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Logarithms and the Topology of the Complement of a Hypersurface

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Abstract. This paper is devoted to analysing the relation between the logarithm of a non-constant holomorphic polynomial $Q(z)$ and the topology of the complement of the hypersurface defined by $Q(z) = 0$.

1 Introduction

Let $Q(z)$ be a given non-constant holomorphic polynomial in $\mathbb{C}^n$, and H_Q the hypersurface defined by $Q(z) = 0$. The complement $\mathbb{C}^n \setminus H_Q$ is a Stein manifold of complex dimension n; hence, the singular homology group $H_k(\mathbb{C}^n \setminus H_Q, \mathbb{Z})$ vanishes for all $k > n$, see [3, p. 26]. Moreover, the group $H_n(\mathbb{C}^n \setminus H_Q, \mathbb{Z})$ is generally non-trivial and plays an important part in residue theory and other issues of complex analysis in several variables; see for example the works of Poincaré [10] and Griffiths [5, 6].

The main objective of this paper is to deduce simple geometrical conditions which imply that a given n-dimensional singular cycle Γ_s is homologous to zero in the complement of H_Q. In particular, we are interested in conditions related to the existence of the logarithm $\ln Q(z)$ on Γ_s; see for example Propositions 1.4 and 3.2 which are the main results of this paper.

Properly speaking, any cycle Γ_s in $\mathbb{C}^n \setminus H_Q$ is a formal finite sum $\sum_k m_k f_k$ of continuous functions f_k defined from the standard compact n-real simplex Δ^n into $\mathbb{C}^n \setminus H_Q$, see for example [1, 12]. Hence, any cycle Γ_s can be represented by a compact set Γ defined by the finite union $\bigcup_{m_k \neq 0} f_k(\Delta^n)$. A very important case happens when Γ is a compact manifold without boundary. We may ask, for example, whether a given non-trivial element of $H_n(\mathbb{C} \setminus H_Q, \mathbb{Z})$ can be represented by a cycle Γ_s whose associated set Γ is a simply connected manifold. If such a manifold exists, we shall see later that Propositions 1.4 and 3.2 give us strong conditions over Γ.

Definition 1.1 We say that the logarithm $\ln Q(z)$ is well defined (or exists) on Γ if there exists a continuous function $h(z)$ defined on Γ such that $Q(z) = \exp(h(z))$. Recall that Γ does not meet H_Q.

Notice that $h(z)$ can actually be defined on an open neighbourhood W of Γ in such a way that h is holomorphic and $Q(z) = \exp(h(z))$ in W, for $Q(z)$ is a holomorphic

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polynomial. On the other hand, working on the complex plane $\mathbb{C}^1$, we have that the zero locus H_Q of $Q(z)$ is a finite set. And it is easy to see that the existence of the logarithm $\ln Q(z)$ on Γ implies that Γ_s is itself homologous to zero in the complement of H_Q; moreover, Γ has an open neighbourhood in $\mathbb{C}^1 \setminus H_Q$ diffeomorphic to $\mathbb{C}^1$.

We may ask here whether the existence of the logarithm $\ln Q(z)$ on Γ could imply that Γ_s is homologous to zero in $\mathbb{C}^n \setminus H_Q$ for any $n \geq 2$. We get a positive answer if Q is a weighted homogeneous polynomial.

Lemma 1.2 *Let P be a weighted homogeneous holomorphic polynomial P in $\mathbb{C}^n$, that is, the equation $P(t^{\beta_1} z_1, \dots, t^{\beta_n} z_n) = tP(z_1, \dots, z_n)$ holds for each $t \in \mathbb{C}$ and some fixed rational numbers β_k. Considering the zero locus H_P of P, we have that any given n-dimensional cycle Γ_s is homologous to zero in $\mathbb{C}^n \setminus H_P$, whenever the logarithm $\ln P(z)$ is well defined on the associated set Γ.*

We shall prove this lemma in the second section of this paper. Unfortunately, we cannot generalise previous lemma in a straightforward way to consider any arbitrary holomorphic polynomial Q. The existence of the logarithm $\ln Q(z)$ on Γ is not a sufficient condition which could imply that Γ_s is homologous to zero. A very nice counterexample was given by Nemirovskiĭ in [9]. Working with the hypersurface H_F associated to the Fermat polynomial $1 + z_1^q + \cdots + z_n^q = 0$, for $n \geq 3$ and $q \geq 3$, Nemirovskiĭ built a smooth sphere $\mathbb{S}^n$ which is not homologous to zero in $\mathbb{C}^n \setminus H_F$.

We need to use the following result in order to generalise Lemma 1.2, see for example Verdier [13], Broughton [2] or Hà Huy Vui [8].

Proposition 1.3 *Let Q be a non-constant holomorphic polynomial on $\mathbb{C}^n$. Then there exists a finite set $\Lambda_Q \subset \mathbb{C}$ such that the fibres of Q induce a locally trivial fibre bundle of $\mathbb{C}^n \setminus Q^{-1}(\Lambda_Q)$ with base on $\mathbb{C} \setminus \Lambda_Q$.*

Now we can state one of the main results of this work.

Proposition 1.4 *Let H_Q be the zero locus of a non-constant holomorphic polynomial Q in $\mathbb{C}^n$. Given an n-dimensional singular cycle Γ_s represented by a compact set Γ in $\mathbb{C}^n \setminus H_Q$ and recalling the finite set Λ_Q defined in Proposition 1.3, the following two statements hold.*

(1) *If the logarithm $\ln Q(z)$ exists on Γ and Λ_Q is contained in the unbounded connected component of $\mathbb{C} \setminus Q(\Gamma)$ union the connected component which contains the origin, then Γ_s is homologous to zero in the complement of H_Q.*
(2) *If Γ is a connected and locally arcwise connected space whose first singular cohomology group $H^1(\Gamma, \mathbb{Z})$ vanishes, and the sets Λ_Q and $Q(\Gamma)$ are disjoint, then Γ_s is homologous to zero in $\mathbb{C}^n \setminus H_Q$.*

This proposition will be proved in the second section of this paper as well. Notice that the open set $\mathbb{C} \setminus Q(\Gamma)$ contains the origin and has only one unbounded connected component because $Q(\Gamma)$ is compact. Besides, we point out that the logarithm $\ln Q(z)$ exists on Γ, when $H^1(\Gamma, \mathbb{Z})$ vanishes and Γ is a topological manifold. First, the singular and the Čech cohomology groups are isomorphic, $H^1(\Gamma, \mathbb{Z}) \cong \check{H}^1(\Gamma, \mathbb{Z})$, see for example [12, pp. 334, 340] or [11, pp. 166–167]. Considering the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{\eta} \mathbb{C}^* \to 0,$$

where $\mathbb{Z}$ and $\mathbb{C}$ are groups under standard addition, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is a group under multiplication, and $\eta(t) = \exp(2\pi i t)$; we have the following induced long exact sequence, see for example [11, p. 145],

$$\cdots \to \check{H}^0(\Gamma, \mathbb{Z}) \to \check{H}^0(\Gamma, \mathbb{C}) \xrightarrow{\eta} \check{H}^0(\Gamma, \mathbb{C}^*) \to \check{H}^1(\Gamma, \mathbb{Z}) \to \cdots .$$

Whence, recalling that $\check{H}^0(\Gamma, \mathcal{A})$ is the set of all possible continuous functions from Γ into $\mathcal{A}$, we have that $Q(z)$ is an element of $\check{H}^0(\Gamma, \mathbb{C}^*)$; so there exists a continuous function h defined on Γ and such that $\exp(2\pi i h(z)) = Q(z)$, whenever $H^1(\Gamma, \mathbb{Z}) \cong \check{H}^1(\Gamma, \mathbb{Z})$ vanishes.

Coming back to the sphere $\mathcal{S}^n$ constructed by Nemirovskiĭ in $\mathbb{C}^n \setminus H_F$. We have that the group $H^1(\mathcal{S}^n, \mathbb{Z})$ vanishes whenever $n \geq 2$, so the logarithm of the Fermat polynomial $F(z) = 1 + z_1^q + \cdots + z_n^q$ indeed exists on $\mathcal{S}^n$, but this sphere is not homologous to zero in the complement of H_F. Moreover, it is easy to calculate that the set Λ_F defined in Proposition 1.3 is composed only of the point $z = 1$, so the critical fibre $\{F(z) = 1\}$ must meet $\mathcal{S}^n$ according to Proposition 1.4. On the other hand, suppose that a non-trivial element of $H_n(\mathbb{C} \setminus H_Q, \mathbb{Z})$ can be represented by a cycle Γ_s whose associated set Γ is a simply connected manifold, then this manifold must satisfy the conditions of Propositions 1.4 or 3.2; we only need to observe that $H^1(\Gamma, \mathbb{Z})$ vanishes when Γ is simply connected, see for example [1, 12].

Finally, the last section of this paper is devoted to proving several consequences of Proposition 1.4.

2 Proofs of Lemma 1.2 and Proposition 1.4

We point out that Lemma 1.2 can be deduced from Proposition 1.4 as a corollary. Nevertheless, we wanted to present Lemma 1.2 as an independent issue because it inspired the main results of this paper: Propositions 1.4 and 3.2.

Definition 2.1 Fixing $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we say that a given point $b \in \mathbb{C}^*$ can be joined by a smooth arc $\Upsilon_k \subset \mathbb{C}^*$ to the origin (respectively, the point at infinity) if there exists an injective smooth function g_k defined from the real line into $\mathbb{C}^*$ such that $g_k(0) = b$ and the limit of $|g_k(t)|$ is equal to zero (resp., infinity) when $t \to +\infty$. The smooth arc Υ_k is then defined as the image of the infinite interval $[0, +\infty)$ under g_k.

Notice that Υ_k is a closed subset of $\mathbb{C}^*$ with only one end point b.

Proof of Proposition 1.4 (1) Since Λ_Q is contained in the unbounded connected component of $\mathbb{C} \setminus Q(\Gamma)$ union the connected component which contains the origin, we may find an open set $D_1 \subset \mathbb{C}^*$ diffeomorphic to the annulus $\mathbb{C}^*$ such that $Q(\Gamma) \subset D_1$, the intersection $D_1 \cap \Lambda_Q$ is empty, and the origin is contained in the bounded connected component of $\mathbb{C} \setminus D_1$. Notice that the complement of D_1 has only one bounded connected component because D_1 is diffeomorphic to an annulus.

We can build the open set D_1 as follows. Considering the given hypotheses, we have that every point of $\Lambda_Q \setminus \{0\}$ can be joined by a smooth arc Υ_k to either the origin or the point at infinity, in such a way that every arc Υ_k is contained inside $\mathbb{C}^* \setminus Q(\Gamma)$ and every two different arcs Υ_j and Υ_k are disjoint. Thus, the open set D_1 defined as the complement of the finite union $\bigcup_k \Upsilon_k \cup \{0\}$ indeed satisfies the desired properties. Besides, one can also verify that the open set D_2 composed of all the points $x \in \mathbb{C}$ with $\exp(x) \in D_1$ is diffeomorphic to the whole plane $\mathbb{C}$, for the origin is contained in the unique bounded connected component of $\mathbb{C} \setminus D_1$. Consider the space M composed of all the points $(x, z) \in \mathbb{C} \times \mathbb{C}^n$ with $\exp(x) = Q(z)$; and the basic projections $\rho_1(x, z) = x$ and $\rho_2(x, z) = z$ defined from M onto $\mathbb{C}$ and $\mathbb{C}^n$, respectively. We have the following commutative diagram,

$$\begin{array}{ccc} \rho_1^{-1}(D_2) & \xrightarrow{\rho_2} & Q^{-1}(D_1) \\ {\scriptstyle \rho_1}\downarrow & & \downarrow{\scriptstyle Q} \\ D_2 & \xrightarrow{\exp} & D_1. \end{array}$$

Notice that the fibres of Q induce a locally trivial fibre bundle of $Q^{-1}(D_1)$ with base on D_1, by Proposition 1.3 and because the intersection $D_1 \cap \Lambda_Q$ is empty. Hence, the fibres of ρ_1 induce a locally trivial fibre bundle of $\rho_1^{-1}(D_2)$ with base on D_2 as well. Recalling that D_2 is diffeomorphic to the plane $\mathbb{C}$, we can then deduce that $\rho_1^{-1}(D_2)$ is diffeomorphic to the product $D_2 \times Z_0$, where Z_0 is the fibre $\{Q(z) = \exp(x_0)\}$ for some point $x_0 \in D_2$, see [3, p. 27]. The fibre Z_0 is a Stein manifold of complex dimension $n - 1$, so both homology groups $H_n(Z_0, \mathbb{Z})$ and $H_n(\rho_1^{-1}(D_2), \mathbb{Z})$ vanish.

Finally, let $\Gamma_s = \sum_k m_k f_k$ be an n-dimensional singular cycle in the complement of H_Q, and suppose there exists a continuous function h defined from the associated set Γ into $\mathbb{C}$ such that $\exp(h(z)) = Q(z)$. Notice that every point $(h(z), z)$ is contained in M, for $z \in \Gamma$, so the sum $T_s = \sum_k m_k(h(f_k), f_k)$ is indeed an n-dimensional cycle in M. Moreover, recall that $Q(z) \in D_1$ and $h(z) \in D_2$, for any $z \in \Gamma$, so T_s is a cycle in $\rho_1^{-1}(D_2)$. Thus, there exists an $(n+1)$-dimensional singular chain Δ_s in $\rho_1^{-1}(D_2)$ whose boundary $\partial\Delta_s = T_s$. We have that Γ_s is homologically trivial, because it is equal to the boundary $\partial\rho_2(\Delta_s)$.

Proof of Lemma 1.2 The main idea is to prove that the set Λ_P defined in Proposition 1.3 is either empty or the singleton $\{0\}$, whenever $P(z)$ is a weighted homogeneous holomorphic polynomial.

Let Z_1 be the fibre $\{P(z) = 1\}$. We can deduce by simple derivation that the formula $P(z) = \sum \beta_k z_k \frac{dP(z)}{dz_k}$ always holds, after recalling the definition of weighted homogeneous. Thus, the differential $dP(y_0)$ is different from zero for any point y_0 with $P(y_0) = 1$, and so Z_1 is a Stein manifold of complex dimension $n - 1$. Consider the holomorphic function η from $\mathbb{C} \times Z_1$ into $\mathbb{C}^n$ defined by $\eta(x, z) = (e^{x\beta_1} z_1, \cdots, e^{x\beta_n} z_n)$, it is easy to see that the equation $P \circ \eta(x, z) = e^x$ always holds. Hence, we have that η is a covering projection from $\mathbb{C} \times Z_1$ onto $\mathbb{C}^n \setminus P^{-1}(0)$, and that the fibres of P induce a locally trivial fibre bundle of $\mathbb{C}^n \setminus P^{-1}(0)$ with base on

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$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. That is, the set Λ_p is either empty or equal to the singleton $\{0\}$. The conclusion of Lemma 1.2 can be deduced following the proof of Proposition 1.4 (1).

Proof of Proposition 1.4 (2) Let μ be a smooth universal covering projection from $\mathbb{C}$ onto $D_3 = \mathbb{C} \setminus (\Lambda_Q \cup \{0\})$, and M be the space composed of all the points $(x, z) \in \mathbb{C} \times \mathbb{C}^n$ such that $\mu(x) = Q(z)$. Considering the projection $\rho_1(x, z) = x$ defined from M onto $\mathbb{C}$, recalling Proposition 1.3 and working as in the proof of Proposition 1.4 (1), we have that M is diffeomorphic to the product $\mathbb{C} \times Z_2$, where Z_2 is the fibre $\{Q(z) = \mu(x_2)\}$ for some point $x_2 \in \mathbb{C}$. Hence, the group $H_n(M, \mathbb{Z})$ vanishes.

Now let $\Gamma_s = \sum_k m_k f_k$ be an n-dimensional singular cycle in the complement of H_Q whose associated set Γ is compact, connected and locally arcwise connected; moreover, suppose that $H^1(\Gamma, \mathbb{Z})$ vanishes. Given any finitely generated free group $\mathcal{G}$, we are going to prove that every homomorphism from the fundamental group $\pi_1(\Gamma)$ into $\mathcal{G}$ is trivial. The following short exact sequence is a consequence of the universal coefficient theorem, see [1, p. 282] or [4, p. 133],

$$\cdots \to H^1(\Gamma, \mathbb{Z}) \to \mathrm{Hom}(H_1(\Gamma, \mathbb{Z}), \mathbb{Z}) \to 0,$$

so every homomorphism from $H_1(\Gamma, \mathbb{Z})$ into $\mathbb{Z}$ is trivial.

Since $H_1(\Gamma, \mathbb{Z})$ is isomorphic to the quotient $\pi_1(\Gamma)/\mathcal{E}$, where $\mathcal{E}$ is the commutator subgroup of $\pi_1(\Gamma)$, then, every element of $H_1(\Gamma, \mathbb{Z})$ can be seen as an equivalence class $[a\mathcal{E}]$ for some a in $\pi_1(\Gamma)$, see for example [1, pp. 173–174]. We assert that every homomorphism f_1 from $\pi_1(\Gamma)$ into $\mathbb{Z}$ is trivial. Recalling that $\mathcal{E}$ is generated by all the commutators $aba^{-1}b^{-1}$ with a and b in $\pi_1(\Gamma)$, we obviously have that $f_1(aba^{-1}b^{-1}) = 0$ because $\mathbb{Z}$ is abelian; so $\mathcal{E}$ is contained in the kernel of f_1. We may then define a homomorphism f_2 from $H_1(\Gamma, \mathbb{Z})$ into $\mathbb{Z}$ by the formula $f_2([a\mathcal{E}]) = f_1(a)$, for any element $[a\mathcal{E}]$ of $H_1(\Gamma, \mathbb{Z})$. The homomorphism f_2 is well defined because $f_1(ab) = f_1(a)$ for each b in $\mathcal{E}$, and f_2 is trivial because $H^1(\Gamma, \mathbb{Z})$ vanishes. Moreover, supposing f_1 is non-trivial, there exists a_0 in $\pi_1(\Gamma)$ such that $f_1(a_0) \neq 0$, so $a_0 \notin \mathcal{E}$ and $f_2([a_0\mathcal{E}]) \neq 0$. We have a contradiction. Whence, we can conclude that every homomorphism from $\pi_1(\Gamma)$ into $\mathbb{Z}$ is trivial whenever the cohomology group $H^1(\Gamma, \mathbb{Z})$ vanishes.

Finally, suppose there exists a non-trivial homomorphism f_3 from $\pi_1(\Gamma)$ into the free group $\mathcal{G}$, then, the image $f_3(\pi_1(\Gamma))$ is a non-trivial free subgroup of $\mathcal{G}$, see for example the Nielsen–Schreier theorem [7, p. 96]. Let b_1 be any generator of $f_3(\pi_1(\Gamma))$, different from the identity in $\mathcal{G}$. We can build a homeomorphism f_4 from $\mathcal{G}$ into $\mathbb{Z}$ defined by the following condition: given any word w in $\mathcal{G}$, the integer $f_4(w)$ is equal to the sum of all the exponents of b_1 in w. Obviously, if the word w contains no letter b_1, then $f_4(w) = 0$. Notice that f_4 is a homeomorphism, so $f_4 \circ f_3$ is a non-trivial homeomorphism from $\pi_1(\Gamma)$ into $\mathbb{Z}$ as well, because $f_4(b_1) = 1$ and b_1 is in $f_3(\pi_1(\Gamma))$. We have a contradiction. Therefore, we may conclude that every homomorphism from $\pi_1(\Gamma)$ into $\mathcal{G}$ is trivial whenever the group $H^1(\Gamma, \mathbb{Z})$ vanishes.

Coming back to our original proof, we already have a pair of continuous functions $\mu \colon \mathbb{C} \to D_3$ and $Q|_\Gamma \colon \Gamma \to D_3$, notice that $Q(\Gamma) \subset D_3$ because $Q(\Gamma)$ and $\Lambda_Q \cup \{0\}$ are disjoint. We obviously have that the induced homomorphism $\pi^*\mu \colon \pi_1(\mathbb{C}) \to \pi_1(D_3)$

is trivial; and moreover, $\pi^* Q|_\Gamma$ defined from $\pi_1(\Gamma)$ into $\pi_1(D_3)$ is trivial as well by the analysis above and because $\pi_1(D_3)$ is a finitely generated free group. Hence, the composition $\pi^*\mu(\pi^* h)$ is identically equal to $\pi^* Q|_\Gamma$ for every homomorphism $\pi^* h\colon \pi(\Gamma) \to \pi(D_3)$. The lifting theorem implies the existence of a continuous function h from Γ into $\mathbb{C}$ such that $\mu \circ h(z) = Q(z)$ for any $z \in \Gamma$, see [1, p. 143] or [12, p. 76]. Finally, proceeding as in the proof of Proposition 1.4 (1), we have that there exists an $(n+1)$-dimensional singular chain Δ_s in M whose boundary $\partial\Delta_s$ is equal to $\sum_k m_k(h(f_k), f_k)$. Considering now the projection $\rho_2(x, z) = z$ defined from M into $\mathbb{C}^n$, we automatically have that the boundary $\partial\rho_2(\Delta_s)$ is equal to Γ_s.

3 Final Results

Recalling the hypotheses of Proposition 1.4, it is very easy to see that the logarithm $\ln Q(z)$ is well defined on Γ whenever the origin is contained in the unbounded connected component of $\mathbb{C} \setminus Q(\Gamma)$. Amazingly, this seems to be a very strong condition.

Proposition 3.1 *Let H_Q be the zero locus of a non-constant holomorphic polynomial $Q(z)$ in $\mathbb{C}^n$, for $n \geq 2$. Suppose that the origin is not contained in the finite set Λ_Q defined in Proposition 1.3. Given an n-dimensional singular cycle Γ_s represented by the set Γ in $\mathbb{C}^n \setminus H_Q$, we have that Γ_s is homologous to zero whenever the origin is in the unbounded connected component of $\mathbb{C} \setminus Q(\Gamma)$.*

Proof The given hypotheses allow us to build a smooth injective function g defined from the real line into $\mathbb{C} \setminus (Q(\Gamma) \cup \Lambda_Q)$ such that $g(1) = 0$ and the limit of $|g(t)|$ is equal to infinity when $t \to +\infty$. Now fix the infinite interval $R_0 = [1, +\infty)$ of the real line and define the arc Υ to be the image $g(R_0)$. Proposition 1.3 automatically implies that the fibres of $Q(z)$ induce a locally trivial fibre bundle of $Q^{-1}(\Upsilon)$ with base on Υ. Therefore, since the arc Υ is obviously contractible and $H_Q = Q^{-1}(0)$, we may find a diffeomorphism F defined from $R_0 \times H_Q$ onto $Q^{-1}(\Upsilon)$ such that $Q \circ F(t, x) = g(t)$. We have the following commutative diagram, where $\rho(t, x) = t$,

$$\begin{array}{ccc} R_0 \times H_Q & \xrightarrow{F} & Q^{-1}(\Upsilon) \\ \Big\downarrow{\scriptstyle\rho} & & \Big\downarrow{\scriptstyle Q} \\ R_0 & \xrightarrow{g} & \Upsilon. \end{array}$$

Define $\mathbb{C}^n \cup \{\infty\}$ to be the one point compactification of $\mathbb{C}^n$; obviously, we have that $\mathbb{C}^n \cup \{\infty\}$ is diffeomorphic to the sphere $\mathbb{S}^{2n}$. We assert that the one point compactification $Q^{-1}(\Upsilon) \cup \{\infty\}$ is contractible, analysing it as a closed subset of $\mathbb{C}^n \cup \{\infty\}$. Consider the inverse of F as a pair of smooth functions (f_1, f_2) defined from $Q^{-1}(\Upsilon)$ onto $R_0 \times H_Q$. We have for example that $f_1(z)$ is equal to $g^{-1} \circ Q(z)$. We may construct the homotopy,

$$G(z, s) = \begin{cases} \infty & \text{if } z = \infty \text{ or } s = 0, \\ F(f_1(z)/s, f_2(z)) & \text{otherwise.} \end{cases}.$$

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It is easy to see that $G(z,s)$ is a continuous function for all $0 \leq s \leq 1$ and z in $Q^{-1}(\Upsilon) \cup \{\infty\}$. Moreover, $G(z,1) = z$ is the identity and $G(z,0) = \infty$ is a constant function. Hence, $Q^{-1}(\Upsilon) \cup \{\infty\}$ is contractible, and so the duality theorem of Alexander yields,

$$H_k(\mathbb{C}^n \setminus Q^{-1}(\Upsilon)) = \check{H}^{2n-k-1}(Q^{-1}(\Upsilon) \cup \{\infty\}) = 0,$$

for $1 \leq k \leq 2n-2$. Thus, Γ_s is homologically trivial in $\mathbb{C}^n \setminus Q^{-1}(\Upsilon)$. The result then follows by recalling that H_Q is contained in $Q^{-1}(\Upsilon)$. ■

Finally, given the right conditions, we may even *push* the points of Λ_Q to the unbounded connected component of $\mathbb{C} \setminus Q(\Gamma)$. Recall Proposition 1.3 and Definition 2.1.

Proposition 3.2 *Let H_Q be the zero locus of a non-constant holomorphic polynomial Q in $\mathbb{C}^n$, and Γ_s be an n-dimensional singular cycle represented by a smooth manifold Γ in $\mathbb{C}^n \setminus H_Q$. If the logarithm $\ln Q(z)$ is well defined on Γ and every point of $\Lambda_Q \setminus \{0\}$ can be joined by a smooth arc $\Upsilon_k \subset \mathbb{C}^*$ to the point at infinity in such a way that every set $Q^{-1}(\Upsilon_k) \cap \Gamma$ is connected and every two different arcs Υ_j and Υ_k are disjoint. Then Γ_s is homologous to zero in the complement of H_Q.*

Proof Recall that $H_n(\mathbb{C}^n, \mathbb{Z})$ vanishes, so there exists an $(n+1)$-dimensional singular chain Δ_s in $\mathbb{C}^n$ whose boundary $\partial\Delta_s$ is equal to Γ_s. We may even choose Δ_s in such a way that it is represented by a compact smooth manifold Δ whose boundary (as a manifold) is equal to Γ as well. Notice that we shall have finished if Δ is contained in $\mathbb{C}^n \setminus H_Q$, so we are supposing from now on that $Q(z)$ has indeed a zero inside Δ.

Notice that each arc Υ_k is a closed subset of the complex plane, and that $Q^{-1}(\Upsilon_k) \cap \Gamma$ is connected, so we may define the set E_k to be the compact connected component of $Q^{-1}(\Upsilon_k) \cap \Delta$ which contains $Q^{-1}(\Upsilon_k) \cap \Gamma$. We already know that there exists a continuous function h_1 from Γ into $\mathbb{C}$ with $\exp(h_1(z)) = Q(z)$. Hence, we can extend h_1 to a continuous function h_2 defined from the finite union $\bigcup_k E_k \cup \Gamma$ into $\mathbb{C}$ such that $\exp(h_2(z)) = Q(z)$, because every intersection $E_k \cap \Gamma$ is connected, the origin is not contained in any Υ_k, and every two different arcs Υ_j and Υ_k are disjoint. We may even go a step further. We can find an open neighbourhood V of the union $\bigcup_k E_k \cup \Gamma$, and a continuous function h_3 defined from V into $\mathbb{C}$ such that V is disjoint from H_Q and $\exp(h_3(z)) = Q(z)$ for every point $z \in V$.

On the other hand, since Δ is a smooth manifold with boundary, every compact $E_k \subset \Delta$ has a small enough open neighbourhood W_k such that the closure $\overline{W_k}$ is contained in V, every two different sets $\overline{W_j}$ and $\overline{W_k}$ are disjoint, the set E_k is equal to $Q^{-1}(\Upsilon_k) \cap \Delta \cap \overline{W_k}$, and the smooth boundary ∂W_k meets Δ transversally. That is, the compact sets $\Delta \cap \overline{W_k}$ and $\Delta \setminus W_k$ are all smooth manifolds with piecewise smooth boundary. Define T to be the boundary of the smooth manifold $\Delta \setminus (\bigcup_k W_k)$.

Each manifold $\Delta \cap \overline{W_k}$ is contained in $\mathbb{C}^n \setminus H_Q$ and Γ is the boundary of Δ, so we can find an n-dimensional singular cycle T_s which is represented by T and is homologous to Γ_s in $\mathbb{C}^n \setminus H_Q$. Besides, every point of T is contained in Γ or in some boundary $\partial W_k \subset V$. Whence, we have that T is completely contained in V, and so the logarithm $\ln Q(z)$ is well defined on T. Finally, if there exists a point x in $Q^{-1}(\Upsilon_k) \cap T$, then x must be contained in $Q^{-1}(\Upsilon_k) \cap \Delta \cap \overline{W_k}$. That is, the point x is contained in $E_k \subset W_k$. However, it is easy to see that T and W_k are disjoint. We have a contradiction. Hence, the set T does not meet any inverse image $Q^{-1}(\Upsilon_k)$. We can rewrite the previous statement as follows: every point of $\Lambda_Q \setminus \{0\}$ is the end point of an arc Υ_k which does not meet $Q(T)$ and goes to infinity, so $\Lambda_Q \setminus \{0\}$ is contained in the unbounded connected component of $\mathbb{C} \setminus Q(T)$. We only need to apply Proposition 1.4 to deduce that Γ_s and T_s are both homologous to zero in the complement of H_Q. ■

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