FUNCTIONAL EQUATIONS OCCURRING IN THE THEORY
OF DELAYED DIFFERENTIAL EQUATIONS

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This paper is devoted to duscussion of some functional equations
obtained in the theory of delayed differential equations. By
means of the Laplace transform distribution solutions of the
considered equations are constructed.

1. Introduction

In this paper we are going to discuss some functional equations which
are obtained in connection with considerations on delayed differential
equations. The functional equations considered here are not especially
complicated and they are special cases of the functional equation of the
form $x(t) = F\big(t,\, x(\varphi(t))\big)$, where $x(t)$ is an unknown function. This
equation was examined in several papers (*cf.* [3], [1]). In principle the
general aim of the paper is to show how we can obtain functional equations
in the theory of delayed differential equations. Moreover, one can show
that the knowledge of solutions of those functional equations enables us to
find some generalized solutions of considered delayed differential
equations. Let us mention that we will consider only analytic solutions of
functional equations although they also have nonanalytic solutions which
will be shown (*cf.* [3], [7]). The tool which allows us to carry over
delayed differential equations to functional equations is the method of
Laplace transforms. This method is very often used for solving linear
ordinary and partial differential equations (see, for example, [4], [6]).

2. Delayed differential equation of the first order

Let us consider the simple delayed differential equation

$$(1) \qquad x'(t) = Ax(at) , \quad t \geq 0 ,$$

with the initial condition

$$(2) \qquad x(0) = x_0 ,$$

(*cf.* [5]). We shall assume that A and a are real constants and $0 < a \leq 1$. It is easy to show that the problem (1)-(2) has the following analytic solution

$$x(t) = x_0 \sum_{n=0}^{\infty} \frac{A^n a^{(n-1)n/2}}{n!} t^n .$$

On the other hand we can show (*cf.* Section 3) that the solution of (1)-(2) is an exponential order, so that we may apply to that problem the Laplace transformation [4] defined by the formula

$$L[x] = \int_0^{\infty} e^{-zt} x(t) dt = f(z) ,$$

where z is complex number. Thus we obtain the following functional equation

$$(3) \qquad zf(z) - (A/a)f(z/a) = x_0 .$$

In the sequel we shall seek an analytic solution of equation (3). First, putting $z = 0$, we have $f(0) = -ax_0/A$. Further, differentiating both sides of (3) and taking $z = 0$ we have $f'(0) = -x_0 a^3/A^2$. In the same way, by induction we calculate that

$$(4) \qquad f^{(n)}(0) = -x_0 n! \frac{a^{(n+1)(n+2)/2}}{A^{n+1}}$$

for $n = 0, 1, 2, \ldots$. Now writing the solution of equation (3) as the Taylor series expansion we get

$$f(z) = -x_0 \sum_{n=0}^{\infty} \frac{a^{(n+1)(n+2)/2}}{A^{n+1}} z^n .$$

It is easy to check that the above series converges uniformly on the whole finite complex plane so that the function $f(z)$ is the analytic solution of equation (3).

Now let us observe that by means of solution (4) we can find a solution of the problem (1)-(2). To do it we have to apply to the series (4) the inverse Laplace transform [6]. Indeed, taking into account that $L[\delta^{(n)}] = z^n$, $n = 0, 1, \ldots$, where $\delta(t)$ denotes the Dirac distribution and applying the theorem on convergence of series of generalized functions [6] we obtain

$$x(t) = -x_0 \sum_{n=0}^{\infty} \frac{a^{(n+1)(n+2)/2}}{A^{n+1}} \delta^{(n)}(t) \ ,$$

for $t \geq 0$. The function $x(t)$ given by the above formula is the solution of (1)-(2) in a generalized sense.

It is worthwhile mentioning that we can also obtain a solution of equation (3) in another way. In order to find it we apply the Laplace transform to the analytic solution of (1)-(2) obtained in the beginning. Thus, because of the well-known formula

$$\int_0^{\infty} t^n e^{-zt} dt = \frac{n!}{z^{n+1}} \ , \quad n = 0, 1, 2, \ldots \ ,$$

we obtain

$$f(z) = x_0 \sum_{n=0}^{\infty} \frac{A^n a^{(n-1)n/2}}{z^{n+1}} \ .$$

Notice that this series does not converge on the whole finite complex plane and this solution is well defined for all $z \neq 0$. Actually the function $f(z)$ satisfies equation (3) what we can check by a simple calculation.

3. Another functional equation

In this section we will consider the delayed differential equation of the second order

(5) $x''(t) = Ax(at) \ , \quad t \geq 0 \ ,$

assuming the following initial conditions

(6) $x(0) = x_1$, $x'(0) = x_2$.

Similarly to the previous section we shall assume that a and A are real constants and $0 < a \leq 1$.

Let us notice that putting $x_1(t) = x(t)$, $x_2(t) = x'(t)$ we can transform equation (6) to the following system,

$$x_1'(t) = x_2(t) ,$$

$$x_2'(t) = Ax_1(at) ,$$

with the initial condition $\big(x_1(0), x_2(0)\big) = \big(x_1, x_2\big)$.   The above problem can be rewritten in the following integral form

$$\begin{cases} x_1(t) = x_1 + \int_0^t x_2(s)ds , \\[2em] x_2(t) = x_2 + \int_0^t Ax_1(as)ds . \end{cases}$$

Hence

$$\begin{cases} |x_1(t)| \leq |x_1| + \int_0^t |x_2(s)|ds , \\[2em] |x_2(t)| \leq |x_2| + |A| \int_0^t |x_1(as)|ds . \end{cases}$$

Further, denoting $\varphi_i(t) = \max\big[|x_i(s)| : 0 \leq s \leq t\big]$, $i = 1, 2$, from the last system of the integral inequalities we obtain

(7)
$$\begin{cases} \varphi_1(t) \leq |x_1| + \int_0^t \varphi_2(s)ds , \\[2em] \varphi_2(t) \leq |x_2| + |A| \int_0^t \varphi_1(s)ds . \end{cases}$$

If we denote

$$\varphi(t) = \begin{bmatrix} \varphi_1(t) \\ \varphi_2(t) \end{bmatrix} , \quad C = \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} , \quad A = \begin{bmatrix} 0 , 1 \\ |A|, 0 \end{bmatrix} ,$$

then, from (8), we have the following inequality

$$\varphi(t) \leq C + \int_0^t A\varphi(s)ds \ .$$

Hence we get

$$\|\varphi(t)\| \leq \|C\| + \int_0^t \|A\|\|\varphi(s)\|ds$$

and in view of Gronwall's lemma [2] we obtain

$$\|\varphi(t)\| \leq \|C\|\exp\left(\int_0^t \|A\|ds\right) = \|C\|\exp(\|A\|t) \ ,$$

where, for example, $\|(x_1, x_2)\| = \max[|x_1|, |x_2|]$.

Finally we obtain the following inequality

$$|x(t)| \leq \|C\|\exp(\|A\|t)$$

which means that a solution of (5)-(6) is of exponential order. This assertion permits us to apply to the problem (5)-(6) the Laplace transformation.

Thus we have

$$z^2 f(z) - zx_1 - x_2 = (A/a)f(z/a) \ .$$

Writing the above functional equation in the form

(8) $$f(z) = a_1 z^2 f(az) - a_2 z - a_3$$

where $a_1 = a^3/A$, $a_2 = a^2 x_1/A$, $a_3 = ax_2/A$ and looking for an analytic solution of (8) we can - via the standard procedure - obtain

$$f^{(2k)}(0) = -(2k)!a^{(k-1)k}a_1^k a_3 \ ,$$

$$f^{(2k+1)}(0) = -(2k+1)!a^{k^2}a_1^k a_2 \ ,$$

for $k = 0, 1, 2, \ldots$. Thus we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k = -\left(\sum_{k=0}^{\infty} a^{(k-1)k} a_1^k a_3 z^{2k} + \sum_{k=0}^{\infty} a^{k^2} a_1^k a_2 z^{2k+1} \right) .$$

It is easy to check that the above series converges uniformly on the whole funite complex plane so that the function $f(z)$ is an analytic solution of equation (8).

Now, as in Section 2, applying to the function $f(z)$ the inverse Laplace transform and using the formula $L\left[\delta^{(n)}\right] = z^n$, $n = 0, 1, \ldots$ we obtain

$$x(t) = -\left(\sum_{k=0}^{\infty} a^{(k-1)k} a_1^k a_3 \delta^{(2k)}(t) + \sum_{k=0}^{\infty} a^{k^2} a_1^k a_2 \delta^{(2k+1)}(t) \right) ,$$

for $t \geq 0$. This function is a generalized solution of the problem (5)-(6).

Let us notice that by the same method as in Section 2 we can also obtain a nonanalytic solution of the functional equation (8). Namely it suffices to find an analytic solution of the problem (5)-(6) and then apply the Laplace transformation to it. We omit the details.

Finally, let us observe that a similar method may be used for the more complicated delay differential equation, of an arbitrary order n ,

$$x^{(n)}(t) = A_1 x^{(n-1)}(t) + A_2 x^{(n-2)}(a_2 t) + \ldots + A_n x(a_n t) ,$$

$$x^{(i-1)}(0) = x_{i-1} , \quad i = 1, 2, \ldots, n ,$$

where A_i, a_i are real constants and $a_i \in (0, 1)$ for $i = 1, 2, \ldots, n$.

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