|^2 \leq \|T\|^2 \|\sum a_n x_n\|^2$, and therefore (by the above) that

$$\|Ux\|^2 \leq 1 - \left\|\sum a_n x_n\right\|^2 \leq 1 - \|T\|^{-2}\|Ux\|^2.$$

That is, $(1 + \|T\|^{-2})\|Ux\|^2 \leq 1$, implying $\|Ux\|^2 \leq (1 + \|T\|^{-2})^{-1}$, and since x was an arbitrary unit vector in H it follows that $\|U\| < 1$.

Conversely, suppose that the operator U mapping $\{x_n + f_n\}$ to $\{f_n\}$ is such that $\|U\| < 1$. If we define $S = I - U$, then $\|I - S\| = \|U\| < 1$ and, as is well known, S is invertible on H. Since $S(x_n + f_n) = (I - U)(x_n + f_n) = x_n$, where the basis $\{x_n + f_n\}$ is Besselian (Theorem 1), it follows that $\{x_n\}$ is also Besselian,thereby completing the proof of Theorem 3.

THEOREM 4. *Let $\{x_n, f_n\}$ be a normalized basis for a Hilbert space H and U the corresponding operator on H that maps the basis $\{x_n + f_n\}$ to the basis $\{f_n\}$. Then $\|U\| \geq 1/2$, where $1/2$ is the best possible lower bound for all such operators U, and where equality holds if and only if $\{x_n\}$ is an orthonormal basis for H.*

Proof. Given a normalized basis $\{x_n, f_n\}$ and the operator U mapping the basis $\{x_n + f_n\}$ to $\{f_n\}$, let $x = (x_1 + f_1)/\|x_1 + f_1\|$, a unit vector in H. Then

$$\|Ux\|^2 = (\|f_1\|^2)(\|x_1 + f_1\|^{-2}) = (\|f_1\|^2)(\|f_1\|^2 + 3)^{-1} = (1 + 3\|f_1\|^{-2})^{-1}.$$

Since $\|f_1\|^2 \geq 1$ this last is $\geq 1/4$, implying $\|Ux\| \geq 1/2$, where $\|x\| = 1$, and it follows that $\|U\| \geq 1/2$.

In the case in which $\{x_n\}$ is an orthonormal basis, $x_n = f_n$ for all n, so that U maps $2x_n$ to x_n, for all n, and is therefore simply $\frac{1}{2}I$, an operator of norm $1/2$ on H, from which it follows that the number $1/2$ is the best possible lower bound over all such operators U. Conversely, if $\{x_n\}$ is a normalized basis for H for which the associated operator U has norm $1/2$, then from the above (with x_1 and f_1 replaced by x_n and f_n, respectively, for arbitrary n) we see that

$$1/4 = \|U\|^2 \geq (1 + 3\|f_n\|^{-2})^{-1} \quad \text{for all } n.$$

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Since $\|f_n\| \geq 1$, it follows that $\|f_n\| = 1$, for all n, and hence that $\{x_n\}$ is an orthonormal basis since

$$\|x_n - f_n\|^2 = \|x_n\|^2 + \|f_n\|^2 - 2 = 0,$$

implying that $f_n = x_n$ for all n, from which it easily follows that $\{x_n\}$ is an orthonormal basis.

REFERENCE

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