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ON THE REPRESENTATION OF STRICTLY CONTINUOUS LINEAR FUNCTIONALS

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1. Introduction

Let X be a topological space, E a real or complex topological vector space, and $C(X, E)$ the vector space of all bounded continuous E-valued functions on X; when E is the real or complex field this space will be denoted by $C(X)$. The notion of the strict topology on $C(X, E)$ was first introduced by Buck (**1**) in 1958 in the case of X locally compact and E a locally convex space. In recent years a large number of papers have appeared in the literature concerned with extending the results contained in Buck's paper. In particular, a number of these have considered the problem of characterising the strictly continuous linear functionals on $C(X, E)$; see, for example, (**2**), (**3**), (**4**) and (**8**). In this paper we suppose that X is a completely regular Hausdorff space and that E is a Hausdorff topological vector space with a non-trivial dual E'. The main result established is Theorem 3.2, where we prove a representation theorem for the strictly continuous linear functionals on the subspace $C_{tb}(X, E)$ which consists of those functions f in $C(X, E)$ such that $f(X)$ is totally bounded.

Throughout, we use the notation and terminology introduced in (**5**).

2. Preliminaries

Let $\mathscr{B}$ be the σ-algebra of Borel subsets of X, and $M(X)$ the Banach space of all bounded regular Borel measures on X. The topology τ of E may be determined by a family $\mathscr{C}$ of $\mathscr{F}$-semi-norms, $\{\nu_i : i \in I\}$ say (see (**7**), p. 2)), and without loss of generality we can assume that $\mathscr{C}$ is full in the sense that, if $\nu_{i_1}, \ldots, \nu_{i_m}$ is any finite collection of members of $\mathscr{C}$, then $\max_{1 \leqq k \leqq m} \nu_{i_k}$ is also in $\mathscr{C}$, and $\lambda\nu \in \mathscr{C}$ for all $\lambda > 0$ and $\nu \in \mathscr{C}$. For each $i \in I$, let $M_i(X, E')$ denote the set of all finitely additive E'-valued set functions μ on $\mathscr{B}$ which have the following properties:

(i) for each $a \neq 0$ in E, $\mu_a(F) = \mu(F)(a)(F \in \mathscr{B})$ defines an element μ_a of $M(X)$;

(ii) there exists a constant k such that $|\mu|_i(X) \leqq k$, where, for each $F \in \mathscr{B}$, we define $|\mu|_i$ by

$$|\mu|_i(F) = \sup \left| \sum_j \mu_{a_j}(F_j) \right|,$$

the supremum being taken over all finite partitions $\{F_j\}$ of F into members of $\mathscr{B}$ (henceforth referred to as a $\mathscr{B}$-partition) and all finite collections $\{a_j\}$ of points in E such that $\nu_i(a_j) \leqq 1$.

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Let $M(X,E')=\bigcup_{i\in I} M_i(X,E')$. We now suppose that $m\in M_i(X,E')$, $F\in\mathscr{B}$, and $f\in C_{tb}(X,E)$. For each $F\in\mathscr{B}$, let $\mathscr{D}_F$ be the collection of all $\alpha=\{F_1,\ldots,F_n;\ x_1,\ldots,x_n\}$, where $\{F_j\}$ $(j=1,\ldots,n)$ is a $\mathscr{B}$-partition of F and $x_j\in F_j$. If $\alpha_1,\alpha_2\in\mathscr{D}_F$, define $\alpha_1\geqq\alpha_2$ if and only if each set which appears in α_1 is contained in some set in α_2. In this way $\mathscr{D}_F$ becomes an indexing set. Let $\omega_\alpha=\sum_{j=1}^{n} m(F_j)(f(x_j))$. We then have the following

Lemma 2.1. *$\{\omega_\alpha\}$ $(\alpha\in\mathscr{D}_F)$ is a Cauchy net.*

Proof. Let $\varepsilon>0$ (and without loss of generality suppose that $\varepsilon<1/4$). Then the set $V=\{x\in E: \nu_i(x)\leqq\varepsilon\}$ is a τ-neighbourhood of 0 in E. $f(X)$ is totally bounded and so there exist points $y_1,\ldots,y_n$ in X such that $f(X)\subseteq\bigcup_{j=1}^{n}(f(y_j)+V)$. Let $V_j=\{x\in X: f(x)-f(y_j)\in V\}$. Each V_j is closed, and so is in $\mathscr{B}$. Let $F_j'=V_j\cap F$ $(1\leqq j\leqq n)$ and define $G_1=F_1'$, $G_j=F_j'\backslash\bigcup_{k=1}^{j-1}F_k'$ $(2\leqq j\leqq n)$. By keeping those G_j's which are non-empty we get a $\mathscr{B}$-partition, $\{G_1,\ldots,G_{n_0}\}$ say, of F. Choose $x_j\in G_j$ and let $\alpha_0=\{G_1,\ldots,G_{n_0};x_1,\ldots,x_{n_0}\}$. Note that $\nu_i(f(x)-f(y))\leqq 2\varepsilon$ if x,y are in the same G_j. Then for $\alpha_1,\alpha_2\geqq\alpha_0$, we have

$$|\omega_{\alpha_1}-\omega_{\alpha_2}|\leqq|\omega_{\alpha_1}-\omega_{\alpha_0}|+|\omega_{\alpha_0}-\omega_{\alpha_2}|.$$

Now

$$\begin{aligned}|\omega_{\alpha_1}-\omega_{\alpha_0}|&=\left|\sum_k m(F_k)f(y_k)-\sum_{j=1}^{n_0} m(G_j)f(x_j)\right|\\&=\left|\sum_{j=1}^{n_0}\left(\sum_{\substack{k\\F_k\subseteq G_j}} m(F_k)f(y_k)-\sum_{\substack{k\\F_k\subseteq G_j}} m(F_k)f(x_j)\right)\right|\\&=\left|\sum_{j=1}^{n_0}\sum_{\substack{k\\F_k\subseteq G_j}} m(F_k)(f(y_k)-f(x_j))\right|.\end{aligned}$$

Note that

$$\nu_i\left(\left[\frac{1}{2\varepsilon}\right](f(y_k)-f(x_j))\right)\leqq\left[\frac{1}{2\varepsilon}\right]\nu_i(f(y_k)-f(x_j))\leqq\frac{1}{2\varepsilon}\,\nu_i(f(y_k)-f(x_j))\leqq 1,$$

where $[t]$ denotes the integer part of t. It follows that

$$\begin{aligned}\left[\frac{1}{2\varepsilon}\right]\left|\sum_{j=1}^{n_0}\sum_{\substack{k\\F_k\subseteq G_j}} m(F_k)(f(y_k)-f(x_j))\right|&=\left|\sum_{j=1}^{n_0}\sum_{\substack{k\\F_k\subseteq G_j}} m(F_k)\left(\left[\frac{1}{2\varepsilon}\right](f(y_k)-f(x_j))\right)\right|\\&\leqq|m|_i(F),\end{aligned}$$

and so

$$|\omega_{\alpha_1}-\omega_{\alpha_0}|\leqq\frac{1}{\left[\dfrac{1}{2\varepsilon}\right]}|m|_i\,(F)<4\varepsilon\,|m|_i\,(F)$$

since $0<\varepsilon<1/4$.

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Similarly we can prove that $|\omega_{\alpha_2}-\omega_{\alpha_0}|<4\varepsilon\,|m|_i\,(F)$. Thus $|\omega_{\alpha_1}-\omega_{\alpha_2}|<8\varepsilon\,|m|_i\,(F)$, and since ε is arbitrary the result follows.

In view of the above lemma, we can now make the following

Definition 2.2. Let $\mu\in M(X,E')$ and let $f\in C_{tb}(X,E)$. The *integral of f with respect to μ is defined by*

$$\int_X d\mu f=\lim_\alpha w_\alpha,$$

where the limit is taken over the indexing set $\mathcal{D}_X$.

Let $C(X)\otimes E$ denote the vector space spanned by the set of all functions of the form $\phi\otimes a$, where $\phi\in C(X)$, $a\in E$, and $(\phi\otimes a)(x)=\phi(x)a$ $(x\in X)$. It is straightforward to show that, if $\phi\in C(X)$ and $a\in E$, then $\int_X d\mu(\phi\otimes a)=\int_X\phi\,d\mu_a$. Also it is easy to show that the equation

$$\Phi(f)=\int_X d\mu f\quad (f\in C_{tb}(X,E))$$

defines a linear functional Φ on $C_{tb}(X,E)$.

Every topological vector space has a base of closed, balanced, shrinkable neighbourhoods of 0 (**6**). (A neighbourhood W of 0 in a *TVS* is said to be shrinkable if $\lambda\bar{W}\subseteq\operatorname{int}W$ for $0\leqq\lambda\leqq 1$.)

If $\mathcal{W}$ is a base of closed, balanced, shrinkable τ-neighbourhoods of 0 in E, then the Minkowski functional ρ_W of each $W\in\mathcal{W}$ is continuous (**6**, Theorem 5). We also note that, for each $W\in\mathcal{W}$, $W=\{x\in E:\rho_W(x)\leqq 1\}$, and that ρ_W is positive homogeneous.

Lemma 2.3. *Let $m\in M_i(X,E')$. Then*

(a) $|m|_i\in M(X)$;

(b) *there exists a $W_i\in\mathcal{W}$ such that*

$$\left|\int_X dmf\right|\leqq\int_X(\rho_{W_i}\circ f)\,d\,|m|_i\leqq\|f\|_i\,|m|_i\,(X)\quad (f\in C_{tb}(X,E)),$$

where $\|f\|_i=\sup_{x\in X}\rho_{W_i}(f(x))$.

Proof. (a) It follows immediately from the definition that $|m|_i$ is a bounded non-negative-valued set function on X. We show that $|m|_i$ is countably additive, as follows.

It is straightforward to show that $|m|_i$ is finitely additive. Let $\{A_k\}$ $(k=1,2,\dots)$ be a sequence of disjoining sets in $\mathcal{B}$ and suppose that $\bigcup_{k=1}^{\infty}A_k=A$. For any positive integer n,

$$|m|_i\,(A)\geqq|m|_i\left(\bigcup_{k=1}^{n}A_k\right)=\sum_{k=1}^{n}|m|_i\,(A_k),$$

and so

$$|m|_i\,(A)\geqq\sum_{k=1}^{\infty}|m|_i\,(A_k).\tag{1}$$

Let $\varepsilon > 0$. Then there exist a $\mathscr{B}$-partition $\{F_j\}$ $(1 \leqq j \leqq m)$ of A and a collection of points $\{a_j\}$ $(1 \leqq j \leqq m)$ with $\nu_i(a_j) \leqq 1$ such that

$$|m|_i\,(A) \leqq \left|\sum_{j=1}^{m} m_{a_j}(F_j)\right| + \varepsilon.$$

Since each m_{a_j} is countably additive and $\{F_j \cap A_k\colon k = 1, 2, \ldots\}$ is a partition of F_j, we have $m_{a_j}(F_j) = \sum_{k=1}^{\infty} m_{a_j}(F_j \cap A_k)$ $(1 \leqq j \leqq m)$. Hence

$$|m|_i\,(A) \leqq \left|\sum_{j=1}^{m}\sum_{k=1}^{\infty} m_{a_j}(F_j \cap A_k)\right| + \varepsilon \leqq \sum_{k=1}^{\infty} |m|_i\,(A_k) + \varepsilon. \tag{2}$$

Since ε is arbitrary, it follows from (1) and (2) that $|m|_i$ is countably additive.

To complete the proof of (a) we show that $|m|_i$ is regular. Let $\varepsilon > 0$ and $F \in \mathscr{B}$. There exist a $\mathscr{B}$-partition $\{F_j\}$ $(1 \leqq j \leqq m)$ of F and a collection $\{a_j\}$ $(1 \leqq j \leqq m)$ of points with $\nu_i(a_j) \leqq 1$ such that

$$|m|_i\,(F) \leqq \sum_{j=1}^{m} |m_{a_j}|\,(F_j) + \varepsilon.$$

Since each m_{a_j} is regular, there exist compact sets K_j $(j = 1, \ldots, m)$ such that $K_j \subseteq F_j$ and $|m_{a_j}|\,(F_j) < |m_{a_j}|\,(K_j) + \varepsilon/2^j$. Let $K = \bigcup_{j=1}^{m} K_j$. Then $K \subseteq F$ and

$$|m|_i\,(F) \leqq \sum_{j=1}^{m} |m_{a_j}|\,(K_j) + 2\varepsilon.$$

Moreover, for each $j = 1, \ldots, m$, there exists a $\mathscr{B}'$-partition of K_j, $\{G_{j,1}, \ldots, G_{j,t_j}\}$ say, such that

$$|m_{a_j}|\,(K_j) < \sum_{l=1}^{t_j} |m_{a_j}(G_{j,l})| + \varepsilon/2^j. \tag{$*$}$$

If $m_{a_j}(G_{j,l}) \neq 0$, we can write $|m_{a_j}(G_{j,l})| = m(G_{j,l})(a'_{j,l})$, where

$$a'_{j,l} = \frac{m_{a_j}(G_{j,l})}{|m_{a_j}(G_{j,l})|}\, a_j,$$

and we note that $\nu_i(a'_{j,l}) \leqq 1$ for all j and l.

If $m_{a_j}(G_{j,l}) = 0$ for some j and l, then the contribution of such terms to the summation in $(*)$ is zero, and so we define $a'_{j,l} = 0$ for these terms.

Thus

$$\begin{aligned}|m|_i\,(F) &< \sum_{j=1}^{n}\sum_{l=1}^{t_j} m(G_{j,l})(a'_{j,l}) + 3\varepsilon\\ &\leqq |m|_i\,(K) + 3\varepsilon.\end{aligned}$$

Since ε is arbitary it follows that $|m|_i\,(F) = \sup_{K \subseteq F} |m|_i\,(K)$, where K is compact. Similarly we can prove that $|m|_i\,(F) = \inf_{F \subseteq G} |m|_i\,(G)$, where G is open. Thus $|m|_i$ is regular, and so is an element of $M(X)$.

(b) Let W_i be a closed, balanced, shrinkable τ-neighbourhood of 0 in E such that $\{x \in E: \nu_i(x) \leqq 1\} \supseteq W_i = \{x \in E: \rho_{W_i}(x) \leqq 1\}$. For any $\varepsilon > 0$, there exist a $\mathcal{B}$-partition, $\{F_j: 1 \leqq j \leqq m\}$ say, of X, and points $x_j \in F_j$ such that

$$\left| \int_X dmf \right| \leqq \left| \sum_{j=1}^{m} m(F_j) f(x_j) \right| + \varepsilon$$

and

$$\left| \sum_{j=1}^{m} (\rho_{W_i} \circ f)(x_j) \, |m|_i \, (F_j) \right| \leqq \int_X (\rho_{W_i} \circ f)(x) \, d \, |m|_i + \varepsilon.$$

Let H_1 (resp. H_2) be the set of $j \in \{1, \ldots, m\}$ such that $\rho_{W_i}(f(x_j)) \neq 0 (\rho_{W_i}(f(x_j)) = 0)$. We note that, if $j \in H_2$, then $\nu_i(tf(x_j)) \leqq 1$ for all $t > 0$. Then

$$\begin{aligned} \left| \int_X dmf \right| &\leqq \sum_{j \in H_1} (\rho_{W_i} \circ f)(x_j) \left| m(F_j) \left(\frac{f(x_j)}{\rho_{W_i} \circ f(x_j)} \right) \right| \\ &\quad + \sum_{j \in H_2} \frac{\varepsilon}{|m|_i \, (X)} \left| m(F_j) \left(\frac{|m|_i \, (X) f(x_j)}{\varepsilon} \right) \right| + \varepsilon \\ &\leqq \sum_{j \in H_1} (\rho_{W_i} \circ f)(x_j) \left| m(F_j) \left(\frac{f(x_j)}{\rho_{W_i} \circ f(x_j)} \right) \right| + 2\varepsilon. \end{aligned}$$

We note that, if $|m|_i \, (X) = 0$, then the inequality we are seeking to establish holds trivially.

It follows that

$$\begin{aligned} \left| \int_X dmf \right| &\leqq \sum_{j \in H_1} (\rho_{W_i} \circ f)(x_j) \, |m|_i \, (F_j) + 2\varepsilon \\ &\leqq \int_X (\rho_{W_i} \circ f)(x) \, d \, |m|_i + 3\varepsilon, \end{aligned}$$

and so, since ε is arbitrary,

$$\left| \int_X dmf \right| \leqq \int_X (\rho_{W_i} \circ f) \, d \, |m|_i.$$

The other inequality is straightforward to prove.

3. The representation theorem

Definition 3.1. The pair (X, E) is said to have the β-density property if $C(X) \otimes E$ is β-dense in $C(X, E)$.

It has been proved in (5) that $C(X) \otimes E$ has the β-density property in each of the following cases:

(a) if X is a completely regular Hausdorff space of finite covering dimension and E is any topological vector space;

(b) if X is any completely regular Hausdorff space and E is a locally convex space.

In the sequel we shall assume that X is a completely regular Hausdorff space and that (X, E) has the β-density property.

Theorem 3.2. *For each $\mu \in M(X, E')$, the equation*

$$\Phi(f) = \int_X d\mu f \quad (f \in C_{tb}(X, E)) \tag{3}$$

defines a β-continuous linear functional Φ on $C_{tb}(X, E)$. Conversely, if Φ is a β-continuous linear functional on $C_{tb}(X, E)$, then there exists a unique μ in $M(X, E')$ such that Φ is given by (3).

Proof. Let $\mu \in M(X, E')$ and suppose that Φ is the linear functional on $C_{tb}(X, E)$ defined by (3). Now $\mu \in M_i(X, E')$ for some $i \in I$, and so, by Lemma 2.3(a), $|\mu|_i \in M(X)$. It follows from (**3**, Lemma 4.2) that the equation

$$\Phi_i(\phi) = \int_X \phi \, d|\mu|_i \quad (\phi \in C(X))$$

defines a β-continuous linear functional Φ_i on $C(X)$. Thus, using the notation of (**5**), there exists a function ψ in $B_0(X)$, $0 \leqq \psi \leqq 1$, such that $|\Phi_i(\phi)| \leqq 1$ whenever $\phi \in C(X)$ and $\|\psi\phi\| \leqq 1$. Let W_i be a closed, balanced, shrinkable τ-neighbourhood of 0 defined as in the proof of Lemma 2.3(b) and let $f \in U(\psi, W_i)$. Then, since $\|\psi(\rho_{W_i} \circ f)\| = \|\rho_{W_i}(\psi f)\| \leqq 1$, it follows from Lemma 2.3(b) that

$$|\Phi(f)| \leqq \int_X (\rho_{W_i} \circ f) \, d|\mu_i| \leqq 1.$$

Thus Φ is β-continuous.

Conversely, let Φ be a β-continuous linear functional on $C_{tb}(X, E)$. Then there exist a $\nu_i \in \mathscr{C}$ and a $\psi \in B_0(X)$ such that $|\Phi(f)| \leqq 1$ for all $f \in U(\psi, V_i)$, where $V_i = \{x \in E \colon \nu_i(x) \leqq 1\}$. For each $a \neq 0$ in E, let $\Phi_a(\phi) = \Phi(\phi \otimes a)(\phi \in C(X))$. It is straightforward to prove that Φ_a is a β-continuous linear functional on $C(X)$, and so, by (**3**, Lemma 4.5), there exists a unique μ_a in $M(X)$ such that

$$\Phi_a(\phi) = \int \phi \, d\mu_a \quad (\phi \in C(X)).$$

For each $F \in \mathscr{B}$, the functional $\mu(F)$, defined by

$$(\mu(F))(a) = \mu_a(F) \quad (a \in E),$$

is an element of E', as follows. It is straightforward to show that $\mu(F)$ is linear. Since Φ is β-continuous it is continuous with respect to the uniform topology on $C(X, E)$ and so there exists a closed, balanced, shrinkable τ-neighbourhood W of 0 in E, such that $|\Phi(\phi)| \leqq 1$ whenever $\phi \in U(1, W)$. Consider $h \in C(X)$, with $0 \leqq h \leqq 1$. Then $\rho_W(h(x)a) = h(x)\rho_W(a) \leqq \rho_W(a)$ for all $x \in X$, and so $h \otimes a \in U(1, W)$ whenever $\rho_W(a) \leqq 1$. Thus $|\Phi_a(h)| = |\Phi(h \otimes a)| \leqq 1$ whenever $\rho_W(a) \leqq 1$, which implies that $|\Phi_a(h)| \leqq \rho_W(a)$ for all $a \in E$. If $h \in C(X)$ and $\|h\| \leqq 1$, then $|\Phi_a(h)| \leqq 4\rho_W(a)$. Thus $\|\Phi_a\| \leqq 4\rho_W(a)$, and so from the inequalities

$$|\mu(F)(a)| = |\mu_a(F)| \leq \|\mu_a\| = \|\Phi_a\| \leqq 4\rho_W(a),$$

the continuity of $\mu(F)$ follows.

Thus $\mu: \mathscr{B} \to E'$, defined by

$$(\mu(F))(x) = \mu_x(F) \quad (F \in \mathscr{B}, x \in E),$$

is a finitely additive E'-valued set function on $\mathscr{B}$ with property (i). Moreover $|\mu|_t\,(X)$ is finite for some $t \in I$, as we now show.

There exists an $\mathscr{F}$-semi-norm ν_t in $\mathscr{C}$ such that

$$\{x \in E: \nu_t(x) \leqq 1\} \subseteq W = \{x \in E: \rho_W(x) \leqq 1\}.$$

Let $\{F_j\}$ $(1 \leqq j \leqq m)$ be a $\mathscr{B}$-partition of X and let $\{a_j\}$ be any collection of points in E such that $\nu_t(a_j) \leqq 1$ $(1 \leqq j \leqq m)$. We now proceed by using the same argument as the one given in (**8**, Lemma 4). Let $\varepsilon > 0$. Each μ_{a_j} is regular and so there exist compact sets $K_j \subseteq F_j$ such that $|\mu_{a_j}|\,(F_j \backslash K_j) < \varepsilon/2m$, and open sets $V_j \supseteq K_j$ such that $|\mu_{a_j}|\,(V_j \backslash K_j) < \varepsilon/2m$ for $j = 1, \ldots, m$; since the K_j's are disjoint compact sets and X is completely regular, the V_j's may be chosen so that $V_j \cap V_{j'} = \phi$ $(j \neq j')$. Choose functions g_j $(1 \leqq j \leqq m)$ in $C(X)$, $0 \leqq g_j \leqq 1$, such that $g_j(x) = 1$ for $x \in K_j$ and supp $g_j \subseteq V_j$. Let $h = \sum_{j=1}^{m} g_j \otimes a_j$. Then $h \in C(X, E)$ and $\nu_t(h(x)) \leqq 1$ for all $x \in X$, and so $|\Phi(h)| \leqq 1$. By using the above inequalities as in the proof of (**8**, Lemma 4) we have that

$$\left| \sum_{j=1}^{m} \mu(F_j) a_j \right| < \varepsilon + |\Phi(h)| \leqq \varepsilon + 1.$$

Since ε is arbitrary, it follows that μ satisfies condition (ii).

Let g be any function in $C(X) \otimes E$. Then $g = \sum_{j=1}^{p} \phi_j \otimes b_j$, where $\phi_j \in C(X)$, and $b_j \in E$, and so

$$\Phi(g) = \sum_{j=1}^{p} \Phi(\phi_j \otimes b_j) = \sum_{j=1}^{p} \int_X \phi_j \, d\mu_{b_j} = \sum_{j=1}^{p} \int_X d\mu(\phi_j \otimes b_j) = \int_X d\mu g.$$

Since $C(X) \otimes E$ is β-dense in $C_{tb}(X, E)$, it follows from the above that $\Phi(f) = \int_X d\mu f$ for all $f \in C_{tb}(X, E)$.

Finally, μ is unique, as we now show. Suppose that there is an m in $M(X, E')$ such that $\Phi(f) = \int_X dmf$ for all $f \in C_{tb}(X, E)$. In particular, for any $\phi \in C(X)$ and $x \in E$, $\int_X d\mu(\phi \otimes x) = \int_X dm(\phi \otimes x)$. Hence $\int \phi \, d\mu_x = \int \phi \, dm_x$ for all $\phi \in C(X)$, and so, by (**3**, Lemma 4.5), $\mu_x = m_x$. Thus, for any Borel set F and any x in E, $\mu(F)(x) = \mu_x(F) = m_x(F) = m(F)(x)$. It follows that $\mu(F) = m(F)$, and so $\mu = m$, as required.

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