This department welcomes short notes and problems believed to be new. Contributors should include solutions where known, or background material in case the problem is unsolved. Send all communications concerning this department to G. D. Findlay, Department of Mathematics, McGill University, Montreal, P. Q.

ALMOST ALL TOURNAMENTS ARE IRREDUCIBLE

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(received August 28, 1961)

Given a set of n points, with each pair of distinct points joined by a line that is oriented towards exactly one of the points, then the resulting configuration is called a (round-robin) tournament. A tournament is reducible if the points can be separated into two non-empty subsets, A and B, such that every line that joins a point in A to a point in B is oriented towards the point in B. If a tournament is not reducible it is called irreducible. The object of this note is to derive an approximation for $P(n)$, the probability that a tournament on n points, chosen at random from the set of $2^{\binom{n}{2}}$ possible ones, will be irreducible. $P(1) = 1$, by definition.

If a given tournament is reducible, then there exists a unique, maximal, proper subset of the points with the property that these points, together with the lines joining points within this subset, form an irreducible sub-tournament, and all lines, which join points not in this subset with points which are in it, are oriented towards the latter points. To see that this is so, consider the subset of points B, as defined above, along with the lines joining points in B. If this is an irreducible sub-tournament we are finished. If it is reducible further, separate the points of B into two classes which satisfy the conditions for reducibility and apply the criterion again. This process, repeated as often as necessary, will ultimately yield the desired subset.

The probability that this subset consists of j points, $1 \leq j \leq n-1$, equals

$$\binom{n}{j} \frac{P(j)\, 2^{\binom{j}{2}}\, 2^{\binom{n-j}{2}}}{2^{\binom{n}{2}}} = \binom{n}{j} \frac{P(j)}{2^{j(n-j)}}$$

since $P(j)\, 2^{\binom{j}{2}}$ irreducible tournaments may be formed on each of the $\binom{n}{j}$ subsets of j points, and having chosen one of these there remain only $\binom{n-j}{2}$ lines whose direction needs to be chosen.

As these cases are mutually exclusive and exhaustive, summing over j gives

$$(1) \qquad P(n) = 1 - \sum_{j=1}^{n-1} \binom{n}{j} \frac{P(j)}{2^{j(n-j)}} .$$

In general, the main contribution to the sum arises from the two extreme terms, so to approximate $P(n)$ we need bounds for the remaining terms.

LEMMA 1:

$$\frac{n+1}{n(n+1-j)\, 2^{j-1}} < \frac{1}{n+1} \quad \text{for } n \geq 13 \text{ and } j = 2, 3, \ldots, n-2.$$

For fixed admissible values of n, the case $j = 2$ is equivalent to showing that $0 < n^2 - 4n - 1$, which is obviously so. Since the left member of the inequality is a decreasing function for $j = 2, 3, \ldots, n-2$, the statement follows.

LEMMA 2:

$$\binom{n}{j} \frac{1}{2^{j(n-j)}} < \frac{1}{n\, 2^{n-1}} \quad \text{for } n \geq 13 \text{ and } j = 2, 3, \ldots, n-2.$$

https://doi.org/10.4153/CMB-1962-010-4 Published online by Cambridge University Press

When $n = 13$ the inequality may be verified directly for $j = 2, 3, \ldots, 11$. Now assume that the lemma has been established for each value of n, and all admissible values of j, up to $n = k$. Then, for $n = k+1$ and $j = 2, \ldots, k-2$, we have

$$\binom{k+1}{j} \frac{1}{2^{j(k+1-j)}} = \frac{k+1}{(k+1-j)2^j} \cdot \binom{k}{j} \frac{1}{2^{j(k-j)}}$$

$$< \frac{k+1}{(k+1-j)2^j} \cdot \frac{1}{k\, 2^{k-1}} \qquad \text{by hypothesis}$$

$$< \frac{1}{(k+1)2^k} \qquad \text{by Lemma 1.}$$

The only value of j, for $n = k+1$, not included in this argument is $j = k-1$. But this case is equivalent to showing that $k(k+1)^2 < 2^{k-1}$, which is easily seen to be so for the values of k involved.

Therefore, for $n \geq 13$,

$$\text{(2)} \qquad P(n) > 1 - \frac{2n}{2^{n-1}} - \sum_{j=2}^{n-2} \binom{n}{j} \frac{1}{2^{j(n-j)}}$$

$$> 1 - \frac{2n}{2^{n-1}} - \frac{n-3}{n\, 2^{n-1}} \qquad \text{(by lemma 2)}$$

$$> 1 - \frac{2n+1}{2^{n-1}} .$$

And, for $n-1 \geq 13$,

$$\text{(3)} \qquad P(n) < 1 - \frac{n}{2^{n-1}} - \frac{n\, P(n-1)}{2^{n-1}}$$

https://doi.org/10.4153/CMB-1962-010-4 Published online by Cambridge University Press

$$< 1 - \frac{n}{2^{n-1}} - \frac{n}{2^{n-1}}\left[1 - \frac{2(n-1)+1}{2^{n-2}}\right] \qquad \text{(by (2))}$$

$$= 1 - \frac{2n}{2^{n-1}} + \frac{n(2n-1)}{2^{n-2}} \cdot \frac{1}{2^{n-1}}$$

$$< 1 - \frac{2n-1}{2^{n-1}}, \quad \text{since} \quad \frac{n(2n-1)}{2^{n-2}} < 1 \quad \text{for} \quad n \geq 10.$$

Hence, if $Q(n) = 1 - P(n)$, we have the following

THEOREM 1:

$$\left| Q(n) - \frac{2n}{2^{n-1}} \right| < \frac{1}{2^{n-1}} \quad \text{for} \quad n \geq 14 .$$

Direct calculation, using (1) indicates that Theorem 1 is actually valid for $n \geq 8$.

Similar arguments could be applied to generalizations of tournaments in which the points are split into several classes and only those points which are in different classes are joined by an oriented line.

By virtue of the fact that in a tournament irreducibility is equivalent to the existence of an oriented circuit through all the points, from Camion (1) and Roy (2), Theorem 1 provides an approximate answer to the following problem:

n players of equal strength play a round-robin tournament in which there are no draws. What is the probability that after the tournament it will be possible to label the players $A_1, A_2, \ldots, A_n$, in such a way that A_1 has beaten A_2, A_2 has beaten A_3, ..., A_{n-1} has beaten A_n, and A_n has beaten A_1 ?

REFERENCES

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