LIMIT CIRCLE CRITERIA FOR 2*n*th ORDER DIFFERENTIAL OPERATORS

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1. Introduction

A formally self-adjoint differential operator L is said to be of limit circle type at infinity if its highest order coefficient is zero-free and all solutions x of $L(x)=0$ are square-integrable on $[a,\infty)$. (We will drop reference to "at infinity" in what follows.)

For the second-order case

$$L(x)\equiv(rx')'+qx \tag{1.1}$$

Dunford and Schwartz (**3**) p. 1409 prove that *given*

$$\int_a^\infty |\tfrac{1}{4}(rq)'^2(rq)^{-3/2}q^{-1}+\{(rq)'(rq)^{-1/2}q^{-1}\}'|\,dt<\infty \tag{1.2}$$

then L is of limit circle type if and only if

$$\int_a^\infty (rq)^{-1/2}\,dt<\infty. \tag{1.3}$$

This generalised a number of previously known results. Knowles (**8**) showed that this is one of a family of similar conditions obtained by making different unitary transformations of variable in (1.1).

Few explicit general criteria are known for higher-order equations. Zettl (**13**) constructs limit circle operators of arbitrary order by taking powers of limit circle operators of type (1.1). Eastham (**4**) and Hadid (**6**) obtain explicit criteria (see Examples 3 and 4 of Section 3 below) using a non-asymptotic method of Kuptsov. Eastham (**4**) p. 257–258 observes that for two special cases a necessary and sufficient condition of type (1.3) holds, given certain other integrability conditions. Using asymptotic-type methods Read (**11**) has derived conditions for $L(x)=\sum_{m=0}^{n}(p_m(t)x^{(m)})^{(m)}$ with all the p_m eventually positive, to be limit circle. We will refer to such results where the only restrictions are that the p_m be positive and that certain order conditions hold as *positive-coefficient* results. The results of Eastham and Hadid referred to above require further restrictions and we will refer to them as *restricted-coefficient* results.

For operators with polynomial coefficients

$$L(x)=(t^{\gamma}x^{(2)})^{(2)}+a(t^{\alpha}x')'+bt^{\beta}x \tag{1.4}$$

Walker (**12**) shows that (1.4) is limit circle if a, $b>0$ and

$$\begin{aligned} \alpha-2<\beta<2\alpha-\gamma \\ \alpha+\beta>2. \end{aligned} \tag{1.5}$$

Devinatz (**2**) (and also Eastham (**5**)) discusses the case $\gamma=0$, $\beta=2\alpha$, showing that if $a^2>4b$ then (1.5) implies L is limit circle (a restricted-coefficient result). Eastham (**5**) and Kogan and Rofe-Beketov (**9**) discuss the case $\alpha-2=\beta$ which is also restricted-coefficient. (See the remarks at the end of Example 6 Section 3 below). Read (**11**) uses his results mentioned above to generalise (1.5) to the 2nth order case (again a positive-coefficient result).

In this paper we will establish a criterion similar to (1.2) and (1.3) for 2nth order formally self-adjoint operators (Theorem 2 of Section 1). The results allow two of the coefficients of L to be arbitrary and the rest are then determined. Further, the results are of restricted coefficient type. They are obtained by using a shearing transformation due to Hinton (**7**) after a standard substitution, and applying Levinson's theorem (see Theorem 1) to obtain asymptotic expansions of the solutions. In the examples of Section 3, we show that this result includes those of Eastham and Hadid (Examples 3 and 4), the restricted coefficient result of Devinatz (Example 6) extended to include the case $\gamma\neq 0$, and a generalisation of the latter to the 2nth order case (Example 7). The restricted-coefficient results for $\alpha-2=\beta$ (referred to above) and positive-coefficient results of Walker, Devinatz and Read are not included.

Applications of asymptotic theory to obtain limit circle criteria for higher order equations with oscillatory coefficients are not discussed here, and will be treated elsewhere.

2. Main theorems

We will assume familiarity with Naimark (**10**) §§15, 16 which deals with quasiderivatives and their application to formally self-adjoint differential operators. We consider

$$\begin{aligned} L(x) &\equiv (r^{2n}x^{(n)})^{(n)}+((a_{n-1}r^{2n-2}q^2+k_{n-1}r^{2n-1}q)x^{(n-1)})^{(n-1)}+\dots \\ &\quad+((a_1r^2q^{2n-2}+k_1r^3q^{2n-3})x')'+(a_0q^{2n}+k_0rq^{2n-1})x \\ &\equiv \sum_{m=0}^{n}(p_{n-m}x^{(m)})^{(m)} \quad \text{(say)} \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} p_m &= (a_mr^{2m}q^{2(n-m)}+k_mr^{2m+1}q^{2(n-m)-1}) \quad (m=1,\dots,2n) \\ p_0 &= r^{2n}. \end{aligned} \tag{2.2}$$

We assume

$$\begin{aligned} &r(t), q(t)\in C^2[a,\infty) \text{ are real-valued and positive} \\ &k_i(t)\in L[a,\infty) \quad (i=0,\dots,n-1) \\ &a_i \text{ are real constants } (i=0,\dots,n-1). \end{aligned} \tag{2.3}$$

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The $(2n+1)$ quasiderivatives relative to $L(x)$ are defined by

$$\begin{aligned} x^{[0]} &= x \\ x^{[m]} &= x^{(m)} \\ x^{[n]} &= p_0 x^{(n)} \\ x^{[n+m]} &= p_m x^{(n-m)} + \frac{d}{dt} x^{[n+m-1]} \quad (1 \leqq m \leqq n). \end{aligned}$$

Then $L(x) \equiv x^{[2n]}$.

Let x be a solution of $L(x)=0$, and let $y=(x^{[0]}, \ldots, x^{[2n-1]})^T$. Then y is a solution of $dy/dt = Ay$ where

$$\left[\begin{array}{cccc|cccc} 0 & 1 & & & & & & 0 \\ & \ddots & \ddots & & & & \iddots & \\ & & \ddots & 1 & 0 & \iddots & & \\ & & & 0 & \dfrac{1}{p_0} & & & \\ \hline & & & -p_1 & 0 & 1 & & \\ & & \iddots & & & 0 & \ddots & \\ & \iddots & & & & & \ddots & \ddots \\ & -p_{n-1} & & & & & \ddots & 1 \\ -p_n & & & & & & & 0 \end{array}\right] \quad \text{(unfilled entries 0).} \tag{2.4}$$

Conversely, a solution of $dy/dt = Ay$ has a first component x which is a solution of $Lx=0$.

We now make a shearing transformation following Hinton (7). Let

$$\begin{aligned} Q &= (rq^{2n-1})^{1/2} \operatorname{diag}(1, r/q, \ldots, (r/q)^{n-1}, r^{-2n}(r/q)^n, \ldots, r^{-2n}(r/q)^{2n-1}) \\ &= \operatorname{diag}(s_1, \ldots, s_{2n}) \quad \text{(say)} \\ &= \operatorname{diag}(q^{\alpha_1} r^{\beta_1}, \ldots, q^{\alpha_{2n}} r^{\beta_{2n}}). \end{aligned} \tag{2.5}$$

Then

$$s_m s_{2n-m+1} = 1 \quad (m=1, \ldots, n)$$

so that

$$\alpha_m + \alpha_{2n-m+1} = \beta_m + \beta_{2n-m+1} = 0 \quad (m = 1, \ldots, n). \tag{2.6}$$

Let $z = Qy$. Then

$$z' = (QAQ^{-1} + Q'Q^{-1})z \tag{2.7}$$

where

$$(QAQ^{-1})_{ij} = s_i [A]_{ij} s_j^{-1}$$

so that

$$QAQ^{-1}=\left(\frac{q}{r}\right)\left[\begin{array}{cccc|cccc} 0 & 1 & & & & & & 0\\ & \ddots & \ddots & & & & & \\ & & \ddots & 1 & & 0 & & \\ & & & 0 & 1 & & & \\ \hline & & & -a_{n-1} & 0 & 1 & & \\ & & \cdot & & & \ddots & \ddots & \\ & -a_1 & & & & & \ddots & 1\\ -a_0 & & & & & & & 0 \end{array}\right]+\left[\begin{array}{cccc|c} & & 0 & & 0\\ \hline & & & -k_{n-1} & \\ & & \cdot & & \\ & -k_1 & & & 0\\ -k_0 & & & & \end{array}\right]$$

$$=\left(\frac{q}{r}\right)A_1+K(t) \quad \text{(say)}. \tag{2.8}$$

Also

$$Q'Q^{-1}=\frac{r'}{r}D_1+\frac{q'}{q}D_2 \tag{2.9}$$

where

$$D_1=\operatorname{diag}(\beta_1,\ldots,\beta_{2n}) \quad \text{and} \quad D_2=\operatorname{diag}(\alpha_1,\ldots,\alpha_{2n}). \tag{2.10}$$

Finally, we make the transformation of independent variable

$$d\tau=\frac{q}{r}\,dt. \tag{2.11}$$

Then

$$\frac{dz}{d\tau}=\left(A_1+\left(\frac{r}{q}\right)K(t)+\left(\frac{r}{q}\right)\left(\frac{r'}{r}D_1+\frac{q'}{q}D_2\right)\right)z. \tag{2.12}$$

We will also use the notation

$$\Delta(\mu)=\mu^n+a_{n-1}\mu^{n-1}+\ldots+a_1\mu+a_0 \tag{2.13}$$

and we will denote by

$$\lambda_1,\ldots,\lambda_{2n} \tag{2.14}$$

the roots λ of $\Delta(\lambda^2)=0$.

Theorem 1. *Let* (2.3) *hold and let* $q/r\notin L[a,\infty)$. *Suppose that*

$$\left[\frac{r}{q}\frac{r'}{r}\right]',\ \left[\frac{r}{q}\frac{q'}{q}\right]',\ \frac{r}{q}\left(\frac{r'}{r}\right)^2 \quad \textit{and} \quad \frac{r}{q}\left(\frac{q'}{q}\right)^2\in L[a,\infty).$$

Let the zeros of $\Delta(\mu)=0$ *be real and distinct and let* $a_0\neq 0$.

Then there are independent solutions x_i $(i=1,\dots,2n)$ *of* $L(x)=0$ *satisfying*

$$x_i(t)=(rq^{2n-1})^{-1/2}\exp\left\{\lambda_i\int_a^t \frac{q}{r}\,ds\right\}(1+o(1)).$$

Proof. We apply Levinson's theorem (Coddington and Levinson (**1**), Theorem 8.1 of Chapter 3) to the system (2.12). The hypotheses (2.3) on the k_i imply

$$\int_{\tau(a)}^{\infty}\left|\frac{r}{q}K(\tau)\right|d\tau=\int_a^{\infty}|K(t)|\,dt<\infty.$$

Let

$$\alpha=\frac{r}{q}\frac{r'}{r},\quad \beta=\frac{r}{q}\frac{q'}{q}.$$

Then

$$\int_{\tau(a)}^{\infty}\left|\frac{d}{d\tau}\alpha\right|d\tau=\int_a^{\infty}\left|\frac{d}{dt}\alpha\right|dt<\infty$$

and similarly,

$$\int_{\tau(a)}^{\infty}\left|\frac{d}{d\tau}\beta\right|d\tau<\infty.$$

So the integrability conditions of Levinson's theorem are satisfied.

It is clear that there exist functions $\gamma_i(\tau)$ such that $\lim_{\tau\to\infty}\gamma_i(\tau)=0$ and such that the functions

$$\nu_i(\tau)=\lambda_i+\gamma_i(\tau)\quad(i=1,\dots,2n)$$

are the distinct eigenvalues of $A_1+\alpha D_1+\beta D_2$. We will show that $\gamma_i(\tau)\in L(d\tau)$, and this will imply that the remaining "dichotomy" condition of Levinson's theorem is satisfied. We will need the fact that

$$\int_{\tau(a)}^{\infty}(\alpha^2(\tau)+\beta^2(\tau))\,d\tau<\infty \tag{2.15}$$

which follows from the integrability hypotheses of the theorem.

Let

$$\delta_i=[\alpha D_1+\beta D_2]_{ii}\quad(i=1,\dots,2n).$$

Then by (2.6) we have

$$\delta_m+\delta_{2n-m+1}=0 \quad (m=1,\dots,n). \tag{2.16}$$

$$\det(A_1+\alpha D_1+\beta D_2-\mu I)=\det\left[\begin{array}{cccc|cccc}(\delta_1-\mu) & 1 & & & & & & \\ & (\delta_2-\mu) & \ddots & & & & & \\ & & \ddots & 1 & & & & \\ & & & (\delta_n-\mu) & 1 & & & \\ \hline & & & -a_{n-1} & (\delta_{n+1}-\mu) & & & \\ & & \iddots & & & \ddots & \ddots & \\ & -a_1 & & & & & \ddots & 1 \\ -a_0 & & & & & & & (\delta_{2n}-\mu)\end{array}\right]$$

$$=a_0+(\delta_1-\mu)(\delta_{2n}-\mu)\det\left[\begin{array}{ccc|ccc}(\delta_2-\mu) & 1 & & & & \\ & \ddots & 1 & & & \\ & & (\delta_n-\mu) & 1 & & \\ \hline & & -a_{n-1} & (\delta_{n+1}-\mu) & 1 & \\ & \iddots & & & \ddots & 1 \\ -a_1 & & & & & (\delta_{2n-1}-\mu)\end{array}\right] \tag{2.17}$$

Then it follows by induction using (2.17) (setting $a_n=1$) that

$$\Phi(\mu)\overset{\text{defn}}{=}\det(A_1+\alpha D_1+\beta D_2-\mu I)=a_0+\sum_{m=1}^{n}a_m\prod_{i=1}^{m}(\delta_i-\mu)(\delta_{2n-i+1}-\mu). \tag{2.18}$$

We may regard $\Phi(\mu)$ as a polynomial in the δ_i. The constant term is $\Delta(\mu^2)$ and the linear term vanishes by (2.16). Hence

$$0=\Phi(\lambda_i+\gamma_i)=\Delta((\lambda_i+\gamma_i)^2)+\{\text{polynomial in the } \delta_i \text{ of degree } \geqq 2\} \tag{2.19}$$

Using Taylor's theorem

$$\Delta_1(\lambda_i+\gamma_i)\overset{\text{defn}}{=}\Delta((\lambda_i+\gamma_i)^2)=\Delta_1(\lambda_i)+\Delta_1'(\lambda_i)\gamma_i+\dots+(\Delta_1^{(n)}(\lambda_i)/n!)\gamma_i^n$$

$$=\gamma_i(\Delta_1'(\lambda_i)+o(1)) \quad \text{as } \tau\to\infty.$$

Since $\Delta(\mu)$ has distinct zeros by hypothesis and $a_0\neq 0$, it follows that $\Delta_1(\mu)$ has distinct zeros, so $\Delta_1(\lambda_i)=0$ and $\Delta_1'(\lambda_i)\neq 0$. Substituting for $\Delta((\lambda_i+\gamma_i)^2)$ in (2.19) and using the fact that the δ_i are square integrable by (2.15), we see that $\gamma_i\in L(d\tau)$.

We now apply Levinson's theorem to obtain independent solutions z_i $(i=1,\ldots,2n)$ of (2.12) satisfying

$$z_i(\tau)=\exp\int^{\tau}(\lambda_i+\gamma_i)\,d\tau(1+o(1))e_i$$
$$=\text{const.}(\exp\lambda_i\tau)(1+o(1))e_i \tag{2.20}$$

where e_i is an eigenvector of A_1 corresponding to λ_i. It is then easily seen that if $(A_1-\lambda_i I)e=0$ and the first component of e is zero, then $e=0$. Thus each e_i has non-zero first component. Finally,

$$z=Qy=Q(x,x',\ldots)^{T}$$

so from (2.20) we have the existence of x_i as in the statement of this theorem.

Theorem 2. *Let* (2.5) *hold and let* $q/r\notin L[a,\infty)$. *Suppose that*

$$\left[\frac{r}{q}\frac{r'}{r}\right]',\ \left[\frac{r}{q}\frac{q'}{q}\right]',\ \frac{r}{q}\left(\frac{r'}{r}\right)^2\quad\textit{and}\quad\frac{r}{q}\left(\frac{q'}{q}\right)^2\in L[a,\infty).$$

Let the zeros of $\Delta(\mu)$ *be real, distinct and negative. Then L of* (2.1) *is of limit circle type at infinity if and only if*

$$\int_a^\infty\frac{dt}{rq^{2n-1}}<\infty. \tag{2.21}$$

Proof. Since the a_i are real, the λ_i occur in conjugate pairs and since the zeros of Δ are negative, the λ_i are pure imaginary. Hence if we order the λ_i suitably, there are n pairs of solutions of $L(x)=0$, z_m and $\tilde z_m$ $(m=1,\ldots,n)$ satisfying

$$z_m=(rq^{2n-1})^{-1/2}(\cos i\lambda_m\tau+o(1))$$
$$\tilde z_m=(rq^{2n-1})^{-1/2}(\sin i\lambda_m\tau+o(1))$$

and these solutions are independent. Hence (2.21) implies that L of (2.1) is of limit circle type. Conversely, if (2.1) is limit circle, then it follows from the relations

$$z_i^2+\tilde z_i^2=(rq^{2n-1})^{-1}(1+o(1))\geqq\tfrac{1}{2}(rq^{2n-1})^{-1}\quad(\text{all } t \text{ sufficiently large})$$

that (2.21) holds.

Remarks. (1) If the conditions of the statement of Theorem 2 (or Theorem 1) hold then we may take

$$k_i(t)=\left[\frac{r}{q}\left(\frac{q'}{q}\right)\right]'\quad(\text{all } i)$$

for example, since the k_i are then integrable,

(2) If $r\equiv1$, the conditions of Theorem 2 become

$$\frac{q''}{q^2},\ \frac{(q')^2}{q^3}\in L[a,\infty).$$

(3) If we write $r_1 = r^{2n}$, $q_1 = q^{2n}$ then since $q_1'/q_1 = q'/q$, and similarly for r_1 and r, we may rewrite the integrability conditions as

$$\left[\left(\frac{r_1}{q_1}\right)^{1/2n}\left(\frac{r_1'}{r_1}\right)\right]', \left[\left(\frac{r_1}{q_1}\right)^{1/2n}\left(\frac{q_1'}{q_1}\right)\right]', \left(\frac{r_1}{q_1}\right)^{1/2n}\left(\frac{r_1'}{r_1}\right)^2, \left(\frac{r_1}{q_1}\right)^{1/2n}\left(\frac{q_1'}{q_1}\right)^2 \in L[a, \infty).$$

If $r \equiv 1$ these become

$$\frac{q_1''}{q_1^{1+1/2n}}, \quad \frac{(q_1')^2}{q_1^{2+1/2n}} \in L[a, \infty).$$

(4) It can be shown that

$$q \notin L[a, \infty) \quad \text{and} \quad \frac{q''}{q^2} \in L[a, \infty) \quad \text{implies} \quad \frac{(q')^2}{q^3} \in L[a, \infty)$$

(see Hinton (7), Corollary).

3. Applications

Example 1. We consider the second-order case, $n = 1$. In this case $\alpha_1 = -\alpha_2 = \beta_1 = -\beta_2 = \frac{1}{2}$ and

$$\alpha D_1 + \beta D_2 = \frac{1}{2}\frac{r}{q}\left(\frac{r'}{r} + \frac{q'}{q}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

The integrability conditions of Theorem 1 or 2 can be weakened to

$$\left[\frac{r}{q}\left(\frac{r'}{r} + \frac{q'}{q}\right)\right]' \quad \text{and} \quad \frac{r}{q}\left(\frac{r'}{r} + \frac{q'}{q}\right)^2 \in L[a, \infty). \tag{3.1}$$

Taking $a_1 = 1$ we get $\Delta(\mu) = \mu + 1$ which has a negative zero. Hence if (3.1) holds and $q/r \in L[a, \infty)$ then the equation

$$(r^2x')' + q^2x = 0 \tag{3.2}$$

is limit circle if and only if

$$\int_a^\infty \frac{dt}{rq} < \infty. \tag{3.3}$$

This is close to the result of Dunford and Schwartz (see (1.2) above) which states that if

$$\frac{1}{4}\frac{r}{q}\left(\frac{r'}{r} + \frac{q'}{q}\right)^2 + \left[\frac{r}{q}\left(\frac{r'}{r} + \frac{q'}{q}\right)\right]' \in L[a, \infty) \tag{3.4}$$

then (3.2) is limit circle if and only if (3.3) holds. (The equivalence of (3.4) with the Dunford and Schwartz condition follows readily.)

For $r \equiv 1$, the condition $q''/q^2 \in L[a, \infty)$ and $q \notin L[a, \infty)$ imply (3.1) (see Remark 4 above).

We can also conclude that given (3.1) and $q/r \notin L[a, \infty)$ then the equation

$$(r^2x')' + (q^2 + krq)x = 0 \tag{3.5}$$

is limit circle for any $k \in L[a, \infty)$ if and only if (3.3) holds.

Example 2. We consider the fourth-order case $n=2$ i.e.

$$(r^4x^{(2)})^{(2)}+((ar^2q^2+k_1r^3q)x')'+(bq^4+k_0rq^3)x=0. \tag{3.6}$$

$\Delta(\mu)=\mu^2+a\mu+b$ and this has distinct negative zeros if and only if

$$a, b>0 \quad \text{and} \quad a^2>4b. \tag{3.7}$$

The integrability conditions are

$$\frac{q}{r}\notin L[a,\infty) \quad \text{and} \quad \left[\frac{r}{q}\frac{r'}{r}\right]', \left[\frac{r}{q}\frac{q'}{q}\right]', \frac{r}{q}\left(\left(\frac{r'}{r}\right)^2+\left(\frac{q'}{q}\right)^2\right)\in L[a,\infty). \tag{3.8}$$

Hence if (3.7) and (3.8) hold and k_0, $k_1\in L[a,\infty)$ then (3.6) is limit circle if and only if

$$\int_a^\infty \frac{dt}{rq^3}<\infty. \tag{3.9}$$

If $r\equiv 1$ then (3.8) can be simplified (Remark 4) to

$$q\notin L[a,\infty) \quad \text{and} \quad \frac{q''}{q^2}\in L[a,\infty). \tag{3.8)'$$

If $q_1=q^4$ then

$$\frac{q''}{q^2}=\frac{q_1''}{q_1^{5/4}}. \tag{3.10}$$

We will now show that Example 2 contains a number of results obtained in the literature by other means.

Example 3. Using a non-asymptotic method, Hadid (**6**) Example 1 (a) (generalising a result of Eastham (**4**)) considers

$$x^{(4)}+a(a_1^\gamma x')'+\{c(\gamma-1)q_1^{\gamma-2}q_1'^2+cq_1''q_1^{\gamma-1}+bq_1^{2\gamma}\}x=0 \tag{3.11}$$

where a, b and c are positive and satisfy

$$0<b<c(a-c/\gamma)/\gamma; \tag{3.12}$$

$$q_1\in C^2,\ q_1>0,\ q_1'\geqq 0;\ q_1'^2q_1^{-(\gamma/2)-2} \quad \text{and} \quad q_1''q_1^{-(\gamma/2)-1}\in L[a,\infty). \tag{3.13}$$

He shows that all solutions x of (3.11) then satisfy

$$x=0(q^{-k}) \tag{3.14}$$

where $k=\frac{1}{4}[2\gamma-\{(c^3/b-3c\gamma^2+a\gamma^3)/(a\gamma-\gamma^2b/c-c)^{1/2}]$. In the notation we have been using,

$$q=q_1^{\gamma/2}.$$

Then

$$q_1'^2q_1^{-(\gamma/2)-2}=\left(\frac{q_1'}{q_1}\right)^2q_1^{-\gamma/2}=\left(\frac{q'}{q}\right)^2q^{-1}=\frac{q'^2}{q^3}$$

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and

$$q_1''q_1^{-(\gamma/2)-1}=\frac{2}{\gamma}\frac{q''}{q^2}+\frac{2}{\gamma}\left(\frac{2}{\gamma}-1\right)\frac{q'^2}{q^3}.$$

Hence (3.13) implies (3.8)$'$.

Let $c/\gamma=\delta$. Then (3.12) states that

$$0<b<\delta(a-\delta).$$

Then if (3.12) holds,

$$a^2-4b>a^2-4\delta(a-\delta)=(a-2\delta)^2\geqq 0$$

so that $a^2-4b>0$. Further,

$$(q_1')^2q_1^{\gamma-2}/q^3=q_1'^2q_1^{-(\gamma/2)-2}\in L[a,\infty)$$

and

$$q_1''q_1^{\gamma-1}/q^3=q_1''q_1^{-(\gamma/2)-1}\in L[a,\infty)$$

so that

$$c(\gamma-1)q_1^{\gamma-2}q_1'^2+cq_1''q_1^{\gamma-1}=k_0q^3 \quad \text{where} \quad k_0\in L[a,\infty).$$

It follows from Example 2 above with $k_1=0$ that all solutions of (3.11) are square integrable if and only if

$$\int_a^\infty \frac{dt}{q^3}<\infty \quad \left(\text{or } \int_a^\infty \frac{dt}{q_1^{3\gamma/2}}<\infty\right). \tag{3.15}$$

It also follows from Theorem 1 that all solutions of (3.11) satisfy

$$x=O(q_1^{-3\gamma/4}) \tag{3.16}$$

whether or not (3.15) holds. (Compare this with (3.14)).

We will generalise this in Example 5 below where, in particular, the coefficients c and $c(\gamma-1)$ in the bracket of (3.11) can be chosen arbitrarily if $a^2>4b$ is assumed.

Example 4. Hadid (6) Example 4 proves the following:
Consider

$$x^{(4)}+(\{\gamma q_1''q_1^{-1}+aq_1^{\gamma}-\gamma(\gamma+1)q_1'^2q_1^{-2}\}x')x'+bq_1^{2\gamma}x=0 \tag{3.17}$$

subject to (3.13) and

$$0<b<a-1. \tag{3.18}$$

It is shown that all solutions x of (3.17) then satisfy

$$x=O(q^{-3\gamma/4}). \tag{3.19}$$

By (3.18) we see that

$$a^2-4b\geqq a^2-4(a-1)=(a-2)^2\geqq 0$$

so $a^2-4b>0$.

In our notation we have again $q=q_1^{\gamma/2}$. Further,

$$(\gamma q_1''q_1^{-1}-\gamma(\gamma+1)q_1'^2q_1^{-2})/q=\gamma q_1''q_1^{-(\gamma/2)-1}-\gamma(\gamma+1)q_1'^2q_1^{-(\gamma/2)-2}\in L[a,\infty) \text{ by (3.13).}$$

As before we can use Example 2 with $k_0\equiv 0$ to deduce that all solutions of (3.17) satisfy

$$x=O(q^{-3/2})=O(q_1^{-3\gamma/4})$$

which is the same as (3.19). We now give an example generalising Examples 3 and 4.

Example 5. Using Remark 4 above and Example 2 with $r\equiv 1$ we obtain the following:

Let

$$a,b>0,\quad a^2>4b,\quad q\in L[a,\infty),\quad q''/q^2\in L[a,\infty) \tag{3.20}$$

and let c_i $(i=1,\dots,4)$ be arbitrary constants. Consider

$$x^{(4)}+((aq^2+c_1q'^2/q^2+c_2q''/q)x')'+(bq^4+c_3q'^2+c_4q''q)x=0. \tag{3.21}$$

Then all solutions of (3.21) satisfy

$$x=O(q^{-3/2})$$

and (3.21) is limit circle if and only if

$$\int_a^\infty \frac{dt}{q^3}<\infty.$$

Example 6. We consider

$$(t^\gamma x^{(2)})^{(2)}+a(t^\alpha x')'+bt^\beta x=0. \tag{3.22}$$

To apply our theorems we need

$$2\alpha=\beta+\gamma.$$

Then

$$\left[\frac{r}{q}\frac{r'}{r}\right]',\ \left[\frac{r}{q}\frac{q'}{q}\right]',\ \frac{r}{q}\left(\frac{r'}{r}\right)^2 \quad\text{and}\quad \frac{r}{q}\left(\frac{q'}{q}\right)^2 \text{ are all } O(t^{[(\alpha-\beta)/2]-2}).$$

Hence the integrability conditions are satisfied if

$$\frac{\alpha-\beta}{4}<1 \quad\text{i.e.}\quad \gamma<\beta+4$$

or

$$\alpha-\beta<2.$$

We have

$$rq^3\equiv t^{(\gamma+3\beta)/4}.$$

Hence

$$\int^{\infty}\frac{dt}{rq^3}<\infty \quad \text{if and only if} \quad \frac{\gamma+3\beta}{4}>1$$

i.e.

$$\gamma+3\beta>4 \quad \text{or} \quad \alpha+\beta>2.$$

Thus we have the limit circle case if

$$\alpha-2<\beta=2\alpha-\gamma \tag{3.23}$$

$$\alpha+\beta>2 \tag{3.24}$$

and

$$a^2>4b; \quad a, b>0. \tag{3.25}$$

This result in the case $\gamma=0$ is due to Devinatz (**2**). See also Eastham (**4**) p. 267. In fact, our results imply that if (3.23) and (3.25) hold then (3.22) is limit circle if and only if (3.24) holds. This also applies to the equation

$$(t^{\gamma}x^{(2)})^{(2)}+((at^{(\beta+\gamma)/2}+u(t))x')'+(bt^{\beta}+v(t))x=0$$

provided that

$$u(t)=O(t^{(3\gamma+\beta)/4-1-\varepsilon}) \quad \text{and} \quad v(t)=O(t^{(3\beta+\gamma)/4-1-\varepsilon}) \quad (\text{some } \varepsilon>0).$$

It should be noted that results of Walker (**12**) show that (3.22) is limit circle if

$$\alpha-2<\beta<2\alpha-\gamma$$
$$\alpha+\beta>2$$

irrespective of a and b. Also results of Eastham (**5**) show that if

$$\alpha-\beta=2 \quad \text{and} \quad \beta>\gamma$$

then (3.22) is limit circle if

$$(b/a)^{1/2}>(3\beta^2-2\beta\gamma+8\beta+4-\gamma^2)/8(\beta-\gamma).$$

Finally, Kogan and Rofe-Beketov (**9**) show that (3.22) is limit circle if

$$\alpha=\gamma=\beta+2 \quad \text{and} \quad \frac{b}{a}>\tfrac{1}{2}\beta+\tfrac{1}{4}.$$

These results of Walker, Eastham and Rofe-Beketov do not seem to follow from our analysis.

Example 7. We consider

$$\begin{aligned}(t^{\gamma}x^{(n)})^{(n)}&+((a_{n-1}t^{\gamma[1-(1/n)]+\beta/n}+v_{n-1})x^{(n-1)})^{(n-1)}\\&+((a_{n-2}t^{\gamma[1-(2/n)]+2\beta/n}+v_{n-2})x^{(n-2)})^{(n-2)}\\&+\ldots+((a_1t^{(\gamma/n)+\beta[1-(1/n)]}+v_1)x')'+(a_0t^{\beta}+v_0)x=0.\end{aligned} \tag{3.26}$$

Suppose that

$$\Delta(\mu) = \mu^n + a_{n-1}\mu^{n-1} + \dots + a_1\mu + a_0 \tag{3.27}$$

has distinct negative zeros

$$t^{[(\gamma-\beta)/2n]-2} \in L[a, \infty) \quad \text{i.e.} \quad \gamma - \beta < 2n, \tag{3.28}$$

and

$$v_m = O(t^\delta) \quad \text{where} \quad \delta = \gamma\left(\frac{2m+1}{2n}\right) + \beta\left(\frac{2n-2m-1}{2n}\right) - 1 - \varepsilon \quad (\text{some } \varepsilon > 0). \tag{3.29}$$

Then (3.26) is limit circle if and only if

$$\frac{\gamma}{2n} + \beta\left(1 - \frac{1}{2n}\right) > 1. \tag{3.30}$$

Note that (3.30) reduces to (3.24) if $n = 2$ and $\beta = 2\alpha - \gamma$ as in (3.23), and that (3.28) reduces to $\alpha - \beta < 2$ in this case as in (3.23).

This result complements that of Read (**11**) Theorem 2. It follows from (3.26) and (3.28) that (in Read's notation)

$$\alpha_{n-1} - \alpha_n = \alpha_{n-2} - \alpha_{n-1} = \dots = \alpha_0 - \alpha_1 > -2 \tag{3.31}$$

(compare with Read's equation (1.4)) and (3.30) is the same as his (1.5). (For $n = 2$, (3.31) is equivalent to (3.23)). In Read's case his conditions (1.4) and (1.5) imply the equation is limit circle irrespective of the nature of the positive coefficients a_i. See also the remarks in Read (**11**) p. 109 first paragraph.

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