A CONVERGENCE THEOREM FOR DOUBLE L^2 FOURIER SERIES

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1. Our aim in this paper is to extend a known theorem about the convergence of subsequences of the partial sums of the Fourier series in one variable of class L^2 to Fourier series in two variables of the same class, (**1**, p. 396). The theorem asserts that for each function f in L^2, there is a sequence $\{m_\nu\}$ of positive integers of upper density one such that

$$s_{m_\nu}(x;f)$$

converges to f almost everywhere, where $s_m(x;f)$ denotes the mth partial sum of the Fourier series of f. The sequence $\{m_\nu\}$ depends on the function f but not on the point x (**3**, p. 264). The main tools of proof used were the theorem of Kolmogoroff asserting the almost everywhere convergence of lacunary subsequences of partial sums of L^2 Fourier series and the theorem of Kolmogoroff and Seliverstoff (**3**, p. 253). These same tools are available in the two-dimensional case (**2**), but they do not seem to be adequate in themselves to obtain an extension.

Our method of proof is the following. First we extend the one-dimensional theorem so that a single sequence $\{m_\nu\}$ of upper density one will serve for a given sequence $\{f_n\}$ of functions, each f_n belonging to L^2. From this we may generalize to the two-dimensional case by considering first iterated limits of partial sums.

2. The definition of upper density for a sequence $\{m_\nu\}$ of positive integers strictly increasing to ∞ is as follows. Let $\sigma(n)$ be the number of terms of the sequence less than or equal to n. We say the sequence is of upper density β if $\limsup \sigma(n)/n = \beta$.

THEOREM 1. *Let $\{f_n\}$ be a sequence of functions, each of class L^2. Then there is a sequence $\{p_\nu\}$ of upper density one such that*

$$s_{p_\nu}(x;f_n)$$

converges to f_n almost everywhere for each n.

Let $\{\lambda_r\}$ and $\{k_r\}$ be two sequences of integers each strictly increasing to ∞ and such that $k_{r+1} > 2k_r$ and $r+1$ divides k_r. At a later stage we shall

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impose a further restriction relating to the size of k_r. Let $m_k = \lambda_r^k$, $k_r \leqslant k \leqslant 2k_r$, and let

$$\sum_{m=-\infty}^{+\infty} c_m(n) e^{imx}$$

denote the Fourier series of the function $f_n(x)$. For any function $f(x)$ of L^2 with Fourier coefficients c_m, we introduce the following notation:

$$(1) \qquad \epsilon_k = \sum_{|m|=m_k+1}^{m_{k+1}} |c_m|^2, \; k = k_r, k_r + 1 \ldots 2k_r - 1; \; D_r = \sum_{k=k_r}^{2k_r-1} \epsilon_k.$$

If α of the k_r numbers ϵ_k are greater than $\beta D_r/k_r$, then

$$(2) \qquad \alpha\beta \frac{D_r}{k_r} \leqslant D_r \qquad \text{or} \qquad \alpha \leqslant \frac{k_r}{\beta}.$$

Now let $\epsilon_k(n)$ and $D_r(n)$ correspond to $f_n(x)$ as ϵ_k and D_r in (1) correspond to $f(x)$. Fix r, and consider $f_1, f_2, \ldots, f_r$. For each n, $n = 1, 2, \ldots, r$, at least $rk_r/(r+1)$ of the numbers ϵ_k (n) are less than $(r+1)D_r(n)/k_r$ since by (2) the number greater does not exceed $k_r/(r+1)$. There must be some k, say $k(r)$, $k_r \leqslant k(r) \leqslant 2k_r - 1$, such that

$$\epsilon_{k(r)}(n) \leqslant \frac{(r+1)D_r(n)}{k_r}, \qquad n = 1, 2, \ldots, r$$

since among the rk_r numbers $\epsilon_k(n)$, no more than $rk_r/(r+1)$ of them exceed the above. Thus for $k = k(r)$

$$(3) \quad \sum_{|m|=m_k+1}^{m_{k+1}} |c_m(n)|^2 \log(|m|) \leqslant 2k_r (\log \lambda_r)\epsilon_k(n) \leqslant 2(r+1) (\log \lambda_r) D_r(n), \qquad n = 1, 2, \ldots, r.$$

For fixed λ_r and n,

$$\sum_{|m|=\lambda_r^k}^{\infty} |c_m(n)|^2$$

goes to 0 as k increases to ∞. We may thus choose $k = k_r$ subject to the previous conditions and so large that

$$2(r+1) (\log \lambda_r) D_r(n) \leqslant 2^{-r}, \qquad n = 1, 2, \ldots, r.$$

From (3),

$$\sum_{\substack{k=k(r)\\ r \geqslant n}} \sum_{|m|=m_k+1}^{m_{k+1}} |c_m(n)|^2 \log(|m|) \leqslant 2 \sum_{r \geqslant n} (r+1) (\log \lambda_r) D_r(n) < \infty$$

for all n. Now let $\{p_\nu\}$ take on the values m, $m_k \leqslant m \leqslant m_{k+1}$, $k = k(r)$, $r = 1, 2, \ldots$. It is easily seen that $\{p_\nu\}$ is of upper density one and the almost everywhere convergence of each sequence

$$s_{p_\nu}(x; f_n)$$

to f_n follows as in our original proof (**1**, p. 396).

Remark. For convenience we suppose that $6(r+1)$ divides k_r. Among the rk_r numbers $\epsilon_k(n)$, no more that $rk_r/3(r+1)$ of them exceed $3(r+1)D_r(n)/k_r$, for the same reasons as used above. Hence for more than one-half the indices k in the given range,

$$\epsilon_k(n) \leqslant \frac{3(r+1)D_r(n)}{k_r}, \qquad n = 1, 2, \ldots, r,$$

a fact we shall use in the proof of Theorem 2.

3. We first extend the notion of upper density for single sequences of positive integers to double sequences. Let P be a set of ordered pairs of positive integers (p, q), and let $\sigma(m, n)$ be the number of pairs (p, q) from P such that $p \leqslant m$ and $q \leqslant n$. If β is the largest number for which there are sequences $\{m_k\}$ and $\{n_k\}$ of positive integers each strictly increasing to ∞ as k goes to ∞ such that $\lim_{k\to\infty}\sigma(m_k, n_k)/m_k n_k = \beta$ we say that the set P has upper density β. It is easily seen that a largest number must exist. We may state our generalized theorem. Part (i) relates to iterated limits and is used in the proof of part (ii). Let $s_{p,q}(x, y; f)$ denote the pqth partial sum of the Fourier series of an integrable function $f(x, y)$ and let Ω be the square in the xy plane with $(0, 0)$ and $(2\pi, 2\pi)$ as opposite vertices.

THEOREM 2. *Let $f(x, y)$ belong to $L^2(\Omega)$.*

(i) *There exist sequences $\{p_\mu\}$, $\{q_\nu\}$ of positive integers, each separately of upper density one, such that almost everywhere*

$$\lim_{\mu\to\infty}\left[\lim_{\nu\to\infty} s_{p_\mu, q_\nu}(x, y; f)\right] = \lim_{\nu\to\infty}\left[\lim_{\mu\to\infty} s_{p_\mu, q_\nu}(x, y; f)\right] = f(x, y).$$

(ii) *There exists a double sequence P' of positive integers of upper density one such that almost everywhere*

$$\lim_{\substack{p,p\to\infty \\ (p,q)\,\epsilon P}} s_{p,q}(x, y; f) = f(x, y).$$

In the double limit of part (ii), p and q go to ∞ independently except that the pair (p, q) must belong to P. Let $c_{m,n}$ denote the Fourier (exponential) coefficient of $f(x, y)$. Since for each n,

$$\sum_{m=-\infty}^{+\infty} |c_{m,n}|^2 < \infty,$$

the series

$$\sum_{m=-\infty}^{+\infty} c_{m,n} e^{imx}$$

is the Fourier series of a function $f_n(x)$ in $L^2(0, 2\pi)$. We have

$$c_{m,n} = \frac{1}{2\pi i}\int_0^{2\pi} e^{-imx}\left\{\frac{1}{2\pi i}\int_0^{2\pi} f(x, y)e^{-inv}\, dy\right\} dx$$

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so that

$$2\pi i f_n(x) = \int_0^{2\pi} f(x, y)e^{-iny}\,dy.$$

By Theorem 1, there is a sequence $\{p_\mu\}$ of upper density one such that for almost every x and every $n = 0, \pm 1, \ldots$

$$\lim_{\mu\to\infty} \sum_{m=-p_\mu}^{p_\nu} c_{m,n}e^{imx} = f_n(x).$$

Hence, for every fixed q, every y, and almost every x

$$\lim_{\mu\to\infty} s_{p_\mu,q}(x, y; f) = \sum_{n=-q}^{q} f_n(x)e^{iny}. \tag{4}$$

By Parseval's equality

$$2\pi \sum_{n=-\infty}^{+\infty} |f_n(x)|^2 = \int_0^{2\pi} |f(x, y)|^2 dy$$

which is finite for almost every x. Hence, for almost every x, each member of the family, indexed by x, of series

$$\sum_{n=-\infty}^{+\infty} f_n(x)e^{iny} \tag{5}$$

is the Fourier series of an L^2 function of the variable y, that is, $f(x, y)$. The numbers $f_n(x)$ are then the Fourier coefficients.

Let $\{\alpha_r\}$ be a sequence of numbers, $0 < \alpha_r \leqslant 2\pi$, such that $\sum_{r=1}^{\infty}\alpha_r < \infty$. Let $\{\lambda_r\}$ and $\{k_r\}$ be two sequences of positive integers, each strictly increasing to ∞ and such that $k_{r+1} > 2k_r$. A further restriction is needed on $\{k_r\}$. For fixed λ_r, choose k_r so large that

$$\sum_{|n|>\lambda_r^k} \int_0^{2\pi} |f_n(x)|^2 dx \leqslant \frac{2^{-r-1}\alpha_r}{\log \lambda_r}, \qquad k \geqslant k_r. \tag{6}$$

This is possible since the left side of (6) equals

$$2\pi \sum_{|n|>\lambda_r^k} \sum_{m=-\infty}^{+\infty} |c_{m,n}|^2.$$

Let $n_k = \lambda_r^k$, $k_r \leqslant k \leqslant 2k_r$, and let

$$\sum_{|n|=n_k+1}^{n_{k+1}} |f_n(x)|^2 = \epsilon_k(x),\ k = k_r, k_r + 1, \ldots, 2k_r - 1;\ \sum_{k=k_r}^{2k_r-1} \epsilon_k(x) = D_r(x).$$

We set

$$\epsilon_k = \int_0^{2\pi} \epsilon_k(x)\,dx$$

and

$$D_r = \int_0^{2\pi} D_r(x)\,dx = \sum_{k=k_r}^{2k_r-1} \epsilon_k.$$

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Since there are k_r terms ϵ_k, it follows that for at least one k, say $k(r)$,

$$k_r \leqslant k(r) \leqslant 2k_r - 1, \quad \epsilon_{k(r)} \leqslant D_r/k_r.$$

Thus $\epsilon_{k(r)}(x) \leqslant D_r/\alpha_r k_r$ for x outside a set E_r whose measure, $|E_r|$, does not exceed α_r. Since

$$\sum_{r=1}^{\infty} \alpha_r < \infty,$$

for almost every x

$$\epsilon_{k(r)}(x) \leqslant D_r/\alpha_r k_r$$

for all sufficiently large r. For such an x and all sufficiently large r with $k = k(r)$

$$\text{(7)} \qquad \sum_{|n|=n_k+1}^{n_{k+1}} |f_n(x)|^2 \log(|n|) \leqslant (\log \lambda_r^{2k_r})\epsilon_k(x) \leqslant \frac{2D_r \log \lambda_r}{\alpha_r}.$$

Since D_r does not exceed the left side of (6), the left side of (7) does not exceed 2^{-r}. Thus for almost every x

$$\sum_{\substack{k=k(r)\\ r\geqslant 1}} \sum_{|n|=n_k+1}^{n_{k+1}} |f_n(x)|^2 \log(|n|) < \infty$$

since, except for a finite number of r values the terms satisfy (7) and so do not exceed 2^{-r}. This is sufficient to show as in our previous arguments **(1**, p. 396**)** that for almost every x, the $\{q_\nu\}$ partial sums of the series (5) converge to $f(x, y)$ for almost every y where the sequence $\{q_\nu\}$ takes on the values $n, n_{k(r)} \leqslant n \leqslant n_{k(r)+1}, r = 1, 2, \ldots$. The sequence $\{q_\nu\}$ is also of upper density one. This, together with (4), gives the second equality of part (i) of the theorem.

Our first step in proving the first equality of part (i) is to show that the sequences $\{p_\mu\}$ and $\{q_\nu\}$ already chosen in the proof of the second equality may be made the same. The sequence $\{p_\mu\}$ was chosen by the technique of Theorem 1 so as to insure the almost everywhere convergence of each sequence

$$\sum_{m=-p_\mu}^{p_\mu} c_{m,n} e^{imx}$$

to $f_n(x)$. The sequence $\{\lambda_r\}$ used in the proof of Theorem 1 may be taken to be the same as the sequence $\{\lambda_r\}$ already used in the proof of Theorem 2. Moreover the two $\{k_r\}$ sequences may be taken the same since, in each case, k_r was chosen large relative to a condition involving λ_r. By the remark following Theorem 1, for more than one-half the indices k, $k_r \leqslant k \leqslant 2k_r - 1$,

$$\epsilon_k^{(n)} \leqslant \frac{3(r+1)\, D_r(n)}{k_r}, \qquad n = 1, 2, \ldots, r.$$

As in our proof of the present theorem, for more than one-half the indices k,

$$\epsilon_k = \int_0^{2\pi} \epsilon_k(x)\, dx \leqslant 3\, D_r/k_r.$$

Hence, there is at least one index, say $k(r)$, for which both conditions are satisfied. Now the $\{p_\mu\}$ and the $\{q_\nu\}$ are chosen from the same blocks of integers, that is, n such that $\lambda_r^k \leqslant n \leqslant \lambda_r^{k+1}$, $k = k(r)$, $r = 1, 2, \ldots$.

It is easy to see that the same sequences $\{p_\mu\}$, $\{q_\nu\}$ can be chosen for any two functions of $L^2(\Omega)$, in particular for our $f(x, y)$ and for $g(x, y) = f(y, x)$. Since $s_{p,q}(x, y; g) = s_{q,p}(y, x; f)$ we may apply the second equality of part (i) to the sequence

$$s_{p_\mu, q_\nu}(x, y; g)$$

to obtain that almost everywhere

$$g(x, y) = \lim_{\nu\to\infty}\left[\lim_{\mu\to\infty} s_{p_\mu, q_\nu}(x, y; g)\right] = \lim_{\nu\to\infty}\left[\lim_{\mu\to\infty} s_{q_\nu, p_\mu}(y, x; f)\right] = f(y, x).$$

Since the $\{p_\mu\}$ take on the same values as the $\{q_\nu\}$ the first equality of part (i) of Theorem 2 is proved.

Now part (ii) follows easily. The difference

$$\sum_{n=-q_\nu}^{q_\nu} f_n(x)e^{iny} - f(x, y)$$

is smaller in absolute value than $1/s$ for (x, y) outside the set E_s when q_ν takes on values in the rth block, that is, $n_k \leqslant q_\nu \leqslant n_{k+1}$, $k = k(r)$. We may choose $r = r_s$ so large that

$$\sum_{s=1}^{\infty} |E_s| < \infty.$$

Also the difference

$$s_{p_\mu, q_\nu}(x, y; f) - \sum_{n=-q_\nu}^{q_\nu} f_n(x)e^{iny}$$

is smaller in absolute value than $1/s$ for (x, y) outside the set F_s, for q_ν in the rth block and p_μ in the Rth block, that is, $n_k \leqslant p \leqslant n_{k+1}$, $k = k(R)$. Again we may choose $R = R_s$ so large that

$$\sum_{s=1}^{\infty} |F_s| < \infty.$$

Hence, for (x, y) outside both E_s and F_s,

$$|s_{p_\mu, q_\nu}(x, y; f) - f(x, y)| < \frac{2}{s}, \quad \begin{array}{l} n_k \leqslant p_\mu \leqslant n_{k+1},\ k = k(r_s) \\ n_k \leqslant q_\nu \leqslant n_{k+1},\ k = k(R_s). \end{array}$$

Almost every point (x, y) is outside all E_s and F_s for s sufficiently large. Now let P, the double sequence of positive integers, consist of the union of all P_s for all s where P_s is defined as all (p, q) such that p belongs to the R_sth block and q to the r_sth block. For the double sequence P

$$\frac{\sigma(n_{k(R)+1},\, n_{k(r)+1})}{n_{k(R)+1}\, n_{k(r)+1}} \geqslant \left(1 - \frac{1}{\lambda_R}\right)\left(1 - \frac{1}{\lambda_r}\right), \qquad r = r_s, R = R_s$$

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which approaches one as s increases to ∞ so that P is of upper density one. Part (ii) follows from this.

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