OBITUARY

A. C. AITKEN, D.Sc., F.R.S.

Alec Aitken was much more than an outstanding mathematician; his versatile genius ranged far beyond his chosen specialism, and no notice by an individual could do justice to his achievements in the many fields in which he excelled. In the following pages several contributors have written independently on different aspects of his activity; it is hoped that by this illumination of different facets of a personality unique in our time a truer picture may emerge than could be achieved by an artificially edited memoir.

The notice begins with a section containing biographical and personal details prepared by Dr R. Schlapp. These are supplemented by a note, quoted from a letter to Professor Erdélyi from Professor E. T. Copson, who was closely associated with Aitken in the early formative years in Edinburgh. There are appreciations of Aitken's work in Statistics by Professor D. G. Kendall, in Numerical Analysis by Dr J .C. P. Miller, and in Pure Mathematics by Professor W. Ledermann. Finally a list of publications is included, and a remarkable (hitherto unpublished) letter from Professor Aitken to an Edinburgh colleague. The Editors wish to express their thanks to all the contributors, to Paul Shillabeer, F.R.P.S., for permission to use his photograph of Professor Aitken, and especially to Mrs. Aitken who has made much valuable material freely available to them.

Alexander Craig Aitken was born in Dunedin, New Zealand, on 1st April, 1895, the eldest son of the seven children of William and Elizabeth Aitken. On his father's side he was descended from Lanarkshire farming families, his grandfather having emigrated to Otago about 1868. From primary school he went on in 1908 with a scholarship to the Otago Boys' High School in Dunedin, leaving with a Junior University Scholarship to Otago University in 1913. He had originally intended studying languages and mathematics with a view to becoming a teacher, but his University course was interrupted at the end of his second year by the outbreak of war. In April 1915 he enlisted as a private in the New Zealand Expeditionary Force and saw service in Gallipoli, Egypt and France, where he gained a commission in the field in August 1916. In the following month he was wounded in the battle of the Somme, and after three months in hospital was invalided home to New Zealand, returning there in March 1917.

Aitken took up the broken threads of his career as a student of languages and mathematics in 1918, in which year he gained First Class Honours in Latin and French. His mathematical studies, however, were carried on under considerable difficulties. There was at that time no Professor of Mathematics at Otago, and the lectures to first-year students were given by the Professor of French. For an advanced course, (Aitken being the only entrant), it was arranged that the Mathematics master at Otago Boys' High School should tutor him, and later on he got some help by correspondence with Professor D. M. Y. Sommerville of Victoria College. The Final Honours examination in 1919 was an unsatisfactory affair; the questions bore little relation to the instruction given, and the examiners in Britain knew nothing about the candidates. The upshot was that Aitken got II Class Honours. It is idle to speculate whether a more conventional mathematical training would have fostered or stifled his mathematical genius; the fact remains that Aitken was a creative mathematician whose profound insight was innate; his inspiration was not derived from any teacher, but from the study of the original sources.

He graduated in 1920, in which year he married Mary Winifred Betts, daughter of Alfred and Ada Betts of Nelson, New Zealand. From then until 1923 Aitken taught languages at Otago Boys' High School. He used to recall how he was once discovered by his Headmaster rendering the week's football report in the local newspaper into Ciceronian Latin with a senior class. The Headmaster advised him to stick to more conventional methods. During this time his mathematical interests were kept alive by the opportunity of working as a part-time tutor under Professor R. J. T. Bell, who had lately been appointed to the Chair of Mathematics at Otago, and who immediately recognised Aitken's mathematical gifts.

In 1923, with Bell's encouragement, Aitken took the step, decisive for his subsequent career, of coming to Edinburgh, aided by a post-graduate scholarship of the University of New Zealand, to study under Professor E. T. (later Sir Edmund) Whittaker. He had originally entered for the degree of Ph.D., but his thesis, on the graduation of observational data, which he presented two years later, was immediately recognised as of superior merit, and he was awarded the higher degree of D.Sc. In October, 1925 there began a succession of appointments on the staff of the University of Edinburgh, to the service of which he was to devote the the rest of his life. From lectureships in Actuarial Mathematics and in Statistics and Mathematical Economics he went on to a Readership in Statistics in 1936, and ten years later he succeeded Professor Sir Edmund Whittaker in the Chair of Mathematics, holding this position until his retirement in September, 1965 with the title of Emeritus Professor.

In the Chair, Aitken regarded it as his duty to follow the admirable Scottish tradition of lecturing to undergraduate classes at all levels. He was a most inspiring teacher; his lecturing was superb, and his personality made an indelible impression on his students. If in later life they might not remember the entire mathematical content of his lectures, they never forgot their erudition, humanity and wit. Every summer Mrs. Aitken and he would entertain the Honours students, with their teachers, in their beautiful and historic house and garden. On these occasions he would reveal unsuspected virtuosity in such arts as throwing the javelin: in his youth he had been a notable athlete, and until his health began to fail continued to be capable of strenuous physical exertion. To his research students he was most generous in providing ideas and encouragement, and had usually himself worked out beforehand the results he was leading the unsuspecting student to " discover " for himself.

On University Committees he could enliven a heavy piece of business with some arresting and individual turn of thought concisely expressed, for he was a master of extempore speaking. In everything he favoured the direct approach, and was as incapable of the manœuvrings of University politics as he was of hurting anyone's feelings.

Aitken's name has become known to a wider circle through his phenomenal memory. Some instances of this have already passed into popular legend, such as his reconstruction of his platoon roll, lost in battle, complete with full names, regimental numbers and addresses of next-of-kin. Probably not unconnected with this faculty of remembering was the rare gift of rapid mental calculation which he possessed, to a degree possibly exceeding that of any of the lightning calculators for whom authenticated records exist. In his case the faculty was partly hereditary, but he had developed it, along with his phenomenal memory, by constant practice from his boyhood. These extraordinary gifts, of which he would sometimes give striking demonstrations before an audience, enabled him to perform well-nigh incredible feats. Details need not be given here, as fortunately a record of one such occasion, together with his own fascinating account of the mental processes involved, is to be found in a published lecture (**67**) to the Society of Engineers (November, 1954), and an objective study has been undertaken by Professor I. M. L. Hunter in *Brit. J. Psychol.* (1962). None who witnessed them can forget such spectacular demonstrations as his almost instantaneous multiplication, division and extraction of square and even cube roots of numbers proposed to him; or the " reprehensibly useless feat " as he called it, of writing up on the blackboard, from memory, the 707 digits of Shanks's calculation of π, with a speed and regularity reminiscent of a teleprinter. When Shanks was shown by D. F. Ferguson in 1945 to have gone wrong at the 528th place, Aitken easily memorised the corrected value.† The conclusion of this *tour de force* would be followed by a complete silence while the audience were breathlessly checking the figures against the printed sheets which had been handed round, the lecturer meanwhile remaining in a semi-trance, until finally brought back to earth by tumultuous applause. Aitken sometimes belittled his unusual powers as being unconnected with his real mathematical bent. But it seems certain that he was led to many important analytical results by first going through what ordinary mortals would have regarded as very heavy arithmetical work. As he himself once put it, " Familiarity with numbers acquired by innate faculty sharpened by assiduous practice does give insight into the profounder theorems of algebra and analysis ".

Aitken possessed an exceptional knowledge of the theory and practice of music, and for long stretches would appear to be possessed by music to the

† Dr J. C. P. Miller believes that Aitken eventually learnt 2000 places of the value produced on an electronic computer and published in 1949. He may not however have gone beyond 1000 places in giving public demonstrations. Aitken once remarked to Dr Miller that this process of recall was " largely rhythmic ".

exclusion of all else. Although largely self-taught, he was an accomplished violinist and viola player. He composed many pieces—songs, pianoforte preludes and some pieces for orchestra—" mostly rigorously suppressed ", as he tells us. He prepared, in his singularly beautiful musical calligraphy, his own critical version of J. S. Bach's suites for solo violin, an unnecessary task, one might think, since he knew them all by heart!

An instance which illustrates Aitken's musical knowledge and historical sense is his research into the recital given by the composer and pianist Frederic Chopin during his visit to Edinburgh in 1848. Aitken made his own identification of the items, which were inadequately described in the original programme, and contributed a note to the printed programme of the anniversary recital at the Edinburgh Festival in 1948. His own account (from a letter to Professor J. Stewart Deas) shows how other things than mathematics would at times wholly engross him. " Over a long space of years " he tells us, " I endeavoured to catch the spirit of place and time by even visiting the West end of Queen Street (where Chopin's recital was given) by night, with coat collar turned up, and recalling every detail of the recital, with every collateral circumstance—the stay at Calder House, at the Lyszczynski's house in Warriston Crescent, Chopin's letters enshrining his opinion of musical appreciation in these islands, the state of his health, the programmes of his earlier recitals in June and July in London, in August in Manchester, etc., etc. You may say that this effort of mine *à la recherche du temps perdu* bore rather meagre fruit. Well, what of it? I enjoyed doing it; I caught something of Edinburgh's extremely varied past—and I learnt a lot about other matters in the process ".

In humane letters too he was well versed. The effects of his early classical and literary training remained with him all his life—he could repeat from memory long passages from Virgil or Milton; he had a wide knowledge of world literature, and himself wrote several poems, mostly sonnets, as well as humorous verse; but these only a few friends were privileged to see.

Aitken published upwards of 80 papers, mostly in the fields of statistics, numerical analysis and algebra. He collaborated with the late Professor H. W. Turnbull in a book, *The Theory of Canonical Matrices* (Blackie & Son, 1932). He was joint editor, with the late Professor D. E. Rutherford, of the valuable series of University Mathematical Texts published by Oliver & Boyd, himself contributing the first two, and perhaps the most successful volumes of the series, *Determinants and Matrices*, and *Statistical Mathematics*.

Many honours and distinctions came to him. He was elected a Fellow of the Royal Society of London in 1936, and was an Honorary Fellow of the Royal Society of New Zealand; he served the Royal Society of Edinburgh for a period as Secretary to the Ordinary meetings, and for three different terms as a Vice-President. He held the Society's highest award, the Gunning Victoria Jubilee Prize, and also the Makdougall-Brisbane Prize. He was an Honorary member of our own Society, of which he had twice been President. The Faculty of Actuaries, the Society of Engineers and the Royal Society of Literature inscribed his name in their rolls; he was an Honorary Doctor of Science of the University of New Zealand, and an Honorary LL.D. of Glasgow University.

In spite of his transcendent gifts, which might have set him apart from and above his fellows, he was the most approachable of men, for he possessed simplicity of heart and true humility. He took pride in his Scottish ancestry, and loved the Scottish countryside and its literary and historical associations. He knew intimately East Lothian, the Borders and Galloway, from long days spent walking on their hills and coasts. On such walks he was the most delightful of companions; his wide erudition, carried so lightly, made him a fascinating talker, and his penetrating and witty commentary on men and affairs never held a trace of malice. He was beloved by all who knew him.

Aitken's last publication, though non-mathematical, deserves more than a passing mention. Endowed as he was with an exceptionally sensitive and perceptive nature, he had not been granted the healing gift of oblivion. At certain seasons he was oppressed by the memory of the atrocious fighting of the campaigns in Gallipoli and on the Somme in which he had taken part (the adjective, with its classical overtones, is his own). It is a testimony to the stature of his mind and personality that nearly 50 years after the events he could come to terms with that experience, the more terrible because constantly re-lived, and set down in his book *Gallipoli to the Somme—Recollections of a New Zealand Infantryman* a narrative of his war service, compassionate, restrained, yet vivid and intensely moving. For this superb piece of writing Aitken was elected to the Fellowship of the Royal Society of Literature. To those who knew him best it stands as a testament and a memorial.

He died in Edinburgh on 3rd November, 1967, after some years of indifferent health, and is survived by his widow, a son and a daughter.

THE EARLY YEARS IN EDINBURGH

(From a letter from Professor E. T. Copson to Professor Erdélyi, dated 8th November, 1965)

. . . At the time when he taught languages at Otago Boys' High School he was more interested in mathematics, though he knew very little about modern mathematics. He never met a real mathematician until he was 28 and came to work with Whittaker. He brought with him some unpublished work on the Theory of Numbers which appeared later in the " Edinburgh Mathematical Notes ". The work of people like Fermat fascinated him. With his gift of computation he could guess results in the Theory of Numbers from a consideration of particular cases. Like Ramanujan, the integers were his personal friends. His first piece of serious work was the solution of a sixth order difference equation which arose in Whittaker's theory of graduation. Eddy Whittaker and Richard Gwilt had tried to use E. T. Whittaker's original version of the theory. I had tried to solve the difference equation by analytical methods and got nowhere. But

Alec got a solution as an infinite series expressing the inverse of a difference operator as a Laurent series in the operator E. It did not look at all promising; but Alec with his great arithmetical skill was able to calculate the numerical values of the coefficients. Actually the method was not very good for a reason which is explained on p. 310 of " The Calculus of Observations ". But Alec got over that trouble too. . . It was an exceptional effort to do this work and gain his D.Sc. two years after he came to Edinburgh. The problem was one which fascinated him; it made use of his great gifts in classical algebra and his skill in computation.

Even then he knew very little about modern mathematics; but Edinburgh suited him. E. T. W.'s lectures on determinants and matrices were unique; nowhere else in Britain was matrix theory taught at that time, and Aitken took to it at once. It was fascinating to see how he was able to apply techniques of that sort of algebra to numerical analysis. He was not long a research student—he joined the Edinburgh staff in 1925 and never left. It was not from lack of offers. They tried on more than one occasion to persuade him to go to the London School of Economics. But it was no use. Edinburgh had everything he wanted—concerts, musical friends, the hills to walk on, a congenial job. . . . He must have been a very rapid worker; he once told my wife that for 75 per cent of his time he was thinking of music. He must have employed the other 25 per cent to very great purpose. Of course by staying in Edinburgh he avoided all the administrative burdens which beset many of us and which he found rather uncongenial when he was elected Professor. . .

A. C. AITKEN'S CONTRIBUTIONS TO STATISTICS

Perhaps Aitken's chief service to the science of statistics was his writing of *Statistical Mathematics*, which appeared in 1939. This beautifully written little book, containing in its 150 pages an astonishing amount of information, became the practical bible of many thrown willy-nilly into statistical work by the chances of war. Until the first volume of Maurice Kendall's encyclopaedic work appeared in 1943 it was in fact almost the only source of reference to the mathematics of current statistical theories and explained what it set out to cover with admirable lucidity. These were the days, as J. L. Doob has recently recalled, when it was not impossible to find texts on statistics containing *disproofs* of what are now regarded as results basic to the whole theory . Still, the limitations of the age may be discerned in Aitken's book too, which relegated the concept of statistical independence to an appendix, although in the derivation of Student's distribution the lacuna in the proof given (concerning the independence of $\bar{x}$ and s^2) is carefully pointed out.

Of Aitken's statistical papers the one best known to me and I suppose the most influential was " The estimation of statistical parameters ", a joint work with H. Silverstone; this was published in 1942. It concerned one-parameter estimation problems admitting, among the unbiassed estimates of the parameter, one distinguished by the property of minimum variance. The argument is difficult to follow, although the authors say that they have to do with " a minimal problem in the Calculus of Variations, of positive definite type and formally simple ". As those who wrestled with the paper eventually found, and as others found independently, what lies at the root of the matter is what is now most commonly called the Cramér-Rao inequality. It is the fashion nowadays to decry this inequality somewhat, either by pointing to the numerous independent discoveries of it, or by saying that it is just Schwarz plus a gimmick, but the significant fact is I think that a result at once so neat and so central to the matter should have escaped the attention of many who made their reputations during the great days of maximum likelihood. The work of Aitken and Silverstone was continued by Aitken and Solomon, and Aitken in 1948 published a continuation of the earlier paper, correcting an error in it and exploring the many-parameter situation.

Around 1950 Aitken became interested in Markov chains, but he appears never to have published his work in this field. In the Statistical Laboratory at Cambridge there is a bundle of letters to the late John Wishart which indicate the general lines of Aitken's thought. He started with a two-state Markov chain with constant transition probabilities, and supposed that a score S_0, S_1, $S_2, \ldots$ was built up by taking $S_0 = 0$ and letting S increase by an amount x or y according to the identity of the next state visited. He then showed how to write down in elegant matrix form the Fourier characteristic function for the score after n steps, and studied the asymptotic behaviour of this for large n, deriving central limit theorems and so on. Generalisations were made in three directions: (i) to k-state Markov chains, (ii) to Markov chains over a compact interval $[a, b]$ of states, when the transition function has a probability density, and (iii) to the discussion of the joint distribution of m consecutive scores, and thus to the multiple auto-correlations, etc. Doubtless the manuscript notes on these investigations still exist in Edinburgh, and some reader may like to pursue the matter.

Others have given an account of Aitken's extraordinary powers of mental computing. He appears to have considered that these were due to a skilful subliminal combination of algorithmic tricks and the memory of previous computations. The idea that one could make use of previous many-digit multiplications in performing a desired one sounds odd at first, but curiously this last month Professor Donald Michie and Dr. R. J. Popplestone of the University of Edinburgh's Department of Machine Intelligence and Perception have suggested * that function-evaluation in automatic computing may best be performed by exactly such a combination of algorithm and memory, so much so that in reading their reports while writing these notes I was very struck by the impression that Michie and Popplestone were in fact designing an auto-Aitken. I hardly think he can have known of their work. How enchanted he would have

* To be submitted to the I.F.I.P.S. Congress, 1968.

been! (As regards the possibility of the " total recall " necessary for memory to play the desired role, mention should be made of the suggestion put forward recently * by another member of the same Department, that a temporal analogue of optical holography may explain much that is puzzling about global memories.)

In case it is overlooked by others, I should like to mention another less publicised feat of Aitken's; he could throw a pebble an improbable distance, and make it describe an even less probable trajectory, which one would swear was bimodal. He would then state the exact range of the throw, though when he demonstrated this to me at St. Andrews the pebbles always fell in the sea and verification was not possible.

Aitken dismissed his athletic feats as " what all New Zealand boys learnt in their childhood ". Of his New Zealand days he once told me the following charming tale; I hope I record it with reasonable accuracy.

As a boy Aitken often stayed with an uncle in a fairly remote part of the Dominion, and would make private excursions of many days through the wilderness, just to have a look around. On one of these occasions he discovered a penguin colony, but when he told his uncle, he was disbelieved and punished, I think beaten. Years later he revisited that part of New Zealand and found, to his great satisfaction, not only that the penguin colony had become a nature reserve, but also that his uncle had been appointed the warden of it.

One ought to mention that Aitken became an F.R.S. in the days when this was a very unusual occurrence for a statistician or numerical analyst, and thus he was in a position to direct a warm wind of encouragement from the establishment fire on many chilled young men, and did so frequently, thus earning their permanent affection. There were many other reasons, however, for feeling affectionate towards Aitken, perhaps the chief being the fact that he was one of the few mathematicians who knew that there were more interesting things in life than mathematics.

Aitken never seems to have contributed to probabilistic model-building, but he did come near it. He once described to me how he and McKendrick had stood for some time in Princes Street watching with fascination a queue of trams extending into the far distance. " Aitken," said McKendrick, " there is a probability problem here. We ought to do a Fourier analysis!" Apparently they never carried out McKendrick's threat, or the explosion of queueing theory would have occurred even earlier. In that event it seems likely that Aitken would have prediscovered much current work on the combinatorial identities which have grown out of the ballot theorem.

DAVID KENDALL

A. C. AITKEN'S WORK IN NUMERICAL ANALYSIS

I have never lived close enough to Edinburgh to have discussion with A. C. Aitken as frequently as I should have liked, and our meetings were all too rare,

* H. C. LONGUET-HIGGINS, *Nature*, **217** (1968), 104.

and so my familiarity with his work and methods is less than I should have liked it to be. I am therefore confining these remarks to some personal reminiscences, and to comments on some of the ideas that Aitken has injected into numerical analysis, which have had substantial effect on the work of myself and others. This applies most particularly to his liking for a simple, repetitive, algorithmic approach, and his profound studies on determinants and matrices.

One cannot think of Aitken without remembering his phenomenal power of mental calculation, perhaps based largely on his remarkable memory. He once told me that he had to be careful what he read for entertainment, it was so hard to forget it afterwards. His calculating powers were not, however, based on conscious memory alone, he was a mathematician as well—an unusual combination—and interested in analysing his powers and in the way his mind worked. He has told me that results " came up from the murk ", and I have heard him say of a number, that it " feels prime ", as indeed it was. But he also helped out with conscious use of short cuts. This is in contrast with another lightning calculator, Wim Klein, a Dutchman with whom Aitken had a radio " contest " and discussion, and who relies on a good memory and quick but deliberate mental calculation with a relatively limited supply of mathematical formulae.

Aitken was one to whom integers were personal friends—Ramanujan was another—and I can appreciate this well, for I am one myself. I have heard him quote Hardy's tale of Ramanujan; when 1729 was suggested to him as uninteresting, he replied that it was the smallest number expressible as the sum of two cubes in two ways. Aitken introduced me, in correspondence, to 163, which is such that $\exp(\pi\sqrt{163})$ differs from an integer by less than 10^{-12}, and suggested 997 might have a related property. Aitken once suggested 823 to me as of little interest, though he knew of the property that marks it out for myself—it is one of a prime-four, a set of four consecutive primes ending in 1, 3, 7, 9 and sharing the same leading digits.

It was perhaps Aitken's facility with calculation, which extended to duodecimal as well as to decimal arithmetic, that led him to advocate an extension of duodecimal currency, rather than conversion to decimal currency. In fact, I consider he was largely right; both he and I met G. S. Terry, an American advocate of Duodecimals who produced a fairly extensive book of tables *Duodecimal Arithmetic*. I worked on this myself for a time and there is no doubt that, other things being equal, duodecimal arithmetic is in many ways simpler and easier to learn than decimal arithmetic—but other things are not equal, the decimal system is too entrenched. But Aitken thought it a pity to give up the duodecimals we have and are familiar with. Maybe, if one day decimals and duodecimals meet on equal terms, we shall find Aitken justified.

I will now comment on some specific instances of the impact of Aitken's work on computation. His most effective and widespread ideas are simple ones —simple to express, but clearly the distilled essence of wide experience, and often profound knowledge and thought. Most of these ideas rest on Aitken's liking for an algorithmic approach, based on repetition of a simple process, yielding regularly related results which could *then* be further analysed and improved on by combining them suitably. Many rest on one basic process, widely used by Aitken. This is the evaluation of a *linear cross-mean* or *weighted cross-product* of the form

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \div e.$$

This is often used iteratively, almost to the exclusion of any other kind of process in the calculation. This, as it turns out, is ideal for automatic computation, though Aitken was advocating its use well before these computers appeared.

The best-known of these ideas introduced by Aitken is for the *acceleration of convergence*. This applies to a set of three numerical estimates of one quantity, a, of which the major part is of the form

$$u_n = a + br^n$$

available for three values of n in arithmetic progression. Then br^n may be eliminated by using

$$a^* = \begin{vmatrix} u_{n-1} & u_n \\ u_n & u_{n+1} \end{vmatrix} \div \delta^2 u_n.$$

This has very widespread application, and is the subject of a large and growing literature of applications and extensions. Aitken himself invented it in connection with Bernoulli's recurrence method for solving algebraic equations for the largest root, and used it a great deal in his numerical studies and work on the solution of linear algebraic equations by iteration, and in similar work on the evaluation of latent roots and latent vectors of matrices.

This process gives weight to his arguments and liking for systematic methods. In correspondence with an Edinburgh colleague, which I have been privileged to read, he argues against Southwell's " relaxation " approach of " swatting the largest residual " by pointing out that a systematic iteration may take longer in the first place, but that the regularity, besides making satisfactory theoretical discussion of convergence possible (or at least much easier), also allows use of accelerative methods, which give the required result more easily in the long run than the unsystematic work.

The second major use of the *linear cross-mean* is in Aitken's method of *progressive linear interpolation*. This depends on repeated use of a basic step:

If $f(x;\ a, b, c, \ldots, j, k)$ is the Lagrange polynomial that agrees with $f(x)$ at the n points with abscissae $x = a, b, c, \ldots k$, then the Lagrange polynomial through $n+1$ points is obtained from two Lagrange polynomials through n points ($n-1$ in common) by the formula

$$f(x;\ a, b, c, \ldots, j, k, l) = \begin{vmatrix} f(x;\ a, b, c, \ldots, j, k) & x-k \\ f(x;\ a, b, c, \ldots, j, l) & x-l \end{vmatrix} \div (k-l).$$

Then, starting from the constant approximations $f(x;\ a) = f(a)$, etc., we obtain from this formula, repeatedly applied, the unique Lagrange polynomial through the stated points at each stage. This is an effective method of interpolation, particularly on automatic computers, where the recording of intermediate results (rather a nuisance with desk computation) is essential and automatic at each stage.

This basic idea has been very slow in gaining ground over the nearly 40 years since its inception. It needs a small variation due to E. H. Neville (the idea of using successive overlapping sets $a, b, c, \ldots k$; $b, c, \ldots k, l$; $c, \ldots k, l, m$; etc., rather than a "pivotal set" $a, b, c, \ldots, j$ and one further extra k, l or m), an idea due to L. F. Richardson (the deferred approach to the limit), one due to W. Romberg (repeated Richardson error elimination in integration, applied by Neville's variation of Aitken's process) and one to L. Fox (allowing elimination of error terms that are not just successive powers of one variable, and using Romberg's incidental but important choice of arguments $a, b, c \ldots$, here integration intervals, in geometric progression), and a few more ideas to bring the method to full flower, as I hope I have shown in an expository paper to appear shortly in *Phil. Trans. Roy. Soc.* It is Aitken's basic linear cross-mean that makes the whole thing neat and workable.

The linear cross-mean also appears in Aitken's methods for repetitive reduction of determinants and solution of linear equations, here based on the formula for an $n \times n$ determinant

$$|A| = \begin{vmatrix} |A(n, n)| & |A(n-1, n)| \\ |A(n, n-1)| & |A(n-1, n-1)| \end{vmatrix} \div \left| A\begin{pmatrix} n-1, & n-1 \\ n, & n \end{pmatrix} \right|,$$

in which the arguments in ( ) show which rows and columns are missing in the minors.

Used like this in pivotal reduction of determinants, all minors of any order are integers if the original elements are integers. When used for the reduction of a system of linear equations this is almost equivalent to Gaussian eliminations, but with the divisions delayed one stage, so that we always have exact *finite* working (in integers). This is a useful checking feature, and aesthetically satisfying to some. Aitken himself stated, however, that this is rather more laborious than some methods in which division is performed earlier, at the cost of having to approximate, and in any case with large determinants the integers tend to grow inordinately.

Nevertheless, these last remarks do not apply in simple cases, and the division delay can be useful, for example, in simple linear programming problems, which the introduction of Stiefel's "exchange step" converts to obvious matrix reduction problems.

Another application of this determinant evaluation appears in Aitken's papers in *Proc. Roy. Soc. Edin.*, **46** (1925-26) 289-305 and **51** (1930-31) 80-90 to the extension of Bernoulli's process using Hankel or persymmetric matrices. Almost all the ideas used in Rutishauser's QD-algorithm are contained in these papers, including the downward recurrence, starting from the coefficients in the equation, that is so much more stable than the recurrence starting from a sequence generated by the Bernoulli recurrence.

Aitken had a great belief in the virtue of a sound mathematical explanation and analysis of the methods he advocated, and did not like to work in the dark, by experience alone. His theoretical background, particularly with determinants and matrices, was so extensive and his computational experience so great, that he could afford to indulge this liking more than most, and record his work for others to use. This, and the regularity of the regular algorithmic arrangements of calculations that he preferred, pay dividends in actual computational use of error analysis to provide error elimination, as we have seen above.

In a short article, it is not possible to do full justice to Aitken's work; one can only suggest that one must go to his written works and read and re-read with great care. He had a great ability to write in a concise, clear and stimulating style, with interesting choice of material. I myself have received great stimulus, for instance, from his two little books in the series of University Mathematical Texts, *Determinants and Matrices* and *Statistical Mathematics* and still find interesting new points in papers I thought I knew well.

J. C. P. MILLER

A. C. AITKEN'S WORK IN PURE MATHEMATICS

Aitken's contributions to pure mathematics were almost entirely concerned with algebra. His numerous publications in this field range from short notes to elaborate papers and, taken as a whole, strongly reflect his personality, his special talents and his attitude to mathematics.

As has been remarked elsewhere, Aitken's development as a mathematician came about largely through his own efforts. As a young man he was not, in the traditional manner, initiated into research by an experienced supervisor, but instead derived his inspiration from the works of the great masters of the past. This accounts for his remarkable knowledge of historical details with which he often enlivened his mathematical writings; he was particularly fond of stressing the claims of lesser known mathematicians of former times for discoveries erroneously attributed to their more famous contemporaries. Many of his historical references were gleaned from Sir Thomas Muir's monumental work on determinants, for which Aitken had a profound admiration.

To assert that, as a mathematician, Aitken was a formalist is perhaps an over-simplification, but it describes a dominant quality of his work; that is, he sought progress in mathematics mainly through the discovery of hitherto concealed algebraic identities and relations of a formal nature rather than by abstract reasoning. By his superb skill in computation and through his mastery of classical algebra he was particularly endowed to succeed in his quest, and it is not surprising that he found himself in closest affinity with the algebraists of the preceding generation, to whose genius we owe the enormous wealth and power of formal algebra. To gather new fruits from this well-harvested field was surely no mean feat. This is not to say that Aitken lacked interest in the more recent developments of algebra. Indeed, his work on the representation theory of the symmetric and general linear groups, which he undertook during his most creative period, yielded important results. But he was apparently not in sympathy with the ever-increasing trend towards abstraction which, since the third decade of this century, has turned algebra into an almost exclusively conceptual discipline. One can only speculate whether this is the reason why, in later years, he concentrated his efforts more on statistics and numerical analysis which by that time had become more congenial to him.

Aitken was a highly intuitive mathematician. Presumably, like most creative mathematicians, he conjectured general results after a close scrutiny of special cases. But the certainty he derived in his own mind from this inductive procedure was so great that he frequently refrained from working out the details of a proof and instead invited his readers to accept the description of a " typical case " in place of a rigorous and complete argument. Whilst his mathematical style is thus occasionally somewhat sketchy, his English prose is always lucid and pleasing. Each paper bears witness to Aitken's enthusiasm and love for mathematical beauty. Those who shared with him the joys of amateur music-making will recognize the kinship between his mathematical and musical activities. Both displayed a refreshing spontaneity and directness, unfettered by ponderous professionalism, but executed with an individual technique that had been evolved by genuine insight and personal involvement.

Aitken's publications began in 1923 with notes on topics in analysis, geometry and number theory (**(1)** to **(4)**), which were not quite typical of his mathematical style. Later on **(18)** he used his knowledge of number theory to establish some remarkable formulae for the computation of π. All his other papers on pure mathematics are devoted to algebraic problems. His virtuosity in evaluating special determinants was unsurpassed, but there was one type of determinant which played a central role in his researches: let $h_0(=1), h_1, h_2, \ldots$ be the complete symmetric functions of the indeterminates $x_1, x_2, \ldots, x_m$, thus

$$h_s = \Sigma\, x_{k_1}x_{k_2}\ldots x_{k_s},$$

where the summation is extended over all sets of integers satisfying

$$1 \leqq k_1 \leqq k_2 \leqq \ldots \leqq k_s \leqq m,$$

with the convention that $h_{-s} = 0$ when $s>0$. Jacobi had studied the " isobaric " determinant or " bialternant "

$$H(p, q, r, \ldots) = \begin{vmatrix} h_p & h_{q+1} & h_{r+2} & \cdots \\ h_{p-1} & h_q & h_{r+1} & \cdots \\ h_{p-2} & h_{q-1} & h_r & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}, \tag{1}$$

in which $p<q<r< \ldots$ are non-negative integers which may be thought of as a partition of $p+q+r+ \ldots$ or else as describing a tableau consisting of p dots in the first row, q dots in the second row, r dots in the third row, etc. A classical result on symmetric functions is the celebrated duality theorem which states that

$$H(p, q, r, \ldots) = A(p', q', r', \ldots),$$

where $A(p', q', r', \ldots)$ is a determinant analogous to (1), in which each h_s is replaced by the elementary symmetric function

$$a_s = \Sigma x_{k_1} x_{k_2} \ldots x_{k_s} \quad (1 \leqq k_1 < k_2 \ldots < k_s \leqq m)$$

and $(p', q', r', \ldots)$ is the partition conjugate to $(p, q, r, \ldots)$, that is the tableau obtained by interchanging rows and columns. Aitken generalized **(10)** this law to determinants of the type

$$\begin{vmatrix} h_{\alpha+\alpha'} & h_{\alpha+\beta'} & h_{\alpha+\gamma'} & \cdots \\ h_{\beta+\alpha'} & h_{\beta+\beta'} & h_{\beta+\gamma'} & \cdots \\ h_{\gamma+\alpha'} & h_{\gamma+\beta'} & h_{\gamma+\gamma'} & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix},$$

where $\alpha > \beta > \gamma \ldots, \alpha' < \beta' < \gamma' \ldots$. Sixteen years later **(55)** he returned to the subject and developed the relation between bialternants and the theory of partitions.

The bialternant functions constitute the link with the other major problem in algebra that attracted Aitken's attention. I. Schur had classified all algebraic representations of the general linear groups, that is matrix functions $T(A)$ satisfying

$$T(AB) = T(A)T(B), \tag{2}$$

where the elements of the matrix $T(A)$ are polynomials in the elements of A. It turned out that if $T(A)$ is irreducible then its trace is one of the bialternants (1) and that $T(A)$ is characterized by its trace. Special cases of (2) had been known before Schur, notably the Schläflian (or induced) matrix $A^{[s]}$, which is the matrix of the linear transformation induced on the individual terms of h_s, other examples being the compound matrices and the Kronecker powers of A. Aitken computed the latent roots and elementary divisors of these special representations over the complex field **(11, 12, 42, 57)**. In what, I think, is his most profound mathematical paper **(48)** he studies the sequence of induced matrices $A^{[s]}(s = 1, 2, \ldots)$ when A runs through the natural representation of a symmetric group. Although some of his results had been anticipated by D. E. Littlewood and A. R. Richardson, Aitken's exposition is nevertheless of considerable interest.

With his flair for elegant formalism Aitken was quick to realize the usefulness of matrix algebra as a powerful tool in many branches of mathematics. At a time when matrix techniques were not yet widely known he applied matrix algebra with striking success to certain statistical problems.

His interest in matrices was shared by H. W. Turnbull. Their joint book *Canonical Matrices* **(31)** soon became a standard work on the subject and has recently been reprinted in the Dover series on Advanced Mathematics. To this

https://doi.org/10.1017/S0013091500012530 Published online by Cambridge University Press

day it is one of the most accessible sources for the less familar parts of matrix theory, and it contains a number of original contributions not previously published. Aitken's collaboration with Turnbull went far beyond a congenial mathematical partnership. It developed into a warm and life-long friendship between two men whose temperaments and background were quite different (although music was a strong bond), and the exchange of mathematical ideas continued throughout the years.

Aitken and D. E. Rutherford were the originators and first joint editors of the University Mathematical Texts, published by Oliver & Boyd. The success of these text-books is well known and need not be elaborated here. Amongst the initial volumes in the series was Aitken's *Determinants and Matrices* (**50**), which has gone through nine editions since its appearance in 1939. Charmingly written, it is typical of Aitken's style, and the choice of topics is characteristic of his personal taste and his attitude to algebra. It is quite unlike any of the numerous treatises on linear algebra, which have appeared since the 1930's. Although linear equations are treated in some detail, there is no explicit mention of vector spaces or linear mappings and only a few brief paragraphs are devoted to latent roots and vectors. Instead the reader is told, as he should be, about Laplace's expansion of the Cauchy-Binet formula, and he is fully informed of the classical results due to Jacobi, Cauchy, Franke, Sylvester and others concerning compound matrices and their minors. He meets the expansion of the quotient of determinants given by Schweins in 1825, and, not surprisingly, he is introduced to the theory of bialternants mentioned earlier. To be sure, some of this material is now regarded as old-fashioned, yet it includes some beautiful mathematics that is still worth recording, if only for the few but significant occasions when it impinges on modern research. Such occasions do exist and may well continue to occur; for, despite the vast power that abstraction and generality of approach has placed in the hands of algebraists, it happens time and again that the very core of an important advance rests upon an intricate formal relationship that was discovered by our mathematical ancestors. We are grateful to Aitken for preserving these historical gems for the younger generation.

He would have been ideally suited to achieve a synthesis between the old and the new algebra. In a remarkable letter to an Edinburgh colleague in 1944, which is appended to these notes, he described his plans and ideas for modernizing and unifying invariant theory and other branches of classical algebra. Alas for various reasons to which he alludes, his intentions were never carried out, and we are left with the feeling that Aitken's potentiality as an algebraist had not been fully deployed when he decided to apply his versatile talent to other branches of mathematics.

May our science be blessed with many a man like him: an individualist of exquisite taste, intellectual power and originality.

W. LEDERMANN

A LETTER FROM A. C. AITKEN TO AN EDINBURGH COLLEAGUE

23 Stirling Road,
Edinburgh, 5.
23rd December 1944.

My Dear — — —,

Let me essay to write, *currente calamo*, a discourse on invariant theory, Schur matrices, the symmetric group, etc.

The various explorers of this fascinating country are apt to become affected by a kind of mountain-blindness, rendering them incapable of seeing that the features they believe themselves to be discovering are already charted fairly accurately, though under other names and on a different scale, on some other man's map. Indeed the classical invariantists from Sylvester on, the group theorists from Frobenius on, the substitutional analysts from Alfred Young on, the tensorists, and finally topologists like Hodge (with his " k-connexes ") are all engaged on one and the same topic, and it falls, at its widest, under the domain of group theory.

One might begin, arbitrarily, anywhere in this country, and ramify at large until the chart is fairly complete. I shall begin with types of coordinates. From a vector or set of coordinates $x \equiv \{x_1, x_2, x_3, \ldots, x_n\}$, and from cogredient vectors y, z, ... undergoing all the vicissitudes that x does, we have various *derived* sets, e.g. the powers and products sets,

$$\{x_1^2, x_1x_2, x_1x_3, \ldots, x_2^2, \ldots, x_n^2\}, \quad \{x_1^3, x_1^2x_2, \ldots, x_n^3\}, \text{ etc.}$$

The " quantics " or ground forms from which invariant theory begins are linear in these " power and product " coordinates. The laws of transformation of such vectors are easily derived and studied: thus, if

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \text{or } Ax = y \text{ say,}$$

you find easily

$$\begin{bmatrix} a_1^2 & 2a_1b_1 & b_1^2 \\ a_1a_2 & a_1b_2+a_2b_1 & b_1b_2 \\ a_2^2 & 2a_2b_2 & b_2^2 \end{bmatrix}\begin{bmatrix} x_1^2 \\ x_1x_2 \\ x_2^2 \end{bmatrix} = \begin{bmatrix} y_1^2 \\ y_1y_2 \\ y_2^2 \end{bmatrix},$$

which I write $A^{[2]}x^{[2]} = y^{[2]}$, and similarly for any degree of powers I *define* $A^{[k]}$ by $y^{[k]} = [Ax]^{[k]} = A^{[k]}x^{[k]}$, and name $A^{[k]}$ the kth *Schläflian matrix* of A. The set of coordinates or elements of the vector $x^{[k]}$ is sometimes called a " transformable set ", since, as here, it carries over into an exactly similar $y^{[k]}$.

A fundamental property at once emerges. We have, say,

$$BAx = By = z.$$

Then

$$(BA)^{[k]}x^{[k]} = B^{[k]}y^{[k]} = z^{[k]}.$$

https://doi.org/10.1017/S0013091500012530 Published online by Cambridge University Press

But

$$B^{[k]}y^{[k]} = B^{[k]}A^{[k]}x^{[k]}.$$

But x is an arbitrary vector. Hence

$$(BA)^{[k]} = B^{[k]}A^{[k]},$$

a multiplicative property which can be extended to any product of however many matrices.

A second type of transformable set is *determinantal*, familiar in geometry as tangential coordinates, or line coordinates, etc., etc. From three cogredient vectors x, y, z we might construct a vector of determinants

$$\{| x_1y_2z_3 | \; | x_1y_2z_4 | \ldots | x_2y_3z_4 | \ldots\}.$$

Suppose I call the vector, so formed from k vectors $x, \ldots,$ the kth *compound* vector, $x^{(k)}$. The so-called " multiplication theorem of determinantal arrays " (which is nothing more than the theorem for the determinant of a product of two rectangular matrices) then has as its consequence

$$(BA)^{(k)} = B^{(k)}A^{(k)},$$

Yet again, look at the *contragredient* transformation itself. Suppose the inner product ux is kept invariant. Then when x suffers Ax, u suffers $(A')^{-1}u$. Now

$$\{(BA)'\}^{-1} = (A'B')^{-1} = (B')^{-1}(A')^{-1}.$$

Schur, in his Dissertation completed in 1901 (Berlin), working under his master Frobenius, set himself the task of discovering *all* the matrices $T(A)$, having elements of given degree, say k, in the elements a_{ij} of A, and satisfying the multiplicative law,

$$T(BA) = T(B)T(A);$$

satisfying also a condition of *irreducibility*, namely that

$$T(A) = H \begin{bmatrix} T_1(A) & & & \\ & T_2(A) & & \\ & & \ddots & \\ & & & T_s(A) \end{bmatrix} H^{-1}$$

be impossible, where the $T_j(A)$ are in isolated blocks down the diagonal, and H is independent of A. (If this were not so, the separate $T_j(A)$ would themselves possess the property $T_j(BA) = T_j(B)T_j(A)$, and $T(A)$ would be " reducible ").

Let me interpolate a meaning for this. $T(A)$ will transform *a vector* $T(x)$, and we shall have $Ax = y$ and $T(A)T(x) = T(y)$. The elements of $T(x)$ will thus form a " transformable set ", a derived set of coordinates sharing, as I said above, in the vicissitudes of x, genuflecting in their own way whenever x genuflects, and so on. (To anticipate, Hodge's " k-connexes " are simply complete sets of polynomials made out of $T(x)$ and all its polarized forms.)

What Schur found was very remarkable. There were as many irreducible $T(A)$ " of class k " *as there were partitions of the integer* k. Thus if $k = 4$,

https://doi.org/10.1017/S0013091500012530 Published online by Cambridge University Press

there are *five* types of $T(A)$, corresponding to partitions (4), (31), (22), (211), (1111). The (4) type is our friend the 4*th Schläflian*, $A^{[4]}$; the (1111) type, say $A^{[1111]}$ or $A^{[1^4]}$, is the determinantal transformation, the 4*th compound* of A, $A^{(4)}$. (In homogeneous coordinates of a 4-dimensional Cartesian space, these $x^{(4)}$ transformed by $A^{(4)}$ would be *hyperplane coordinates*). The intermediate types (31), (22), (211) were not known before—except in so far as a case of the (22), symmetric in two indices and skew in two, is the transformation of the Riemann-Christoffel tensor. Anyhow, for $k = 3$ we have

$$A^{[3]},\ A^{[21]},\ A^{[1^3]},$$

for $k = 4$,

$$A^{[4]},\ A^{[31]},\ A^{[22]},\ A^{[21^2]},\ A^{[1^4]}$$

and so on.

(It was in regard to the orders of two of these, when the original A is 3×3, that I threw in an interjection in Turnbull's lecture)†.

The *spurs* or *traces* of the $T(A)$ have remarkable properties. The spur of $A^{[k]}$ is the *complete* symmetric function of degree k in the latent roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of A, namely

$$h_k = \alpha_1^k + \alpha_1^{k-1}\alpha_2 + \alpha_1^{k-2}\alpha_2^2 + \ldots + \alpha_1^{k-2}\alpha_2\alpha_3 + \ldots + \alpha_n^k.$$

The spur of $A^{[1^k]}$ or $A^{(k)}$ is the *elementary* symmetric function,

$$a_k = \alpha_1\alpha_2 \ldots \alpha_k + \alpha_2 \ldots \alpha_{k+1} + \ldots.$$

These, and the spurs of the intermediate types, are all examples of the *bialternant symmetric functions studied by Jacobi* in 1841; e.g. for $n = 3$ we have

$$h_3 = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^5 \\ 1 & \alpha_2 & \alpha_2^5 \\ 1 & \alpha_3 & \alpha_3^5 \end{vmatrix} \div \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \\ 1 & \alpha_3 & \alpha_3^2 \end{vmatrix}, \quad \text{spur of } A^{[3]},$$

$$h_{(21)} = \begin{vmatrix} h_1 & h_3 \\ h_0 & h_2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha_1^2 & \alpha_1^4 \\ 1 & \alpha_2^2 & \alpha_2^4 \\ 1 & \alpha_3^2 & \alpha_3^4 \end{vmatrix} \div \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \\ 1 & \alpha_3 & \alpha_3^2 \end{vmatrix}, \quad \text{spur of } A^{[21]},$$

$$h_{(1^3)} = a_3 = \begin{vmatrix} h_1 & h_2 & h_3 \\ h_0 & h_1 & h_2 \\ . & h_0 & h_1 \end{vmatrix} = \begin{vmatrix} \alpha_1 & \alpha_1^2 & \alpha_1^3 \\ \alpha_2 & \alpha_2^2 & \alpha_2^3 \\ \alpha_3 & \alpha_3^2 & \alpha_3^3 \end{vmatrix} \div \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 \\ 1 & \alpha_2 & \alpha_2^2 \\ 1 & \alpha_3 & \alpha_3^2 \end{vmatrix}, \quad (h_0 = 1),$$

spur of $A^{[1^3]}$ or $A^{(3)}$;—in this simple case merely $\alpha_1\alpha_2\alpha_3$.

$A^{[3]}$ is of order 10×10, $A^{[21]}$ of order 8×8, $A^{[1^3]}$ of order 1×1, merely $| a_{11}\, a_{22}\, a_{33} |$.

Schur found that the trace of each $T(A)$, reducible or not, is a *symmetric function* of the latent roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of A; and that if we express this trace *linearly in terms of bialternants* we discover, in so doing, the *irreducible* "*components*" $T_j(A)$ to which such a $T(A)$ may be reduced by $H\,T(A)\,H^{-1}$. There is

† This remark evidently refers to a lecture entitled "The Gordan-Capelli theorem" given by Professor H. W. Turnbull at the meeting of the Edinburgh Mathematical Society on 2nd December 1944.

what I dared to call a " paramorphism " between the behaviour of the traces, tr $T(A)$, and the matrix, $T(A)$. Thus the *sum* of bialternants in tr $T(A)$ corresponds to the *direct sum*

$$\begin{bmatrix} T_1(A) & & \\ & T_2(A) & \\ & & \ddots \end{bmatrix} \equiv T_1(A) \dotplus T_2(A) \dotplus \ldots$$

of the irreducible components $T_j(A)$ possessing those bialternants as their own traces. *Product* of traces, on the other hand, corresponds to " *direct product* " of the $T_j(A)$, say $T_i(A) \times T_j(A)$; or, for that matter, to the direct product of the vectors, vector fields, vector spaces, transformed by the $T_j(A)$. Thus the calculus of symmetric functions, analysing the trace of a $T(A)$ linearly into its bialternant components, arbitrates like a boundary commission upon a composite vector or transformable set, and resolves it into its autonomous independent parts, subsets which transform independently of other subsets.

An example is the direct product of k cogredient vectors. When $x \rightsquigarrow Ax$, $y \rightsquigarrow Ay$, $z \rightsquigarrow Az$, ... the direct product

$$x \times y \times z \times \ldots \rightsquigarrow Ax \times Ay \times Az \times \ldots = A^{\{k\}}(x \times y \times z \times \ldots), \quad \text{say},$$

where $A^{\{k\}}$ may be called the kth direct (or Kronecker) power of A. One easily shows that $(BA)^{\{k\}} = B^{\{k\}}A^{\{k\}}$, so that $A^{\{k\}}$ is a $T(A)$. Further, it is always reducible, and contains as components *every* $T_j(A)$ of class k, usually more than once. For example,

$$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}, \quad \text{say:}$$

$$A^{\{2\}} = \begin{bmatrix} a_1^2 & a_1b_1 & b_1a_1 & b_1^2 \\ a_1a_2 & a_1b_2 & b_1a_2 & b_1b_2 \\ a_2a_1 & a_2b_1 & b_2a_1 & b_2b_1 \\ a_2^2 & a_2b_2 & b_2a_2 & b_2^2 \end{bmatrix}.$$

Since tr $A^{\{2\}} = (\alpha_1+\alpha_2)^2 = (\alpha_1^2+\alpha_1\alpha_2+\alpha_2^2)+\alpha_1\alpha_2 = h_2+a_2$ (a different a_2, symm. fn.), we have $HA^{\{2\}}H^{-1} = A^{[2]} \dotplus A^{[11]}$, by the paramorphism to which I referred earlier. To show you this in action, I shall make *H orthogonal*, producing a " prepared " or " normalized " version of $A^{[2]}$, suitable for work with quadratic or Hermitian forms. Thus:

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1/\sqrt{2} & \cdot & 1/\sqrt{2} \\ \cdot & \cdot & 1 & \cdot \\ \cdot & -1/\sqrt{2} & \cdot & 1/\sqrt{2} \end{bmatrix} A^{\{2\}} \begin{bmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & 1/\sqrt{2} & \cdot & -1/\sqrt{2} \\ \cdot & \cdot & 1 & \cdot \\ \cdot & 1/\sqrt{2} & \cdot & 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} a_1^2 & \sqrt{2}\,a_1b_1 & b_1^2 & \cdot \\ \sqrt{2}\,a_1a_2 & a_1b_2+a_2b_1 & \sqrt{2}\,b_1b_2 & \cdot \\ a_2^2 & \sqrt{2}\,a_2b_2 & b_2^2 & \cdot \\ \cdot & \cdot & & |\,a_1b_2\,| \end{bmatrix}$$

$$= \text{(prepared) } A^{[2]} \dotplus A^{[1^2]}.$$

Now, for the purpose of invariant theory, a ground form should be regarded not as the " quantic " rather particularly envisaged by the earlier invariantists, but as *a linear combination of the variables in an irreducible transformable set* $x^{[\lambda]}$, where $[\lambda]$ is the partition characterizing the set and its $T(A)$, in fact $A^{[\lambda]}$. Thus in fact the ground form is the inner product

$$ax^{[\lambda]},$$

where a is a row vector of coefficients. Thus the vector a will transform *contragrediently to* $x^{[\lambda]}$, *by* $\{(A')^{[\lambda]}\}^{-1}$. It is easy to show (and was shown by D. E. Littlewood) that if

$$bx^{[\mu]}$$

is *any concomitant* of $ax^{[\lambda]}$, the elements of b being, say, of degree s in those of a, then b *transforms under a* $T(A)$ *law*: and indeed, to find all linearly independent concomitants of class s, one merely dissects $(A^{[\lambda]})^{[s]}$ into its irreducible components by symmetric function theory. To each component corresponds an " autonomous " part of b, and a suitable $x^{[\mu]}$, and we have our concomitant

$$bx^{[\mu]}.$$

Thus a census of concomitants can be made, of higher and higher class: and this is what D.E.L. is engaged in doing. He finds some difficulty with the trace of $(A^{[\lambda]})^{[s]}$, which is that rather unhandy thing, the sth complete symmetric function in arguments which are the *single terms* in the expansion of Jacobi's bialternant of type $[\lambda]$. It is rather intractable, and his methods are tentative.

(I myself would take $(A^{[\lambda]})^{\{s\}}$, the sth "direct power": its trace is $(h_{[\lambda]})^s$, specific at once.)

Thus you see that invariant theory, the $T(A)$, the transformable sets, are all fused in one whole. I myself, again, would make the fusion more complete, as follows. Just as the invariantists, Aronhold, Clebsch, Gordan, write a quantic as a *symbolic power of an inner product*, e.g.

$$a_{11}x_1^2+2a_{12}x_1x_2+a_{22}x_2^2 \equiv (a_1x_1+a_2x_2)^2$$

where a_1, a_2 are *symbolic* variables contragredient to x_1, x_2, so I would write all the *equivalent and polarized symbolic representations* of a general ground form $ax^{[\lambda]}$ as *elements of a matrix*

$$(AX)^{[\lambda]},$$

where A is symbolic, and X is comprised of as many columns as may be necessary of cogredient variables. Then all concomitants of class s are components of

$$\{(AX)^{[\lambda]}\}^{\{s\}}.$$

Resolve this into its irreducibles $(AX)^{[\mu]}$, $(AX)^{[\nu]}$, ...; we have then the symbolic representations of all " autonomous " concomitants of class s. The Gordan-Hilbert theorem of the *finiteness* of the census of concomitants of a ground form is probably then a simple consequence of the fact that the set of bialternant symmetric functions is itself finite, all those above a certain degree, being functionally, polynomially, dependent on those of lower degree. I have had these ideas in mind for many years, but various circumstances of anxiety, or duty, or bad health, have prevented me from following them up, and I have observed . . . my talented younger contemporary, D. E. Littlewood, assault and capture most of this terrain. Good luck to him! Latterly he is doing it by *tensors*: because of course the theory is that of the resolution of compound or derived tensors into component tensors with special clusters of *symmetric* or *antisymmetric indices*.

And, lastly, a word about the connexion with group theory. $A^{[3]}$, $A^{[21]}$ and $A^{[1^3]}$, where A is $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, will serve to illustrate. Embedded in $A^{[3]}$ is its most *typical* element

$$a_1b_2c_3+a_1b_3c_2+a_2b_1c_3+a_2b_3c_1+a_3b_1c_2+a_3b_2c_1,$$

from which all other elements a_1^3, $3a_1^2a_2$, ..., $6a_1a_2a_3$ etc., can be regarded as derived by coalescence of indices. This element is in fact the *permanent* $\overset{+}{|}a_1b_2c_3\overset{+}{|}$, and if we write it as

$$a_1b_2c_3 . 1+a_1b_3c_2 . 1+\ldots+a_3b_2c_1 . 1,$$

we may see that 1, 1, 1, 1, 1, 1 is the *scalar representation* of the symmetric group on 3 objects. In the same way, $A^{[1^3]}$ or $A^{(3)}$ is $|\, a_1b_2c_3 \,|$, or

$$a_1b_2c_3 . 1-a_1b_3c_2 . 1+\ldots-a_3b_2c_1 . 1,$$

corresponding to 1, -1, 1, ..., -1, the *alternating representation* of the above symmetric group. $A^{[21]}$ is of order 8×8, and can be characterized by its *central core*, a submatrix of order 2×2, which can be written

$$a_1b_2c_3\begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}+a_1b_3c_2\begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix}+\ldots+a_3b_2c_1\begin{bmatrix} 1 & . \\ -1 & -1 \end{bmatrix},$$

where we recognize

$$\begin{bmatrix} 1 & . \\ . & 1 \end{bmatrix}, \begin{bmatrix} . & 1 \\ 1 & . \end{bmatrix}, \ldots, \begin{bmatrix} 1 & . \\ -1 & -1 \end{bmatrix}$$

as the *irreducible representation of the* 2nd *order*.

In fact if we construct from a matrix representation of the symmetric group a " group matrix " (for that is what it is) of this kind as central core, and then build outwards other such group matrices by allowing all possible coalescences of indices, the construct is one of Schur's $T(A)$; so that, in the final resort, the explicit determination of concomitants depends on suitable representations of the symmetric group. Among these the so-called " natural " representation, the " seminormal ", and the orthogonal representation are the dominant ones: and they are not yet fully explored. Young's substitutional analysis is at bottom a method of constructing a *group algebra* of the symmetric group by

https://doi.org/10.1017/S0013091500012530 Published online by Cambridge University Press

a *system of units containing symmetric and antisymmetric operations*: the operations symbolized by his " tableaux " are in fact these units, but they need modifying and normalizing before they become thoroughly tractable. What H. W. T. is trying to do is to make the Gordan-Capelli polarizing technique do the work either of symmetric function theory on traces of $T(A)$ or, equivalently, of substitutional analysis on the sub-algebras of the symmetric group and their various representations. In my opinion the tools are antiquated; if they had not been, Alfred Young would not have deserted them in order to forge a new calculus. Understand: *c'est mon opinion à moi.*

And now, mathematics has too long delayed my expression of sincerest wishes, from us all, of happiness in your home for 1945.

As for a walk, a circuit from Rachan Mill, a mile beyond Broughton, via Drumelzier Law and round to the starting point may commend itself, or from Peebles via the Gipsy Glen up and round by Glenrath Heights and Hundleshope and back, or from Lochurd, beyond Blyth Bridge, over the Broughton Heights towards Stobo. Much would depend on the weather, not only of the day, but of the few days preceding.

Yours ever,

Alec.

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https://doi.org/10.1017/S0013091500012530 Published online by Cambridge University Press

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[*Note:* The above list does not contain book reviews or obituary articles, several of which are of considerable interest. For periodicals the date given is the period over which the volume was published; in some cases this spans several years. Except for the series (**45a-h**) the list has been arranged in lexicographical order of these dates. In a few instances this may not coincide exactly with the order of appearance of the the papers.]