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A DECOMPOSITION THEOREM FOR m-CONVEX SETS IN R^d

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1. Introduction. Let S be a subset of some linear topological space. The set S is said to be *m-convex*, $m \geqq 2$, if and only if for every m-member subset of S, at least one of the $\binom{m}{2}$ line segments determined by these points lies in S. A point x in S is said to be a *point of local convexity of* S if and only if there is some neighborhood N of x such that if $y, z \in N \cap S$, then $[y, z] \subseteq S$. If S fails to be locally convex at some point q in S, then q is called a *point of local nonconvexity* (lnc point) *of* S.

Several interesting decomposition theorems have been obtained for closed m-convex sets in the plane. Valentine [**7**] has proved that a closed planar 3-convex set is expressible as a union of three or fewer convex sets, and Stamey and Marr [**4**] have obtained conditions under which a closed planar 3-convex set may be written as a union of two convex sets.

In general, for S a closed, planar m-convex set, if $\ker S \neq \emptyset$, then S is a union of $2(m-1)$ convex sets, and without any restriction on $\ker S$, S will be a union of $(m-1)^3 2^{m-3}$ or fewer convex sets for $m \geqq 3$ (Breen and Kay [**2**]).

However, little work has been done on the problem of obtaining decomposition theorems for closed m-convex sets in higher dimensions. The purpose of this paper is to obtain conditions under which an analogue of some of the planar results might be proved in R^d.

The following familiar terminology will be used. For points x, y in S, we say *x sees y via S* if and only if the corresponding segment $[x, y]$ lies in S. Points $x_1, \ldots, x_n$ in S are *visually independent via* S if and only if for $1 \leqq i < j \leqq n$, x_i does not see x_j via S. Throughout the paper, conv S, aff S, cl S, int S, rel int S, and ker S will be used to denote the convex hull, affine hull, closure, interior, relative interior, and kernel, respectively, of the set S. Also if S is convex, dim S will denote the dimension of the affine hull of S.

2. The decomposition theorem. We will prove the following result.

THEOREM 1. *Let* $S = \text{cl}(\text{int}\, S)$ *be an m-convex set in* R^d, $d = \dim \text{aff}\, S$, *and let Q denote the set of points of local nonconvexity of S, with Q a finite union of parallel convex sets—i.e.,* $Q = \bigcup_{i=1}^{n} C_i$, *where C_i is convex and* aff C_i *is a*

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translate of aff C_j, $1 \leqq i \leqq j \leqq n$. *If* $p \in \ker S \sim Q$, *then* S *is a union of* $\sigma(m) = 2(m-1)$ *or fewer convex sets.*

Proof. Without loss of generality, we assume that $Q = \bigcup_{i=1}^{n} C_i$, where each C_i is maximal—i.e., no C_i is properly contained in a convex subset of Q. (If C_i is properly contained in a convex subset C_i' of Q, replace C_i by C_i'. Notice that aff C_i = aff C_i', for otherwise dim $C_i' >$ dim C_i and Q could not be represented as a finite union of convex sets parallel to C_i.) Since Q is closed, each C_i will be closed.

The following series of preliminary lemmas will be important in the proof.

LEMMA 1. *For each* i, $1 \leqq i \leqq n$, dim $C_i = d - 2$.

Proof. For convenience of notation, let $C_i = C$. We will show that the set aff$(\{p\} \cup C)$ has dimension no greater than $d - 1$: Suppose on the contrary that aff$(\{p\} \cup C)$ has dimension d. Clearly C cannot be d-dimensional, so C must have dimension $d - 1$. Let J = aff C, with J_1 and J_2 the distinct open halfspaces determined by the hyperplane J. Certainly $p \notin J$ so we may assume that p lies in J_1. Consider the set

$$\text{cone}(p, C) \equiv \bigcup \{R(p, x) : x \in C\}$$

where $R(p, x)$ denotes the ray emanating from p through x. Since conv$(\{p\} \cup C) \subseteq S$ and $C \subseteq Q$, there are interior points of the cone in $S \cap J_2$, and since $S = \text{cl}(\text{int}\, S)$, interior points of the cone lie in $(\text{int}\, S) \cap J_2$. However, if U is an open set in $(\text{int}\, S) \cap J_2$, then points of C lie interior to conv$(\{p\} \cup U)$, and these points of C cannot be in Q. We have a contradiction, our assumption is false, and dim aff$(\{p\} \cup C) \leqq d - 1$.

Now let H be any hyperplane containing aff$(\{p\} \cup C)$, with H_1 and H_2 the corresponding open halfspaces, and let M be a convex neighborhood of H disjoint from all the C_j sets which do not lie in H. (Clearly such a neighborhood exists since the C_j sets are parallel and there are finitely many of them.) Examine $S \cap M \cap H_1$: For x, y in $S \cap M \cap H_1$, $[p, x] \cup [p, y] \subseteq S \cap M$, no lnc point of S lies in conv$\{p, x, y\}$, so $[x, y] \subseteq S \cap M \cap H_1$ by a lemma of Valentine [**6**, Corollary 1]. Thus $S \cap M \cap H_1$ is convex. Similarly $S \cap M \cap H_2$ is convex. Furthermore, since $S = \text{cl}(\text{int}\, S)$, we have

$$\text{cl}(S \cap M) = \text{cl}(S \cap M \cap H_1) \cup \text{cl}(S \cap M \cap H_2),$$

so $S' \equiv \text{cl}(S \cap M)$ is a union of two convex sets and hence is 3-convex. It is easy to show that every lnc point of a closed 3-convex set lies in the kernel of that set, and clearly the set Q' of lnc points of S' consists of exactly those points of Q which lie in H. Thus Q' is a finite union of convex sets which lie in ker S'.

Also, since $p \in \ker S' \sim Q'$ and $S = \text{cl}(\text{int}\, S)$, it is easy to see that the set $S' \sim Q'$ is connected: If $w \in S' \sim Q'$, then for one of the open halfspaces determined by H, say H_1, w is in $\text{cl}(S \cap M \cap H_1)$. For any point w_0 in

$S \cap M \cap H_1$, $(w, w_0] \cup [w_0, p) \subseteq S \cap M \cap H_1 \subseteq S' \sim Q'$. Hence the set $S' \sim Q'$ is polygonally connected and therefore connected. We conclude that S' satisfies the hypothesis of Lemma 3 in [**1**], so by the corollary to that lemma, $\dim C = d - 2$, finishing the proof of Lemma 1.

LEMMA 2. *For each* i, $1 \leqq i \leqq n$, $(\text{aff } C_i) \cap S = C_i$.

Proof. As in the proof of Lemma 1, let $C = C_i$, let H be a hyperplane containing $\text{aff}(\{p\} \cup C)$ with H_1 and H_2 the corresponding open halfspaces, and let M be a convex neighborhood of H disjoint from all the C_j sets which do not lie in H. Then by our earlier argument $S \cap M \cap H_i$ is convex for $i = 1, 2$, $Q \cap M \subseteq \ker(S \cap M)$, and the set $S' \equiv \text{cl}(S \cap M)$ satisfies the hypothesis of Lemma 3 in [**1**].

Let N be a convex neighborhood of aff C, $N \subseteq M$, with $N \cap C_j = \emptyset$ for all $C_j \nsubseteq \text{aff } C$. First we wish to show that $(\text{aff } C) \cap S \subseteq \ker(S \cap N)$. For $x \in (\text{aff } C) \cap S$, clearly it suffices to show that x lies in the convex set $\text{cl}(S \cap N \cap H_i)$ for $i = 1, 2$: By Lemma 3 in [**1**], the set $(S \cap N) \sim Q$ is connected, and since it is also locally convex, the set is polygonally connected [**5**]. Then by standard arguments, since $S = \text{cl}(\text{int } S)$, $\text{int}(S \cap N)$ is polygonally connected. Hence $H \cap S \cap N$ contains some interior point w of $S \cap N$, and $w \in \text{cl}(S \cap N \cap H_1) \cap \text{cl}(S \cap N \cap H_2) \subseteq \ker(S \cap N)$. Clearly w cannot lie in aff C: Otherwise, for U any neighborhood of w in $S \cap N$, since $C \subseteq \ker(S \cap N)$, the set $\text{conv}(U \cup C) \subseteq S \cap N$ would capture points of C in its interior, contradicting the fact that $C \subseteq Q$. Thus we may select a convex neighborhood V of w, $V \subseteq [\text{int}(S \cap N)] \sim \text{aff } C$.

Since $S = \text{cl}(\text{int } S)$, we may assume that $x \in \text{cl}(S \cap N \cap H_1)$. Select a point z in $V \cap H_2$. Since $w \in \ker(S \cap N)$, we have $[w, x] \cup [w, z] \subseteq S \cap N$. Also, no point of aff C and hence no point of Q is in $\text{conv}\{x, w, z\} \sim [x, z]$, so $[x, z] \subseteq S$ by a generalization of Valentine's lemma [**6**, Corollary 1]. Therefore, $(x, z] \subseteq S \cap N \cap H_2$ and $x \in \text{cl}(S \cap N \cap H_2)$, the desired result. We have $x \in \text{cl}(S \cap N \cap H_i)$ for $i = 1, 2$, so $x \in \ker(S \cap N)$ and our assertion is proved.

Our next goal is the relation $(\text{aff } C) \cap S \subseteq Q$. Let $x \in (\text{aff } C) \cap S \subseteq \ker(S \cap N)$. Select $r \in S \cap N \cap H_1$ and $s \in S \cap N \cap H_2$ so that $[r, s] \nsubseteq S$ and $s \notin \text{aff}(\{r\} \cup C)$. (Clearly this is possible since $S = \text{cl}(\text{int } S)$.) Then since $x \in \ker(S \cap N)$, $[x, r] \cup [x, s] \subseteq S$. Since $[r, s] \nsubseteq S$, by Valentine's lemma there must be some lnc point q of S in $\text{conv}\{x, r, s\} \sim [r, s]$. Note that $q \in Q \cap N \subseteq \text{aff } C$. Now if $x \neq q$, then $q \notin [x, r] \cup [x, s]$, so q would be in $\text{rel int conv}\{x, r, s\}$, and $s \in \text{aff}\{x, r, q\} \subseteq \text{aff}(\{r\} \cup C)$, impossible. Thus $x = q$, $x \in Q$, and we conclude that $(\text{aff } C) \cap S \subseteq Q$.

Moreover, the set $(\text{aff } C) \cap S$ is convex: If $u, v \in (\text{aff } C) \cap S$, then $u, v \in Q \cap N \subseteq \ker(S \cap N)$, so $[u, v] \subseteq (\text{aff } C) \cap S$. Hence $(\text{aff } C) \cap S$ is a convex subset of Q containing C, and since C is maximal, it follows that $\text{aff } C \cap S = C$, finishing the proof of Lemma 2.

COROLLARY. *If* $p \in S \sim Q$, *then* $p \notin \text{aff } C_i$, $1 \leqq i \leqq n$.

LEMMA 3. *If* $H = \text{aff}(\{p\} \cup C_i)$ *for some* i, *the set* $S \cap H$ *is convex.*

Proof. Clearly H is a hyperplane since $\dim C_i = d - 2$ and $p \notin \text{aff } C_i$. As in the proof of Lemma 2, let M be a convex neighborhood of H disjoint from every C_j set which does not lie in H.

Note that since $p \in S \cap H$, the set $S \cap H$ is connected. By a well-known result [**5**], a closed, connected, locally convex set is convex, so to prove the lemma, it suffices to show that $S \cap H$ is locally convex. Clearly any lnc point of $S \cap H$ necessarily would lie in Q, so select $q \in Q \cap H$ to prove that q is not an lnc point for $S \cap H$. Assume that $q \in C_j \equiv C$. By Lemmas 1 and 2, C must be a component of Q having dimension $d - 2$. Let N be any convex neighborhood of q disjoint from the remaining components of Q, $N \subseteq M$, and let $x, y \in S \cap H \cap N$. We wish to show that $[x, y] \subseteq S$.

Now if x and y both belong to the convex set $\text{cl}(S \cap N \cap H_i)$ for either $i = 1$ or $i = 2$, then the argument is finished, so assume $x \in \text{cl}(S \cap N \cap H_1)$, $y \in \text{cl}(S \cap N \cap H_2)$. Also, if x or y were in $\text{aff } C \cap S = C$, then since $C \subseteq \ker(S \cap M)$, $[x, y]$ would lie in S, so we will assume that $x, y \notin \text{aff } C$.

There are two cases to consider: Either x and y are on the same side of the $(d - 2)$-flat aff C in H, or x and y are on opposite sides of aff C in H. Examine the former case. Since $C \subseteq \ker(S \cap M)$, both x and y see every point of C via S, and the convex sets $\text{conv}(\{x\} \cup C)$, $\text{conv}(\{y\} \cup C)$, intersect in some point $z \in (S \cap H \cap N) \sim Q$. In particular, z, x, y are all on the same side of aff C, $[z, x] \cup [z, y] \subseteq S$, no point of C and hence no lnc point of S lies in $\text{conv}\{z, x, y\}$, so by Valentine's useful lemma, $[x, y] \subseteq S$, the desired result.

For the latter case, suppose that x and y are on opposite sides of aff C in H. Since $p \notin C = (\text{aff } C) \cap S$, without loss of generality we may assume that x and p are on the same side of aff C in H. Select a point p' in $\text{conv}(\{p\} \cup C) \cap (N \sim C)$. Since $\{p\} \cup C \subseteq \ker(S \cap M)$, $\text{conv}(\{p\} \cup C) \subseteq \ker(S \cap M)$, and certainly p' sees $S \cap N$ via $S \cap N$. Clearly p' and x are on the same side of aff C in H, so $[x, p'] \cap C = \emptyset$ and hence $[x, p'] \cap Q = \emptyset$. Now since we are assuming that $y \in \text{cl}(S \cap N \cap H_2)$, let $\{y_n\}$ be a sequence in $S \cap N \cap H_2$ converging to y. For each n, $[p', x] \cup [p', y_n] \subseteq S \cap N$, there are no points of C and therefore no lnc points of S in $\text{conv}\{p', x, y_n\}$, so $[x, y_n] \subseteq S$. Then since S is closed, $[x, y] \subseteq S$, finishing this case and completing the proof of Lemma 3.

The final lemma will require the following result by Lawrence, Hare and Kenelly [**3**, Theorem 2].

THEOREM (Lawrence, Hare, Kenelly). *Let T be a subset of a linear space such that for each finite subset $F \subseteq T$, F may be written as a union of k sets $F_1, \ldots, F_k$, where* $\text{conv } F_i \subseteq T$, $1 \leqq i \leqq k$. *Then T is a union of k convex sets.*

LEMMA 4. *Without loss of generality we may assume that S is bounded.*

Proof. For any finite subset F of S, let B be an open d-dimensional ball containing $F \cup \{p\}$, and let $S' = \text{cl}(S \cap B)$. Then $S' = \text{cl}(\text{int } S')$ is an

m-convex set in R^d whose corresponding set Q' of lnc points is exactly $\bigcup_{i=1}^{n} C_i'$, where $C_i' = \text{cl}(C_i \cap B)$, $1 \leqq i \leqq n$. Clearly $p \in \ker S' \sim Q'$. Hence S' is a bounded set satisfying the hypothesis of Theorem 1. By the Lawrence, Hare, Kenelly Theorem, it suffices to prove that F is a union of $\sigma(m) = 2(m-1)$ sets, each having its convex hull in $S' \subseteq S$. Therefore, we need only show that S' is a union of $\sigma(m)$ convex sets, and we may assume that S is bounded.

At last we return to the proof of the theorem.

Since the result is trivial for $d = 1$ and a consequence of [**2**, Theorem 1, Corollary 3] for $d = 2$, we assume that $d \geqq 3$. From Lemmas 1 and 2, each C_i set is a component of Q having dimension $d - 2$. Let Π denote a plane which is orthogonal to aff C_i for each i, and define f to be the projection of R^d onto Π in the direction of aff C. Clearly $f(S)$ is a closed planar m-convex set, and $f(p)$ lies in its kernel. Hence by [**2**, Theorem 1, Corollary 3], $f(S)$ is a union of $\sigma(m) = 2(m-1)$ or fewer convex sets, $B_1, \ldots, B_{2m-2}$.

Define $A_i \equiv \{x : x \in S \text{ and } f(x) \in B_i\}$, $1 \leqq i \leqq 2m - 2$. We assert that the A_i's are convex sets whose union is S, and clearly it suffices to show that for x, y in S, whenever $[f(x), f(y)] \subseteq f(S)$, then $[x, y] \subseteq S$: Suppose on the contrary that the result fails for some pair x, y in S, and without loss of generality assume that $(x, y) \cap S = \emptyset$. By Valentine's lemma, it follows that there must be a point of Q in $\text{conv}\{x, y, p\} \sim [x, y]$. We have two cases to consider.

Case 1. First assume that for some component C of Q, a point of C lies in rel int $\text{conv}\{x, y, p\}$, and let $H = \text{aff}(\{p\} \cup C)$. We assert that $x, y \notin H$: Otherwise, if $x \in H$, then since there are points of C in rel int $\text{conv}\{x, y, p\}$, this would imply that $y \in H$ also. By Lemma 3, $S \cap H$ is convex, so $[x, y]$ would lie in $S \cap H$, impossible.

As in the proof of the lemmas, let M be a convex neighborhood of H disjoint from every component of Q which does not lie in H. Since some point of C lies in rel int $\text{conv}\{x, y, p\}$, $[x, y]$ cuts the set $\text{cone}(p, C) \equiv \bigcup \{R(p, c) : c \in C\}$. Furthermore, since $x, y \notin H$, $[x, y]$ cuts $\text{cone}(p, C)$ at a single point z, and since $(x, y) \cap S = \emptyset$, $z \notin S$. Now $p \in \ker S$ and $z \notin S$, so $z \notin \text{conv}(\{p\} \cup C)$, and (p, z) intersects C.

Recall by an earlier argument that $S \cap M \cap H_i$ is convex for $i = 1, 2$, where H_1 and H_2 denote the open halfspaces determined by H. We assert that we may select points r, s in $S \cap H$ and segments $[r_0, r)$ in $S \cap M \cap H_1$ and $(s, s_0]$ in $S \cap M \cap H_2$ such that f maps $[r_0, r]$ and $[s, s_0]$ into $[f(x), f(y)]$: Clearly $f(x) \in H_1$, $f(y) \in H_2$, and $f(z) \in [f(x), f(y)] \cap H$. Select a sequence $\{b_n\}$ in $[f(x), f(y)] \cap H_1$ converging to $f(z)$, and let $\{r_n'\}$ be a corresponding sequence in $S \cap H_1$, with $f(r_n') = b_n$. By Lemma 4 we may consider S to be bounded, and hence some subsequence $\{r_n\}$ of $\{r_n'\}$ converges to a point r in $S \cap H$. Clearly we may assume that $r_n \in M$ for each n. Then r is in the convex set $\text{cl}(S \cap M \cap H_1)$, so $[r_n, r) \subseteq S \cap M \cap H_1$ for each n. Choose r_0 to be any point r_n. Since f preserves convex sets, f maps $[r_0, r]$ onto a segment in

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$f(S)$, and hence f maps $[r_0, r]$ into $[f(x), f(y)]$. A similar argument may be used to select a point s in $S \cap H$ and a segment $(s, s_0]$ in $S \cap M \cap H_2$ so that f maps $[s, s_0]$ into $[f(x), f(y)]$. Clearly $f(r) = f(s) = f(z)$.

Since (p, z) intersects C, both (p, r) and (p, s) must intersect aff C. Now $\{p\} \cup C \subseteq \ker(S \cap M)$, so each point of $[r_0, r] \cup [s, s_0]$ sees $\text{conv}(\{p\} \cup C)$ via $S \cap M$. But then points of C are captured interior to the d-dimensional set $\text{conv}(C \cup \{p\} \cup [r_0, r]) \cup \text{conv}(C \cup \{p\} \cup [s, s_0]) \subseteq S$, contradicting the fact that $C \subseteq Q$. Our assumption for Case 1 must be false, and no point of Q lies in rel int $\text{conv}\{p, x, y\}$.

Case 2. Since there can be no points of Q in rel int $\text{conv}\{p, x, y\}$, suppose there are points of Q in $(p, x) \cup (p, y)$. Say for some component C of Q, $(p, x) \cap C \neq \emptyset$, and let $H = \text{aff}(\{p\} \cup C)$. Since $S \cap H$ is convex and $x \in H$, it follows that $y \notin H$, and we may assume that y lies in the open halfspace H_2 determined by H. As in Case 1, let M be a convex neighborhood of H disjoint from all components of Q which do not lie in H, and select a point $s \in H$ and a segment $(s, s_0]$ in $S \cap M \cap H_2$ such that f maps $[s, s_0]$ into $[f(x), f(y)]$.

First we show that $x \in \text{cl}(S \cap H_2)$: If $x \in \text{cl}(S \cap H_1)$, then for any point x_0 in the convex set $S \cap M \cap H_1$, $[x_0, x] \subseteq S \cap M$. Using an argument in Case 1 above, since $\{p\} \cup C \subseteq \ker(S \cap M)$, each point of the set $[x_0, x] \cup [s, s_0]$ would see $\text{conv}(\{p\} \cup C)$ via $S \cap M$. Hence points of C would be captured interior to the d-dimensional set $\text{conv}(C \cup \{p\} \cup [x_0, x]) \cup \text{conv}(C \cup \{p\} \cup [s, s_0]) \subseteq S$, mpossible. iWe conclude that $x \notin \text{cl}(S \cap H_1)$, and since $S = \text{cl}(\text{int}\, S)$, it follcws that $x \in \text{cl}(S \cap H_2)$.

Next we select a convex neighborhood U of $\text{conv}\{p, x, y\}$ such that the only components of Q containing points of $U \cap S$ necessarily intersect $[p, x] \cup [p, y]$. (Clearly this is possible: Since $(x, y) \cap S = \emptyset$ we have $Q \cap \text{conv}\{p, x, y\} \subseteq [p, x] \cup [p, y]$, and Q is a finite union of closed convex sets.) Since $x \in \text{cl}(S \cap H_2)$, we may select a sequence $\{x_n\}$ in $S \cap U \cap H_2$ converging to x.

If $(p, y) \cap Q = \emptyset$, then $Q \cap U \subseteq H$, for every n there would be no lnc points in $\text{conv}\{x_n, y, p\} \sim [x_n, y]$, so $[x_n, y] \subseteq S$ and $[x, y] \subseteq S$, impossible. Thus $(p, y) \cap Q \neq \emptyset$, and for some convex component D of Q, (p, y) cuts D. (Note that $D \neq C$ since $y \notin H$.) Let $J = \text{aff}(D \cup \{p\})$. By an earlier argument $x \notin J$, so assume x is in the open halfspace J_1 determined by J. Now it is easy to show that $y \notin \text{cl}(S \cap U \cap J_1)$, for if $\{y_n\}$ were a sequence in $S \cap U \cap J_1$ converging to y, then since $Q \cap U \subseteq H \cup J$, for n sufficiently large there would be no lnc point in $\text{conv}\{x_n, y_n, p\}$, $[x_n, y_n] \subseteq S$, and $[x, y] \subseteq S$, a contradiction. Therefore $y \in \text{cl}(S \cap U \cap J_2)$.

Again as in Case 1, select a point r in J and a segment $[r_0, r)$ in $S \cap J_1$ such that f maps $[r_0, r]$ into $[f(x), f(y)]$. Using earlier arguments, select y_0 in $S \cap J_2$ with $[y, y_0] \subseteq S$. For r_0, y_0 sufficiently close to r and y, respectively, each point of $[r_0, r] \cup [y, y_0]$ sees every point of D via S, and points of D lie interior to the set $\text{conv}(D \cup \{p\} \cup [r_0, r]) \cup \text{conv}(D \cup \{p\} \cup [y, y_0])$, impossible. We have a contradiction, our assumption for Case 2 cannot be true, and $(p, x) \cup (p, y)$ must be disjoint from Q.

From Cases 1 and 2 we conclude that $\text{conv}\{p, x, y\} \sim [x, y]$ contains no points of Q. Hence our original supposition is false and $(x, y) \subseteq S$, the desired result. It follows that each A_i set is convex, $1 \leqq i \leqq 2m - 2$, and S is indeed a union of $2(m - 1)$ or fewer convex sets, finishing the proof of the theorem.

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